POLYNOMIAL ESTIMATES, EXPONENTIAL CURVES AND DIOPHANTINE APPROXIMATION

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ABSTRACT. Let $\alpha \in (0,1)\setminus\mathbb{Q}$ and $K = \{(e^z, e^{\alpha z}) : |z| \le 1\} \subset \mathbb{C}^2$. If P is a polynomial of degree n in $\mathbb{C}^2$, normalized by $\|P\|_K = 1$, we obtain sharp estimates for $\|P\|_{\Delta^2}$ in terms of n, where Δ^2 is the closed unit bidisk. For most α, we show that $\sup_P \|P\|_{\Delta^2} \le \exp(Cn^2 \log n)$. However, for α in a subset $\mathcal{S}$ of the Liouville numbers, $\sup_P \|P\|_{\Delta^2}$ has bigger order of growth. We give a precise characterization of the set $\mathcal{S}$ and study its properties.

1. Introduction

The behavior of polynomials along graphs of entire transcendental functions was recently studied in [CP1, CP2, CP3] and later and in more general situations in [Br]. If f is an entire transcendental function and $P \in \mathbb{C}[z,w]$ is a polynomial, the growth of the function $P(z, f(z))$ can be estimated in terms of its uniform norm on the unit disk and the degree of P. Such an estimate is called a Bernstein inequality and it has important applications (see [CP3], [Br] and references therein). The growth estimate yields bounds on the maximum number of zeros in a fixed disk of the functions $P(z, f(z))$, depending only on the degree of P and f [CP2, CP3]. This was used in [CP3] to derive important properties of the set of algebraic numbers where the values of f are also algebraic.

Let Δ, resp. Δ^2, denote the closed unit disk, resp. bidisk, and let $\mathcal{P}_n$ be the space of polynomials $P \in \mathbb{C}[z,w]$ of degree at most n. The methods introduced in [CP1, CP2, CP3] involve the study of the transcendence measures

$$E_n(f) = \sup \|P\|_{\Delta^2},$$

where $P \in \mathcal{P}_n$ is normalized by $|P(z, f(z))| \le 1$ for $z \in \Delta$. We showed in [CP3] that for any transcendental function f of finite positive order, $\log E_n(f)$ grows like $n^2 \log n$, while the maximum number of zeros in a fixed disk of the functions $P(z, f(z))$, $P \in \mathcal{P}_n$, grows like n^2, at least for an infinite sequence of natural numbers n. Moreover, if f verifies certain growth conditions (and in particular if f is a quasipolynomial), we proved that these estimates hold for every n (see [CP3, Section 7]).

It is an interesting open problem to study the behavior of polynomials along the curve

$$\Gamma = \{(g(z), f(z)) : z \in \mathbb{C}\} \subset \mathbb{C}^2,$$

where g, f are algebraically independent entire functions. Let

$$K = \{(g(z), f(z)) : z \in \Delta\} \subset \mathbb{C}^2.$$

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Note that K is pluripolar. Since the functions g, f are algebraically independent, it follows that the uniform norm $\|\cdot\|_K$ is a norm on each vector space $\mathcal{P}_n$. As $\mathcal{P}_n$ are finite dimensional we have

$$E_n(\Gamma) = E_n(g,f) := \sup\{\|P\|_{\Delta^2} : \ P \in \mathcal{P}_n, \ \|P\|_K \leq 1\} < +\infty, \ \ \forall\, n \geq 0.$$

Once upper bounds on $E_n(\Gamma)$ are known, one can use the classical Bernstein-Walsh inequality as in [CP1] to estimate the growth of any polynomial $P \in \mathcal{P}_n$ at every point in terms of n and $\|P\|_K$, despite the pluripolarity of K:

$$(1) \qquad |P(z,w)| \leq \|P\|_K E_n(\Gamma) \exp(n \log^+ \max\{|z|, |w|\}), \ (z,w) \in \mathbb{C}^2.$$

In some cases when g, f have different orders of growth certain upper bounds on $E_n(\Gamma)$ can be derived using [Br, Theorem 2.3].

In this note we consider the simplest case of the exponential curves

$$\Gamma = \{(e^z, e^{\alpha z}) : \ z \in \mathbb{C}\} \subset \mathbb{C}^2,$$

where α is a real irrational number. The functions e^z and $e^{\alpha z}$ have the same order of growth and the same growth of valencies. We denote in the sequel $E_n(\alpha) := E_n(\Gamma)$. By results of Tijdeman, it is known that, regardless of α, the maximum number of zeros in a fixed disk of the functions $P(e^z, e^{\alpha z})$, $P \in \mathcal{P}_n$, grows like n^2 for all n (see [T], [B]).

We obtain here sharp estimates for $E_n(\alpha)$ and show that these estimates depend on the rate of Diophantine approximation of α. In contrast to the case mentioned above when Γ was the graph of a quasipolynomial, we see that: 1) $E_n(\alpha)$ may grow much faster than the maximal number of zeros in a fixed disk of the functions $P(e^z, e^{\alpha z})$, $P \in \mathcal{P}_n$; 2) transcendental number theory is needed to get estimates on $E_n(g,f)$ for all n.

We now state our results more precisely. Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and

$$e_n(\alpha) = \log E_n(\alpha).$$

Throughout the paper we denote by p_s/q_s the convergents to α given by its continued fractions expansion (see Section 2), and by $[x]$ the greatest integer $\leq x$. We have the following:

Theorem 1.1. *Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and let p_s/q_s, $s \geq 0$, be the convergents to α given by its continued fractions expansion. If $q_s \leq n < q_{s+1}$ then*

$$\max\left\{\frac{n^2 \log n}{2} - n^2, \ \left[\frac{n}{q_s}\right] \log q_{s+1} - n\right\} \leq e_n(\alpha) \leq \frac{n^2 \log n}{2} + 9n^2 + \frac{n}{q_s} \log q_{s+1}.$$

Theorem 1.1 implies a connection between $E_n(\alpha)$ and Diophantine approximation. Namely, $E_n(\alpha)$ provides a lower bound for the rate of approximation of α by rational numbers with denominator at most n.

Corollary 1.2. *Let $\alpha \in (0,1) \setminus \mathbb{Q}$. For every $n \geq 1$ we have*

$$\min_{1 \leq k \leq n} dist(k\alpha, \mathbb{Z}) \geq (2e^n E_n(\alpha))^{-1}.$$

A number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is called Diophantine of order μ, $\mu \geq 2$, if there exists $\varepsilon > 0$ so that $|\alpha - p/q| > \varepsilon q^{-\mu}$, for every rational number p/q. We denote by $\mathcal{D}(\mu)$ the set of such numbers. Then

$$\alpha \in \mathcal{D}(\mu) \Longleftrightarrow q_{s+1} \leq C q_s^{\mu-1},\ \forall\, s \geq 0, \tag{2}$$

for some constant $C > 0$, where p_s/q_s are the convergents to α (see e.g. [Mil, Appendix C]). We let

$$\mathcal{D}(\infty) = \bigcup_{\mu \geq 2} \mathcal{D}(\mu)\ ,\ \mathcal{L} = \mathbb{R} \setminus (\mathbb{Q} \cup \mathcal{D}(\infty)).$$

$\mathcal{L}$ is called the set of Liouville numbers. It has Hausdorff dimension zero (see e.g. [Mil, Lemma C.4]). By a classical theorem of Liouville, any algebraic number of degree μ belongs to $\mathcal{D}(\mu)$. Hence all Liouville numbers are transcendental.

Corollary 1.3. *If $\alpha \in (0,1)$ is Diophantine of order μ then for $n \geq 1$*

$$\frac{n^2 \log n}{2} - n^2 \leq e_n(\alpha) \leq \frac{n^2 \log n}{2} + 9n^2 + Cn,$$

where $C > 0$ is a constant depending on α.

Using Theorem 1.1, it is in fact possible to obtain a precise characterization of the numbers α for which $e_n(\alpha)$ grows like $n^2 \log n$:

Corollary 1.4. *If $\alpha \in (0,1) \setminus \mathbb{Q}$ then*

$$\frac{e_n(\alpha)}{n^2 \log n} = O(1) \Longleftrightarrow \frac{e_{q_s}(\alpha)}{q_s^2 \log q_s} = O(1) \Longleftrightarrow \frac{\log q_{s+1}}{q_s^2 \log q_s} = O(1).$$

Theorem 1.1 and its corollaries are proved in Section 2. We also review there the necessary results about continued fractions and Diophantine approximation.

Corollary 1.4 leads us to consider the following set of irrational numbers:

$$\mathcal{S} = \left\{ \alpha \in (0,1) \setminus \mathbb{Q} :\ \limsup_{s \to +\infty} \frac{\log q_{s+1}}{q_s^2 \log q_s} = +\infty \right\}.$$

If $\alpha \in \mathcal{S}$ then $e_n(\alpha)$ grows faster than $n^2 \log n$ for a sequence of integers $n = q_{s_j}$, where $\log q_{s_j+1}/(q_{s_j}^2 \log q_{s_j}) \to +\infty$.

It follows from (2) that Liouville numbers can be characterized as follows:

$$\alpha \in \mathcal{L} \Longleftrightarrow \limsup_{s \to +\infty} \frac{\log q_{s+1}}{\log q_s} = +\infty.$$

Hence $\mathcal{S} \subset \mathcal{L}$. In fact, we see from the recursive formulas for $\{q_s\}$ (see Section 2) and from Theorem 2.1 that $\mathcal{S}$ is a "small" subset of $\mathcal{L}$ consisting of transcendental numbers which are very well approximated by rationals.

We study the set $\mathcal{S}$ in Section 3. We prove that $\mathcal{S}$ contains a dense G_δ set, hence it is uncountable. We also prove that it has Hausdorff h-measure 0, for a class of rapidly increasing functions h. We also discuss the connection between $\mathcal{S}$ and certain polar sets of Liouville numbers defined in terms of the growth of the denominators q_s given by their continued fractions expansion.

2. Proof of Theorem 1.1

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then α has a unique representation as an (infinite) continued fraction

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}} ,$$

where all a_j are integers and $a_j \geq 1$ for $j \geq 1$ (see e.g. [Khi, Theorem 14]). The rational number

$$\frac{p_s}{q_s} = [a_0; a_1, a_2, \dots, a_s] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots_{a_{s-1} + \frac{1}{a_s}}}}$$

is called the s-th convergent to α. Viewing p_s, q_s as polynomials in the variables $a_0, \dots, a_s$ one has the following recursive formulas [Khi, Theorem 1]:

$$p_s = a_s p_{s-1} + p_{s-2} \ , \ q_s = a_s q_{s-1} + q_{s-2}, \ s \geq 1,$$

where $p_0 = a_0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$. Moreover [Khi, Theorem 2],

$$q_s p_{s-1} - p_s q_{s-1} = (-1)^s,$$

which implies that the fraction $p_s/q_s \in \mathbb{Q}$ is irreducible. For $s \geq 1$, $q_{s+1} > q_s$ and $q_s \geq 2^{(s-1)/2}$ [Khi, Theorem 12]. We now recall a few properties of the convergents p_s/q_s, which will be useful later.

Theorem 2.1. [Khi, Theorems 9 and 13] *For $s \geq 0$,*

$$(2q_{s+1})^{-1} \leq (q_{s+1} + q_s)^{-1} < |q_s \alpha - p_s| < q_{s+1}^{-1}.$$

By a theorem of Lagrange, continued fractions provide the best rational approximations to α:

Theorem 2.2. [Sch, Theorem 5E] *For $s \geq 0$, $|q_s\alpha - p_s| > |q_{s+1}\alpha - p_{s+1}|$. Moreover, if $s \geq 1$, $1 \leq q \leq q_s$, and if $(p,q) \neq (p_s, q_s)$, $(p,q) \neq (p_{s-1}, q_{s-1})$ then $|q\alpha - p| > |q_{s-1}\alpha - p_{s-1}|$.*

Conversely, if $|d\alpha - c| > |b\alpha - a|$ for each integers c, d with $1 \leq d \leq b$, $c/d \neq a/b$ then a/b is a convergent to α ([Khi, Theorem 16]). Another result of this kind is the following theorem of Legendre:

Theorem 2.3. [Sch, Theorem 5C] *If p, q are relatively prime integers, $q > 0$ and $|q\alpha - p| < (2q)^{-1}$ then p/q is a convergent to α.*

Next we develop certain estimates which will be needed in the proof of Theorem 1.1. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let p_s/q_s, $s \geq 0$, be the convergents to α given by its continued fractions expansion. For $k \in \mathbb{N}$ we denote by $(k\alpha)$ the (unique) closest integer to α, so

$$dist(k\alpha, \mathbb{Z}) = |k\alpha - (k\alpha)| < 1/2.$$

Lemma 2.4. *Let $k, x, y \in \mathbb{Z}$, $x \leq y$, $k \geq 1$. Then (with $0^0 := 1$)*

$$\prod_{j=x}^{y} |j - k\alpha| \geq \begin{cases} \frac{1}{2}\left(\frac{y-x}{e}\right)^{y-x}, \text{ if } (k\alpha) \notin [x,y], \\ \left(\frac{y-x}{2e}\right)^{y-x} dist(k\alpha, \mathbb{Z}), \text{ if } x \leq (k\alpha) \leq y. \end{cases}$$

Proof. By Stirling's formula we have

$$e^{7/8} \le \frac{m!}{(m/e)^m \sqrt{m}} \le e, \ m \ge 1,$$

This implies

$$\prod_{j=1}^{m} \left(j - \frac{1}{2} \right) = \frac{(2m)!}{2^{2m} m!} > (m/e)^m.$$

Let $j_0 = (k\alpha)$. If $j \neq j_0$ then

$$|j - k\alpha| \ge |j - j_0| - |j_0 - k\alpha| > |j - j_0| - 1/2.$$

Using this we obtain for $j_0 < x$,

$$\prod_{j=x}^{y} |j - k\alpha| \ge \prod_{j=x}^{y} (j - j_0 - 1/2) = \prod_{j=0}^{y-x} (j + x - j_0 - 1/2) \ge \frac{1}{2} (y - x)!.$$

Similarly, if $y < j_0$,

$$\prod_{j=x}^{y} |j - k\alpha| \ge \prod_{j=x}^{y} (j_0 - j - 1/2) = \prod_{j=0}^{y-x} (j + j_0 - y - 1/2) \ge \frac{1}{2} (y - x)!.$$

We assume now that $x \le j_0 \le y$. Then, as before,

$$\begin{aligned} \prod_{j=x}^{y} |j - k\alpha| &\ge \prod_{j=x}^{j_0-1} (j_0 - j - 1/2) \prod_{j=j_0+1}^{y} (j - j_0 - 1/2) \ \ dist(k\alpha, \mathbb{Z}) \\ &= \prod_{j=1}^{j_0-x} (j - 1/2) \prod_{j=1}^{y-j_0} (j - 1/2) \ \ dist(k\alpha, \mathbb{Z}) \\ &\ge \left(\frac{j_0 - x}{e} \right)^{j_0 - x} \left(\frac{y - j_0}{e} \right)^{y - j_0} dist(k\alpha, \mathbb{Z}). \end{aligned}$$

The function $f(t) = (t - x)\log(t - x) + (y - t)\log(y - t)$ attains its minimum on the interval $[x, y]$ at $t = (x + y)/2$, so

$$f(t) \ge (y - x) \log \left(\frac{y - x}{2} \right).$$

This implies

$$\prod_{j=x}^{y} |j - k\alpha| \ge \left(\frac{y - x}{2e} \right)^{y - x} dist(k\alpha, \mathbb{Z}).$$

□

The following result provides lower estimates for the function

$$D_\alpha(n) = \prod_{k=1}^{n} dist(k\alpha, \mathbb{Z}).$$

Lemma 2.5. *If $q_s \le n < q_{s+1}$ then $D_\alpha(n) \ge (2n)^{-n} q_{s+1}^{-n/q_s}$.*

Proof. We consider the sets

$$S_j = \left\{ k \in \mathbb{N} : \, k \leq n, \, \frac{(k\alpha)}{k} = \frac{p_j}{q_j} \right\}, \, 0 \leq j \leq s, \quad S_{s+1} = ([1, n] \cap \mathbb{N}) \setminus \bigcup_{j=0}^{s} S_j.$$

For $1 \leq k \leq n$, suppose that

$$dist(k\alpha, \mathbb{Z}) = |k\alpha - (k\alpha)| < (2k)^{-1}.$$

Theorem 2.3 implies that $(k\alpha)/k = p_j/q_j$ for some $j \leq s$, so $k \in S_j$. We conclude that for $k \in S_{s+1}$

$$dist(k\alpha, \mathbb{Z}) \geq (2k)^{-1} \geq (2n)^{-1}.$$

Hence

$$\prod_{k \in S_{s+1}} dist(k\alpha, \mathbb{Z}) \geq (2n)^{-|S_{s+1}|}.$$

Since p_j/q_j is irreducible it follows that the sets S_j, $j \leq s$, are disjoint and

$$dist(k\alpha, \mathbb{Z}) = |k\alpha - (k\alpha)| \geq |q_j\alpha - p_j| \geq (2q_{j+1})^{-1}, \; k \in S_j.$$

Here the last inequality follows by Theorem 2.1. Moreover, if $k \in S_s$ then $q_s | k$, so $|S_s| \leq n/q_s$. Hence

$$\prod_{k \in S_j} dist(k\alpha, \mathbb{Z}) \geq (2q_{j+1})^{-|S_j|} \geq (2n)^{-|S_j|}, \; 0 \leq j < s,$$

$$\prod_{k \in S_s} dist(k\alpha, \mathbb{Z}) \geq (2q_{s+1})^{-|S_s|} \geq 2^{-|S_s|} q_{s+1}^{-n/q_s}.$$

Note that $|S_0| + \cdots + |S_{s+1}| = n$. We conclude that

$$D_\alpha(n) = \prod_{j=0}^{s+1} \prod_{k \in S_j} dist(k\alpha, \mathbb{Z}) \geq (2n)^{-n} q_{s+1}^{-n/q_s}.$$

□

Lemma 2.6. *If $q_s \leq n < q_{s+1}$ and $0 \leq m \leq n$ then*

$$D_\alpha(m) D_\alpha(n-m) \geq 2^{-n} n^{-2n} q_{s+1}^{-n/q_s}.$$

Proof. There exist integers j, l so that $q_j \leq m < q_{j+1}$ and $q_l \leq n - m < q_{l+1}$. Note that $m^m (n-m)^{(n-m)} \leq n^n$, so by Lemma 2.5,

$$D_\alpha(m) D_\alpha(n-m) \geq (2n)^{-n} q_{j+1}^{-m/q_j} q_{l+1}^{-(n-m)/q_l}.$$

If $\max\{j, l\} < s$ then

$$q_{j+1}^{-m/q_j} q_{l+1}^{-(n-m)/q_l} \geq q_s^{-m/q_j - (n-m)/q_l} \geq n^{-n}.$$

If $l = s > j$ then

$$q_{j+1}^{-m/q_j} q_{l+1}^{-(n-m)/q_l} \geq n^{-n} q_{s+1}^{-n/q_s}.$$

Finally, if $j = l = s$ then

$$q_{j+1}^{-m/q_j} q_{l+1}^{-(n-m)/q_l} = q_{s+1}^{-n/q_s}.$$

□

Proof of Theorem 1.1. Recall that $\dim \mathcal{P}_n = N + 1$, where $N = (n^2 + 3n)/2$.

We start by proving the upper bound for $e_n(\alpha)$. Let us introduce the following notation. For any polynomial $R(\lambda) = \sum_{j=0}^{m} c_j \lambda^j$ we denote by D_R the constant-coefficient differential operator

$$D_R = R\left(\frac{d}{dz}\right) = \sum_{j=0}^{m} c_j \frac{d^j}{dz^j}\,.$$

Then for any integer $t \geq 0$ and any $a \in \mathbb{C}$ we have

$$D_R[z^t e^{az}]\,\big|_{z=0} = \sum_{j \geq t} c_j \frac{j!}{(j-t)!}\, a^{j-t} = \frac{d^t R}{d\lambda^t}\bigg|_{\lambda=a} = R^{(t)}(a). \tag{3}$$

Fix now $P \in \mathcal{P}_n$, $n \geq 1$, with $\|P\|_K \leq 1$. We write

$$P(z,w) = \sum_{j+k \leq n} c_{jk} z^j w^k,\ f(z) := P(e^z, e^{\alpha z}) = \sum_{j+k \leq n} c_{jk} e^{(j+k\alpha)z}.$$

We will estimate the coefficients c_{lm} of P by using the differential operators given by the polynomials of degree N,

$$R_{lm}(\lambda) = \prod_{j+k \leq n, (j,k) \neq (l,m)} (\lambda - j - k\alpha) = \sum_{t=0}^{N} a_t \lambda^t.$$

Since the coefficients a_t are elementary symmetric functions of the roots of R_{lm} it follows that for $\lambda \geq 0$

$$\sum_{t=0}^{N} |a_t| \lambda^t \leq \prod_{j+k \leq n, (j,k) \neq (l,m)} (\lambda + |j + k\alpha|) \leq (\lambda + n)^N,$$

where for the last inequality we used $|j + k\alpha| \leq j + k \leq n$, since $0 < \alpha < 1$.

By (3) we have

$$D_{R_{lm}} f(z)\,|_{z=0} = c_{lm}\beta_{lm}\,,\quad \beta_{lm} = \prod_{j+k \leq n, (j,k) \neq (l,m)} (l - j + (m-k)\alpha).$$

By Cauchy's estimates $|f^{(t)}(0)| \leq t! \leq N^t$ for $t \leq N$, so we obtain

$$|\, D_{R_{lm}} f(z)|_{z=0}\,| = \left| \sum_{t=0}^{N} a_t f^{(t)}(0) \right| \leq \sum_{t=0}^{N} |a_t| N^t \leq (N+n)^N.$$

Therefore

$$\log(|c_{lm}\beta_{lm}|) \leq N \log(N+n) \leq n^2 \log n + 3.7n^2. \tag{4}$$

Next we obtain lower estimates on $|\beta_{lm}|$. We have

$$|\beta_{lm}| \geq \prod_{k=0, k\neq m}^{n} \prod_{j=0}^{n-k} |l - j + (m-k)\alpha| = A_1 A_2,$$

where

$$A_1 = \prod_{k=0}^{m-1} \prod_{j=0}^{n-k} |j - l - (m-k)\alpha| = \prod_{k=1}^{m} \prod_{j=-l}^{n-m-l+k} |j - k\alpha|,$$

$$A_2 = \prod_{k=m+1}^{n} \prod_{j=0}^{n-k} |l-j-(k-m)\alpha| = \prod_{k=1}^{n-m} \prod_{j=l+m-n+k}^{l} |j-k\alpha|.$$

By Lemma 2.4

$$\begin{aligned} A_1 &\geq D_\alpha(m) \prod_{k=1}^{m} \left(\frac{n-m+k}{2e}\right)^{n-m+k}, \\ A_2 &\geq D_\alpha(n-m) \prod_{k=1}^{n-m} \left(\frac{n-m-k}{2e}\right)^{n-m-k}. \end{aligned}$$

Thus, using Lemma 2.6,

$$\begin{aligned} |\beta_{lm}| &\geq D_\alpha(m) D_\alpha(n-m) \prod_{k=n-m+1}^{n} \left(\frac{k}{2e}\right)^k \times \prod_{k=0}^{n-m-1} \left(\frac{k}{2e}\right)^k \\ &\geq 2^{-n} n^{-2n} q_{s+1}^{-n/q_s} \left(\frac{2e}{n-m}\right)^{n-m} \prod_{k=1}^{n} \left(\frac{k}{2e}\right)^k \\ &\geq 2^{-n} n^{-3n} (2e)^{-n^2} q_{s+1}^{-n/q_s} \prod_{k=1}^{n} k^k. \end{aligned}$$

We have (see e.g. [CP1, Lemma 2.1])

$$\sum_{k=1}^{n} k \log k \geq \frac{n^2 \log n}{2} - \frac{n^2}{4}.$$

This yields

$$\log|\beta_{lm}| \geq \frac{n^2 \log n}{2} - 4.2n^2 - \frac{n}{q_s} \log q_{s+1}.$$

Using (4) we obtain

$$\log|c_{lm}| \leq \frac{n^2 \log n}{2} + 7.9n^2 + \frac{n}{q_s} \log q_{s+1}.$$

Since $\|P\|_{\Delta^2} \leq \sum |c_{jk}| \leq (N+1) \max |c_{jk}|$, we conclude that

$$e_n(\alpha) \leq \frac{n^2 \log n}{2} + 9n^2 + \frac{n}{q_s} \log q_{s+1}.$$

We now proceed to prove the lower bound for $e_n(\alpha)$. There exists a non-trivial polynomial $P \in \mathcal{P}_n$ so that the function $P(e^z, e^{\alpha z})$ has a zero of order at least $N = (n^2+3n)/2$ at 0. Using (1) and repeating the argument in the proof of [CP1, Proposition 1.3], we obtain that

$$e_n(\alpha) \geq N \log n - n^2.$$

Consider the polynomial $P(z,w) = (z^{p_s} - w^{q_s})^{[n/q_s]}$. Since $0 < \alpha < 1$, we have $0 \leq p_s \leq q_s$ for every s, so $P \in \mathcal{P}_n$. Note that $\|P\|_{\Delta^2} = 2^{[n/q_s]}$. If $|z| \leq 1$ we have by Theorem 2.1

$$|(q_s\alpha - p_s)z| \leq q_{s+1}^{-1} \leq 1.$$

Using that $|1-e^{\zeta}|\le 2|\zeta|$ for $|\zeta|\le 1$, we obtain

$$|P(e^z,e^{\alpha z})|\le \left|e^{p_s z}\left(1-e^{(q_s\alpha-p_s)z}\right)\right|^{[n/q_s]}\le e^n(2q_{s+1}^{-1})^{[n/q_s]},\ |z|\le 1.$$

Therefore

$$E_n(\alpha)\ge \|P\|_{\Delta^2}/\|P\|_K\ge q_{s+1}^{[n/q_s]}e^{-n},$$

and the proof is complete. □

Proof of Corollary 1.2. Theorems 2.2 and 2.1 show that if $q_s\le n<q_{s+1}$ then

$$\min_{1\le k\le n} dist(k\alpha,\mathbb{Z})=|q_s\alpha-p_s|\ge 1/(2q_{s+1}),$$

while the lower bound for $e_n(\alpha)$ from Theorem 1.1 implies $\log q_{s+1}\le e_n(\alpha)+n$. It follows that for all $n\ge 1$ we have

$$\min_{1\le k\le n} dist(k\alpha,\mathbb{Z})\ge (2e^nE_n(\alpha))^{-1}.\quad \square$$

Proof of Corollary 1.3. The upper estimate follows immediately from Theorem 1.1, since by (2)

$$\frac{\log q_{s+1}}{q_s}\le \frac{\log C}{q_s}+(\mu-1)\frac{\log q_s}{q_s}\le \log C+\frac{\mu-1}{2}\ .\quad \square$$

Proof of Corollary 1.4. Assume first that $e_{q_s}(\alpha)\le Cq_s^2\log q_s$ for all s, where C is a constant. By the lower estimate in Theorem 1.1 applied for $n=q_s$, we get

$$\log q_{s+1}\le e_{q_s}(\alpha)+q_s\le (C+1)q_s^2\log q_s.$$

Assume now that $\log q_{s+1}\le Cq_s^2\log q_s$ for all s, where C is a constant. Given n, there is a unique s so that $q_s\le n<q_{s+1}$. By Theorem 1.1,

$$e_n(\alpha)\le \frac{n^2\log n}{2}+9n^2+\frac{n}{q_s}\log q_{s+1}\le (C+10)n^2\log n.\quad \square$$

3. The set $\mathcal{S}$

Let $E\subset\mathbb{C}$ and $h(r)$, $0\le r\le r_0$, be a continuous increasing function with $h(0)=0$. Given $\delta>0$ we define

$$\mathcal{H}^h_\delta(E)=\inf\sum_{n=1}^{\infty}h(\operatorname{diam}A_n/2),$$

where the infimum is taken over all coverings $\{A_n\}$ of E with bounded sets A_n of diameter less than δ. As $\delta\searrow 0$ the quantities $\mathcal{H}^h_\delta(E)$ increase, so the limit

$$\mathcal{H}^h(E)=\lim_{\delta\to 0}\mathcal{H}^h_\delta(E)$$

exists and is called the Hausdorff h-measure of E (see e.g. [L, p. 196]). We recall that if $h(r)=1/\log(1/r)$ then $\mathcal{H}^h$ is called the *logarithmic measure*. A set $E\subset\mathbb{C}$ of finite logarithmic measure is polar [L, Theorem 3.14].

We assume now that $h(r)$, $0 \le r \le r_0$, is a continuous increasing function so that

$$\sum_{n=N}^{\infty} n\, h\left(n^{-n^2}\right) < +\infty.$$

An example of such a function is

$$h(r) = \frac{1}{\log \frac{1}{r} \left(\log\log\log \frac{1}{r}\right)^p} \ , \quad p > 1.$$

Proposition 3.1. *If h is as above then $\mathcal{H}^h(S) = 0$. Moreover, $\mathcal{S}$ contains a dense G_δ set, hence it is uncountable.*

Proof. Note that by Theorem 2.1 and the definition of $\mathcal{S}$ we have the following: if $\alpha \in \mathcal{S}$ then there exist infinitely many rational numbers p_s/q_s so that

$$|\alpha - p_s/q_s| < q_{s+1}^{-1} < q_s^{-q_s^2}.$$

Let $r(n) = n^{-n^2}$ and define

$$A_n = \bigcup_{m=1}^{n} \left(\frac{m}{n} - r(n), \frac{m}{n} + r(n) \right).$$

It follows that

$$\mathcal{S} \subset \limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n \ge k} A_n.$$

Fix $\delta > 0$. If k is large enough so that $2r(k) < \delta$, then by the definition of $\mathcal{H}^h_\delta$

$$\mathcal{H}^h_\delta(\mathcal{S}) \le \mathcal{H}^h_\delta\left(\cup_{n\ge k} A_n\right) \le \sum_{n \ge k} n\, h(r(n)).$$

Since $\sum_{n\ge 1} n\, h(r(n)) < +\infty$, it follows that $\mathcal{H}^h_\delta(\mathcal{S}) = 0$ for all $\delta > 0$, so $\mathcal{H}^h(S) = 0$.

We now let $r'(n) = e^{-n^3}$ and define

$$A'_n = \bigcup_{m=1,(m,n)=1}^{n} \left(\frac{m}{n} - r'(n), \frac{m}{n} + r'(n) \right), \ G = \limsup A'_n = \bigcap_{k=1}^{\infty} \bigcup_{n \ge k} A'_n.$$

Here (m,n) denotes the greatest common divisor of m, n. By Baire's theorem, G is a dense G_δ set and hence it is uncountable.

Let us show that $G \subset \mathcal{S}$. If $\alpha \in G$ there exists a sequence of rational numbers m_k/n_k with $(m_k, n_k) = 1$ and $n_k \to +\infty$, so that $|\alpha - m_k/n_k| < r'(n_k)$. Thus

$$|n_k\alpha - m_k| < n_k e^{-n_k^3} < (2n_k)^{-1}.$$

This implies that α is irrational. Indeed, if $\alpha = p/q \in \mathbb{Q}$ with $(p,q) = 1$ then for $n_k > q$ we have

$$q^{-1} \le |n_k\alpha - m_k| < n_k e^{-n_k^3},$$

which yields a contradiction.

Since $|n_k\alpha - m_k| < (2n_k)^{-1}$ we see by Theorem 2.3 that m_k/n_k is a convergent to α, so $m_k = p_s$ and $n_k = q_s$ for some s. Using Theorem 2.1 we obtain

$$(2q_{s+1})^{-1} < |q_s\alpha - p_s| < q_s e^{-q_s^3} \Longrightarrow \frac{\log q_{s+1}}{q_s^2 \log q_s} > \frac{q_s}{\log q_s} - o(1).$$

We conclude that $\alpha \in \mathcal{S}$. $\square$

Remark. An argument similar to the one used to prove $\mathcal{H}^h(\mathcal{S}) = 0$ shows that the above dense G_δ set G has zero logarithmic measure, hence it is polar.

We conclude this section by considering certain polar sets of irrational numbers related to $\mathcal{S}$. Given a sequence $\varepsilon : \mathbb{N} \to (0, +\infty)$ we introduce the sets

$$\begin{aligned} \mathcal{T}(\varepsilon) &= \{\alpha \in (0,1) \setminus \mathbb{Q} : \limsup_{s\to\infty} \varepsilon(q_s) \log q_{s+1} = +\infty\}, \\ \mathcal{U}(\varepsilon) &= \{\alpha \in (0,1) \setminus \mathbb{Q} : \limsup_{n\to\infty} \varepsilon(n) e_n(\alpha) = +\infty\}. \end{aligned}$$

Our interest will be in sequences ε that in some sense decrease rapidly to 0. We have the following:

Proposition 3.2. *(i) If ε satisfies $\sum_{n=1}^{\infty} n\,\varepsilon(n) < \infty$ then the set $\mathcal{T}(\varepsilon)$ is polar.*

(ii) If ε is given by $\varepsilon(n) = \left(x(n) n^2 \log n\right)^{-1}$, $n \geq 1$, where $x(n) \geq 1$ is an increasing sequence, then $\mathcal{T}(\varepsilon) = \mathcal{U}(\varepsilon) \subset \mathcal{S}$.

Proof. (i) Consider the function

$$v(\zeta) = \sum_{n=1}^{\infty} \varepsilon(n) \sum_{m=1}^{n} \log \frac{|\zeta - m/n|}{3}\ , \quad |\zeta| < 2.$$

We have that

$$v(i) \geq -\log 3 \sum_{n=1}^{\infty} n\,\varepsilon(n) > -\infty,$$

so v is a negative subharmonic function in $\{|\zeta| < 2\}$. If $\alpha \in (0,1) \setminus \mathbb{Q}$ it follows from Theorem 2.1 that $|\alpha - p_s/q_s| < q_{s+1}^{-1}$, so $v(\alpha) < -\varepsilon(q_s) \log q_{s+1}$, for every s. Hence if $\alpha \in \mathcal{T}(\varepsilon)$ then $v(\alpha) = -\infty$.

(ii) Clearly $\mathcal{T}(\varepsilon) \subset \mathcal{S}$. Using the lower bound for $e_n(\alpha)$ from Theorem 1.1 with $n = q_s$ we obtain

$$\varepsilon(q_s) e_{q_s}(\alpha) \geq \varepsilon(q_s) \log q_{s+1} - \varepsilon(q_s) q_s.$$

Since $\varepsilon(q_s) q_s \to 0$, we deduce that $\mathcal{T}(\varepsilon) \subset \mathcal{U}(\varepsilon)$.

Conversely, if $\alpha \in \mathcal{U}(\varepsilon)$ there exists a sequence of integers $n_j \to +\infty$ so that $\varepsilon(n_j) e_{n_j}(\alpha) \to +\infty$. We have $q_{s_j} \leq n_j < q_{s_j+1}$ for a unique s_j, so by Theorem 1.1,

$$\begin{aligned} \varepsilon(n_j) e_{n_j}(\alpha) &\leq \frac{10}{x(n_j)} + \frac{\log q_{s_j+1}}{q_{s_j} x(n_j) n_j \log n_j} \\ &\leq 10 + \frac{\log q_{s_j+1}}{x(q_{s_j}) q_{s_j}^2 \log q_{s_j}} = 10 + \varepsilon(q_{s_j}) \log q_{s_j+1}. \end{aligned}$$

We conclude that $\alpha \in \mathcal{T}(\varepsilon)$. $\square$

Remark. There exists a sequence ε which verifies the hypothesis of Proposition 3.2 (i) for which $\mathcal{U}(\varepsilon) = (0,1) \setminus \mathbb{Q}$. Indeed, we let

$$\varepsilon(n) = \begin{cases} n^{-3}, & \text{if } n \in \mathbb{N} \setminus \left\{2^k : k \in \mathbb{N}\right\}, \\ n^{-2}, & \text{if } n \in \left\{2^k : k \in \mathbb{N}\right\}. \end{cases}$$

Clearly $\sum_{n=1}^{\infty} n\,\varepsilon(n) < \infty$. Let $\alpha \in (0,1) \setminus \mathbb{Q}$. By Theorem 1.1 we obtain

$$\varepsilon(n)e_n(\alpha) \geq \varepsilon(n)\frac{n^2 \log n}{2} - \varepsilon(n)n^2 = \frac{\log n}{2} - 1, \text{ if } n = 2^k,\ k \in \mathbb{N}.$$

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