On optimal asymptotic bounds for spherical designs

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Abstract

For each $N \geq c_d t^d$ we prove the existence of a spherical t-design on the sphere S^d consisting of N points, where c_d is a constant depending only on d. This result proves the well-known conjecture of Korevaar and Meyers concerning an optimal order of minimal number of points in a spherical t-design on S^d for a fixed d.

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1 Introduction

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with normalized Lebesgue measure $d\mu_d$ $\big(\int_{S^d} d\mu_d(x) = 1\big)$. The following concept of a spherical design was introduced by Delsarte, Goethals, and Seidel [8].

A set of points $x_1, \ldots, x_N \in S^d$ is called a *spherical t-design* if

$$\int_{S^d} P(x) d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables and of total degree at most t. For each $t \in \mathbb{N}$ denote by $N(d,t)$ the minimal number of points in a spherical t-design. The following lower bounds,

$$(1) \qquad N(d,t) \geq \binom{d+k}{d} + \binom{d+k-1}{d}, \quad t = 2k,$$

$$N(d,t) \geq 2\binom{d+k}{d}, \quad t = 2k+1,$$

are also proved in [8].

Spherical t-designs attaining these bounds are called tight. Exactly eight tight spherical designs are known for $d \geq 2$ and $t \geq 4$. All such configurations of points are highly symmetrical and possess other extreme properties; see Cohn and Kumar [5], and Conway and Sloane [7].

Let us begin by giving a short history of asymptotic upper bounds on $N(d,t)$ for fixed d and $t \to \infty$. First, Seymour and Zaslavsky [15] have proved that spherical designs exist for all d, $t \in \mathbb{N}$. Then, Wagner [16] and Bajnok [2] independently proved that $N(d,t) \leq c_d t^{Cd^4}$ and $N(d,t) \leq c_d t^{Cd^3}$, respectively. Korevaar and Meyers [10] have improved these inequalities by showing that $N(d,t) \leq c_d t^{(d^2+d)/2}$. They have also conjectured that $N(d,t) \leq c_d t^d$. Note that (1) implies $N(d,t) \geq C_d t^d$. In what follows we denote by c_d and b_d sufficiently large constants depending only on d.

The conjecture of Korevaar and Meyers was attacked by many mathematicians. For instance, Kuijlaars and Saff [14] emphasized the importance of this conjecture and revealed its relation to the energy problems. Then, Mhaskar, Narcowich, and Ward [12] have constructed positive quadrature formulas on S^d with $c_d t^d$ points having *almost* equal weights. Very recently, An, Chen, Sloan, and Womersley, see, e.g. [1], [6], have proved the existence of spherical t-designs on S^2 having $(t+1)^2$ points, for $t \leq 100$. In order to prove their result they extensively used numerical methods.

For $d = 2$ there exists even stronger conjecture by Hardin and Sloane [9], that $N(2,t) = \frac{1}{2}t^2 + o(t^2)$ as $t \to \infty$. They also provided a numerical evidence for the conjecture.

Let us briefly explain our previous attempt to prove the conjecture of Korevaar and Meyers. In [3], we have suggested nonconstructive approach for obtaining asymptotic bounds for $N(d,t)$ based on the application of the Brouwer fixed point theorem. This approach led to the following result.

Theorem BV. For each $N \geq c_d t^{\frac{2d(d+1)}{d+2}}$ there exists a spherical t-design on S^d consisting of N points.

However, during the proof we faced technical problems that didn't allow us to obtain an optimal result. In this paper instead of the Brouwer fixed point theorem we will use the following topological theorem.

Theorem OCC. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and Ω be an open bounded subset with the boundary $\partial\Omega$ such that $0 \in \Omega \subset \mathbb{R}^n$. If $(x, f(x)) > 0$ for all $x \in \partial\Omega$, then there exists $x \in \Omega$ satisfying $f(x) = 0$.
This fact is an easy result in the Brouwer degree theory [13, Theorem 1.2.6 and Theorem 1.2.9]. Employing this idea we will prove much stronger result.

Theorem 1. *For each $N \geq c_d t^d$ there exists a spherical t-design on S^d consisting of N points, where c_d is a constant depending only on d.*

This proves the conjecture of Korevaar and Meyers.

2 Preliminaries and the main idea

Let $\mathcal{P}_t$ be the Hilbert space of polynomials P of degree $\leq t$ on S^d such that

$$\int_{S^d} P(x) d\mu_d(x) = 0,$$

equipped with the usual inner product

$$(P, Q) = \int_{S^d} P(x)Q(x) d\mu_d(x).$$

By Riesz representation theorem, for each point $x \in S^d$ there exists a unique polynomial $G_x \in \mathcal{P}_t$ such that

$$(G_x, Q) = Q(x) \ \text{ for all } \ Q \in \mathcal{P}_t.$$

Then a set of points $x_1, \ldots, x_N \in S^d$ forms a spherical t-design if and only if

$$G_{x_1} + \cdots + G_{x_N} = 0. \tag{2}$$

For a differentiable function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}^{d+1}$ denote by

$$\frac{\partial f}{\partial x}(x_0) := \left(\frac{\partial f}{\partial \xi_1}(x_0), \ldots, \frac{\partial f}{\partial \xi_{d+1}}(x_0) \right)$$

the gradient of f at the point x_0.

For a polynomial $Q \in \mathcal{P}_t$ we define the spherical gradient

$$\nabla Q(x) := \frac{\partial}{\partial x} Q(\frac{x}{|x|}),$$

where $| \cdot |$ denotes Euclidean norm in $\mathbb{R}^{d+1}$.

We will apply Theorem OCC to the following open subset of a vector space $\mathcal{P}_t$

$$\Omega = \left\{ P \in \mathcal{P}_t | \int_{S^d} |\nabla P(x)| d\mu_d(x) < 1 \right\}. \tag{3}$$

Now we observe that the existence of a continuous mapping $F : \mathcal{P}_t \to (S^d)^N$ such that for all $P \in \partial\Omega$

$$\sum_{i=1}^{N} P(x_i(P)) > 0, \text{ where } F(P) = (x_1(P), ..., x_N(P)),$$

readily implies the existence of a spherical t-design on S^d consisting of N points. To this end let us consider a mapping $L : (S^d)^N \to \mathcal{P}_t$ defined by

$$(x_1, \dots, x_N) \xrightarrow{L} G_{x_1} + \dots + G_{x_N},$$

and the following composition mapping $f = L \circ F : \mathcal{P}_t \to \mathcal{P}_t$. Clearly

$$(P, f(P)) = \sum_{i=1}^{N} P(x_i(P)),$$

for each $P \in \mathcal{P}_t$. Thus Theorem OCC for the mapping f, vector space $\mathcal{P}_t$, and the subset Ω defined in (3) immediately gives us the existence of a polynomial $P \in \mathcal{P}_t$ such that $f(P) = 0$. Hence, by (2), the components of $F(P) = (x_1(P), ..., x_N(P))$ form a spherical t-design on S^d consisting of N points.

The most naive approach to construct such $F(P) = (x_1(P), \dots, x_N(P))$ is to start with a certain well-distributed collection of points x_i, $i = 1, \dots, N$, put $F(0) := (x_1, \dots, x_N)$, and then move each point along the spherical gradient vector field of P (that is the most greedy way to increase each $P(x_i(P))$ and make $\sum_{i=1}^{N} P(x_i(P))$ positive for each $P \in \partial\Omega$). We will give an explicit construction of F in the next section, which will imply immediately the proof of Theorem 1. To this end we need some auxiliary results.

3 Auxiliary results

We will extensively use the notion of an area-regular partition; see e.g., [4], for the construction of the corresponding mapping F for each $N \geq c_d t^d$. Here is a definition.

Let $\mathcal{R} = \{R_1, \dots, R_N\}$ be a finite collection of closed sets $R_i \subset S^d$ such that $\cup_{i=1}^N R_i = S^d$ and $\mu_d(R_i \cap R_j) = 0$ for all $1 < i \neq j < N$. The partition $\mathcal{R}$ is called area-regular if $\mathrm{vol} R_i := \int_{R_i} d\mu_d(x) = 1/N$, $i = 1, \dots, N$. The partition norm for $\mathcal{R}$ is defined by

$$\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \operatorname{diam} R,$$

where $\operatorname{diam} R$ stands for the maximum geodesic distance between two points in R. We need the following fact stated in [11].

Theorem SK. For each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with $\|\mathcal{R}\| \leq b_d N^{-1/d}$ for some constant b_d.

We will also use a result which is an easy corollary of Theorem 3.1 in [12].

Theorem MNW. There exists a constant r_d such that for each area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$, $i = 1, \dots, N$, and each algebraic polynomial P of total degree m the following inequality,

$$\frac{1}{2} \int_{S^d} |P(x)| d\mu_d(x) < \frac{1}{N} \sum_{i=1}^{N} |P(x_i)| < \frac{3}{2} \int_{S^d} |P(x)| d\mu_d(x), \tag{4}$$

holds.

Remark 1. *Although Theorem 3.1 in [12] was stated for slightly different definition of an area-regular partition the presented proof works for our more general definition as well.*

First we prove the following estimate.

Lemma 1. *For each area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m+1}$, each collection of points $x_i \in R_i$, $i = 1, \dots, N$, and each algebraic polynomial P of total degree m the following inequality holds*

$$\frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)| d\mu_d(x) < \frac{1}{N} \sum_{i=1}^{N} |\nabla P(x_i)| < 3\sqrt{d} \int_{S^d} |\nabla P(x)| d\mu_d(x). \tag{5}$$

Proof. Since $|\nabla P| = \sqrt{P_1^2 + \ldots + P_{d+1}^2}$, where $P_i \in \mathcal{P}_{m+1}$, we obtain the Lemma 1 as a immediate consequence of (4) applied to the polynomials P_i, $i = 1, \ldots, d+1$. ☐

Fix $d, t \in \mathbb{N}$. Take c_d large enough (the exact condition we will write later) and also fix $N \geq c_d t^d$. Now we are ready to give an exact construction of the mapping $F : \mathcal{P}_t \to (S^d)^N$ introduced in section 2. Take an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| \leq b_d N^{-1/d}$ provided by Theorem SK, and chose an arbitrary $x_i \in R_i$ for each $i = 1, \ldots, N$. Put $\varepsilon = \frac{1}{6\sqrt{d}}$, and consider the function

$$h_\varepsilon(s) := \begin{cases} s, \ s > \varepsilon, \\ \varepsilon, \ s \leq \varepsilon. \end{cases}$$

For a polynomial $P \in \mathcal{P}_t$ and each $i = 1, \ldots, N$ let $y_i : [0, \infty) \to S^d$ be the function satisfying a differential equation

$$\frac{dy_i}{ds} = \frac{\nabla P(y_i)}{h_\varepsilon(|\nabla P(y_i)|)} \tag{6}$$

with the initial condition

$$y_i(0) = x_i.$$

For each $i = 1, \ldots, N$, consider a mapping $Y_i : \mathcal{P}_t \times [0, \infty) \to S^d$ with the following action

$$(P, s) \xrightarrow{Y_i} y_i(s),$$

and finally put

$$F(P) = (x_1(P), \ldots, x_N(P)) := (Y_1(P, r_d/3t), \ldots, Y_N(P, r_d/3t)), \tag{7}$$

where r_d is defined by Theorem MNW. Since the function $\frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)}$ is Lipschitz continuous in both P and y, then each mapping Y_i, $i = 1, \ldots, N$, is well defined and continuous in both P and s. Thus, F is continuous in P as well. So, as we explained in section 2, to finish the proof of Theorem 1 it suffices to prove the following lemma

Lemma 2. *Let $F : \mathcal{P}_t \to (S^d)^N$ be the mapping defined by* (7)*. Then, for each $P \in \partial\Omega$,*

$$\frac{1}{N}\sum_{i=1}^{N} P(x_i(P)) > 0,$$

where Ω is given by (3)*.*

Proof. We start with the following easy equation

$$\frac{1}{N}\sum_{i=1}^{N} P(x_i(P)) = \frac{1}{N}\sum_{i=1}^{N} P(y_i(r_d/3t)) = \frac{1}{N}\sum_{i=1}^{N} P(x_i)$$

$$+\int_0^{r_d/3t} \frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right] ds. \tag{8}$$

To prove Lemma 2 first we will estimate the value

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right|$$

from above, and then we will estimate the value

$$\frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right]$$

from below, for each $s \in [0, r_d/3t]$. We have

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right| = \left|\sum_{i=1}^{N}\int_{R_i} P(x_i) - P(x) d\mu_d(x)\right| \le \sum_{i=1}^{N}\int_{R_i} |P(x_i) - P(x)| d\mu_d(x)$$

$$\le \frac{1}{N}\sum_{i=1}^{N} \max_{y_i \in S^d : \operatorname{dist}(y_i, x_i) \le \|\mathcal{R}\|} |\nabla P(y_i)| \max_{x \in R_i} \operatorname{dist}(x, x_i),$$

where $\operatorname{dist}(x, x_i)$ denotes a geodesic distance between x and x_i. Hence, for some $z_i \in S^d$ such that $\operatorname{dist}(z_i, x_i) \le \|\mathcal{R}\|$, $i = 1, \ldots, N$, we obtain

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x_i)\right| \le \frac{1}{N}\|\mathcal{R}\|\sum_{i=1}^{N} |\nabla P(z_i)|.$$

Consider another area-regular partition $\mathcal{R}' = \{R_1', \ldots, R_N'\}$, defined by $R_i' = R_i \cup \{z_i\}$. Clearly $\|\mathcal{R}'\| \le 2\|\mathcal{R}\|$, so by the choice of $\mathcal{R}$ we get $\|\mathcal{R}'\| \le 2b_d c_d^{-1/d} t^{-1}$. Suppose that

$$c_d > (4b_d/r_d)^d. \tag{9}$$

Now we can apply (5) for the partition $\mathcal{R}'$, and obtain immediately

$$\left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| \le 3\sqrt{d} b_d c_d^{-1/d} t^{-1} \int_{S^d} |\nabla P(x)| d\mu_d(x) = 3\sqrt{d} b_d c_d^{-1/d} t^{-1}, \tag{10}$$

for any $P \in \partial\Omega$. On the other hand, the differential equation (6) implies that

$$\frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N} P(y_i(s))\right] = \frac{1}{N}\sum_{i=1}^{N} \frac{|\nabla P(y_i)|^2}{h_\varepsilon(|\nabla P(y_i)|)} \ge$$

$$\ge \frac{1}{N} \sum_{i:\, |\nabla P(y_i)| \ge \varepsilon} |\nabla P(y_i)| \ge$$

$$\ge \frac{1}{N}\sum_{i=1}^{N} |\nabla P(y_i)| - \varepsilon. \tag{11}$$

Since

$$\left| \frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)} \right| \le 1,$$

it follows again from (6) that $\left|\frac{dy_i}{ds}\right| \le 1$. Hence we arrive at

$$\mathrm{dist}(x_i, y_i(s)) \le s.$$

Now for each $s \in [0, r_d/3t]$ consider the area-regular partition $\mathcal{R}'' = \{R_1'', \ldots, R_N''\}$ given by $R_i'' = R_i \cup \{y_i(s)\}$. Clearly $\|\mathcal{R}''\| \le b_d c_d^{-1/d} t^{-1} + r_d/3t$, so if we assume that

$$c_d > (6b_d/r_d)^d, \tag{12}$$

then we can apply (5) for the partition $\mathcal{R}''$. By the inequality (11) we obtain

$$\frac{d}{ds}\left[\frac{1}{N}\sum_{i=1}^{N}P(y_i(s))\right] \geq \frac{1}{N}\sum_{i=1}^{N}|\nabla P(y_i(s))| - \frac{1}{6\sqrt{d}}$$

$$\geq \frac{1}{3\sqrt{d}}\int_{S^d}|\nabla P(x)|d\mu_d(x) - \frac{1}{6\sqrt{d}} = \frac{1}{6\sqrt{d}}, \tag{13}$$

for each $P \in \partial\Omega$ and $s \in [0, r_d/3t]$. Finally, the inequalities (8), (10) and (13) imply

$$\frac{1}{N}\sum_{i=1}^{N}P(x_i(P)) \geq \frac{1}{6\sqrt{d}}\frac{r_d}{3t} - 3\sqrt{d}b_d c_d^{-1/d}t^{-1} > 0. \tag{14}$$

Thus, Lemma 2 holds for all c_d satisfying simultaneously the conditions (9), (12) and (14), say for $c_d > (54db_d/r_d)^d$. □

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