SIMPLE MODULES OF CLASSICAL LINEAR GROUPS WITH NORMAL CLOSURES OF MAXIMAL TORUS ORBITS

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Abstract. Let T be a maximal torus in a classical linear group G. In this paper we find all simple rational G-modules V such that for each vector $v \in V$ the closure of its T-orbit is a normal affine variety. For all other G-modules we present a T-orbit with the non-normal closure. We use a combinatorial criterion of normality formulated in terms of the set of weights of a simple G-module. This work is a continuation of [4], where the same problem was solved in the case $G = SL(n)$.

Introduction

Let G be a connected semisimple algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero. Let $T \subseteq G$ be a fixed maximal torus. Consider a finite-dimensional rational G-module V. Let us check the following property of the module V: whether for each vector $v \in V$ the closure of its T-orbit $\overline{Tv}$ is a normal (affine) algebraic variety.

Earlier this property was checked by J. Morand [5] in the case when G is a simple group and V is its adjoint module; this problem for $G = SL(n)$ was also considered in [6, Ex. 3.7] and [7]. In her previous paper [4] the author checks this property for all simple $SL(n)$-modules. For completeness this result is included in the table below.

The aim of this paper is to investigate this property for the special orthogonal group $SO(n)$, the spinor group $\mathrm{Spin}(n)$, and the symplectic group $Sp(2n)$ for all their simple modules. Recall that a simple G-module is uniquely defined by its highest weight λ. Any dominant weight $a_1\pi_1 + \ldots + a_r\pi_r$ can play the role of λ, where $\pi_1, \ldots, \pi_r$ stand for the fundamental weights, and $a_1, \ldots, a_r$ are nonnegative integers. We enumerate the fundamental weights as in [8, Section 4].

Theorem 1. *The modules below, together with their duals, form the list of all simple modules of classical groups where all maximal torus orbits' closures are normal:*

Group	*Highest weight*	*Checked in*
$SL(n), n \geqslant 2$	π_1	[4]
$SL(n), n \geqslant 2$	$\pi_1 + \pi_{n-1}$	[4]
$SL(2)$	$3\pi_1$	[4]
$SL(2)$	$4\pi_1$	[4]
$SL(3)$	$2\pi_1$	[4]
$SL(4)$	π_2	[4]
$SL(5)$	π_2	[4]
$SL(6)$	π_2	[4]
$SL(6)$	π_3	[4]
$SO(2n+1), n \geqslant 2$	π_1	Case 1

$SO(2n), n \geqslant 4$	π_1	Case 6
$SO(8)$	π_2	Case 7
$\mathrm{Spin}(7)$	π_3	Case 3
$\mathrm{Spin}(8)$	π_3	Case 8
$\mathrm{Spin}(9)$	π_4	Case 3
$\mathrm{Spin}(10)$	π_4	Case 9
$\mathrm{Spin}(12)$	π_5	Case 10
$Sp(2n), n \geqslant 2$	π_1	Cases 3 and 4
$Sp(4)$	$2\pi_1$	Case 2
$Sp(6)$	π_2	Case 5
$Sp(8)$	π_2	Case 5

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The property of normality for the closure of a T-orbit in a module has a well-known combinatorial interpretation. Let $v_1, \dots, v_r$ be vectors of a rational vector space. For any set A of rational numbers we denote by $A(v_1, \dots, v_r)$ the set of all linear combinations of vectors $v_1, \dots, v_r$ with coefficients in A. The set $\{v_1, \dots, v_r\}$ is called *saturated* if

$$\mathbb{Z}_{\geqslant 0}(v_1, v_2, \dots, v_r) = \mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geqslant 0}(v_1, v_2, \dots, v_r).$$

For a simple G-module $V(\lambda)$ with the highest weight λ we denote by $M(\lambda)$ the set of all its weights with respect to the maximal torus T. Now being normal for the closures of all T-orbits in the module $V(\lambda)$ is equivalent to the fact that all subsets in $M(\lambda)$ are saturated.

For every set of weights $M(\lambda)$ from Theorem 1 one needs to check the saturation property for all of its subsets. In the most difficult cases we use properties of unimodular sets of vectors and their generalisations. In the cases which do not appear in Theorem 1 it suffices to construct one non-saturated subset. We can confine ourselves to those sets of weights which are minimal with respect to inclusion and which do not appear in Theorem 1. In each minimal case a nonsaturated subset is constructed explicitly.

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1. Preliminaries

1.1. **Weight decomposition.** Let T be an algebraic torus and let $\Lambda = \Lambda(T)$ be the lattice of its T-characters. For every rational T-module V we have the weight decomposition

$$V = \bigoplus_{\mu \in \Lambda} V_\mu, \quad \text{where} \quad V_\mu = \{v \in V \mid tv = \mu(t)v\}.$$

Denote by $M(V)$ the collection of weights of the module V, i.e. $M(V) = \{\mu \in \Lambda \mid V_\mu \neq 0\}$. For every nonzero vector v we have its weight decomposition $v = v_{\mu_1} + \dots + v_{\mu_s}$, $v_{\mu_i} \in V_{\mu_i}$, $v_{\mu_i} \neq 0$. We consider weights $\mu \in \Lambda$ as points of the rational vector space $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall that an irreducible affine algebraic variety X is called *normal* if its algebra of regular functions $\Bbbk[X]$ is integrally closed in its field of fractions. The following statement is a well-known combinatorial criterion of normality of a T-orbit closure in a T-module, see [3, I, §1, Lemma 1].

Proposition 1. *Let V be a finite dimensional rational T-module and $v = v_{\mu_1} + \dots + v_{\mu_s}$ be the weight decomposition of a vector $v \in V$. The closure $\overline{Tv}$ of the T-orbit of v is normal if and only if the set of characters $\{\mu_1, \dots, \mu_s\}$ is saturated.*

A finite subset M of a rational vector space $\mathbb{Q}^n$ is called *hereditary normal* if all its subsets are saturated.

Corollary 1. *Let V be a finite dimensional rational T-module. The closures of all T-orbits in the module V are normal if and only if the set $M(V)$ is hereditary normal.*

Notice that for the dual module V^* one has

$$M(V^*) = -M(V).$$

This means that the property of hereditary normality for the set $M(V)$ is equivalent to the same property for the set $M(V^*)$.

Let G be a connected simply connected semisimple algebraic group, B be a Borel subgroup in G, and $T \subset B$ be the maximal torus. Denote by Φ the root system of the Lie algebra $\mathrm{Lie}(G)$ associated with the maximal torus T. Let Φ^+ be the set of positive roots, and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be simple roots in Φ^+ corresponding to the Borel subgroup B. Denote by π_i the fundamental weight corresponding to the simple root α_i. It is well-known that the weights

$\pi_1, \dots, \pi_r$ form a basis of the character lattice $\Lambda(T)$ of torus T. The semigroup generated by fundamental weights coincides with the semigroup of dominant weights Λ_+. The subgroup of Λ generated by the root system Φ is called the *root lattice*, we denote it by Ξ. Then Ξ is the sublattice of Λ of finite index, and $\alpha_1, \dots, \alpha_r$ form a basis of Ξ.

Let $V(\lambda)$ be a simple G-module with the highest weight $\lambda \in \Lambda_+$. Recall the following description of the set of T-weights of the module $V(\lambda)$. Let W be the Weyl group of the root system Φ. Then W can be realized as a finite group of linear transformations of the vector space $\Lambda_{\mathbb{Q}}$ generated by reflections. The *weight polytope* $P(\lambda)$ of the module $V(\lambda)$ is the convex hull $\mathrm{conv}\{g\lambda \,|\, g \in W\}$ of the W-orbit of the point λ in $\Lambda_{\mathbb{Q}}$. Then

$$M(\lambda) = (\lambda + \Xi) \cap P(\lambda),$$

see [2, Theorem 14.18]. There is a partial order on the vector space $\Lambda_{\mathbb{Q}}$: $\lambda \succeq \mu$ if and only if $\lambda - \mu$ is a linear combination of simple roots with nonnegative integer coefficients.

Lemma 1. *Let $\lambda,\ \lambda' \in \Lambda_+$. Suppose that $\lambda \succeq \lambda'$, then $M(\lambda) \supseteq M(\lambda')$.*

Proof. Use the criterion from [1, Exercice 1 to Section VIII, §7]: the weight $\lambda' \in \lambda + \Xi$ belongs to $M(\lambda)$ if and only if for all $w \in W$ the weight $\lambda - w\lambda'$ is dominant. First, using this criterion, notice that under our assumptions λ' belongs to $M(\lambda)$. Indeed, for $w = e$ the weight $\lambda - \lambda'$ belongs to Λ_+ due to the assumption, and for $w \neq e$ it is known that $w\lambda' = \lambda' - \mu$, where μ is a sum of positive roots. Hence $\lambda - w\lambda' = \lambda - \lambda' + \mu \in \Lambda_+$ as the sum of two dominant weights. It means that $\lambda' \in M(\lambda)$, and all points of the form $w\lambda'$, where $w \in W$, belong to $M(\lambda)$. Using convexity, we obtain that $M(\lambda') \subseteq M(\lambda)$. □

Corollary 2. *Take $\lambda' \in \Lambda_+$ and assume that $M(\lambda')$ is not hereditary normal. Then for all $\lambda \in \Lambda_+$ such that $\lambda \succeq \lambda'$ the set $M(\lambda)$ is not hereditary normal.*

1.2. **Unimodular sets.** We need some properties of unimodular sets. The proof of the following lemma can be found in [5].

Lemma 2. *Let M be a finite set of vectors in $\mathbb{Q}^n$.*

(i) *If M is linearly independent, then M is saturated.*

(ii) *If M is not saturated and contains both vectors v and $-v$, then either $M \backslash \{v\}$ or $M \backslash \{-v\}$ is not saturated.*

(iii) *Let $v \in \mathbb{Q}_{\geqslant 0}(M)$. Then there exists a linearly independent subset $M' \subseteq M$ such that $v \in \mathbb{Q}_{\geqslant 0}(M')$.*

We often say "points" instead of "elements of M".

We refer to a nonsaturated subset as an *NSS*. By an *extended nonsaturated subset* we mean a nonsaturated subset $\{v_1, \dots, v_r\}$ augmented by a vector v_0 such that

(i) $v_0 \in (\mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geqslant 0}(v_1, v_2, \dots, v_r)) \setminus \mathbb{Z}_{\geqslant 0}(v_1, v_2, \dots, v_r)$,

(ii) there exists a $\mathbb{Q}_{\geqslant 0}$-representation

$$v_0 = q_1 v_{i_1} + \ldots + q_s v_{i_s}, \quad v_{i_j} \in \{v_1, v_2, \dots, v_r\}$$

with linearly independent vectors $v_{i_1}, \dots, v_{i_s}$ and coefficients $q_i \in [0, 1)$.

These subsets will be named *ENSS*es and will be denoted by $\{v_0; v_1, \dots, v_r\}$.

The fractional part of a real value q is denoted by $\{q\}$, the integer part is denoted by $\lfloor q \rfloor$.

Lemma 3. *Suppose that the set $M = \{v_1, \dots, v_r\}$ is not saturated. Then there exists a vector v_0 such that $\{v_0; v_1, \dots, v_r\}$ is an ENSS.*

Proof. Consider any vector $v_0 \in (\mathbb{Z}(M) \cap \mathbb{Q}_{\geqslant 0}(M)) \setminus \mathbb{Z}_{\geqslant 0}(M)$, and the corresponding $\mathbb{Q}_{\geqslant 0}$-combination $v_0 = q_1 v_1 + \ldots + q_r v_r$. By Lemma 2(iii) there exists a linearly independent subset $\{v_{i_1}, \dots, v_{i_s}\} \subseteq \{v_1, \dots, v_r\}$ and the collection of $\mathbb{Q}_{\geqslant 0}$-coefficients q'_j such that $v_0 = q'_1 v_{i_1} + \ldots + q'_s v_{i_s}$. If some $q'_j \geqslant 1$, consider another vector $v'_0 = v_0 - \lfloor q'_1 \rfloor v_{i_1} - \cdots - \lfloor q'_s \rfloor v_{i_s}$

instead of v_0. It is easy to see that it also belongs to $\mathbb{Z}(v_1, \dots, v_r)$ and to $\mathbb{Q}_{\geqslant 0}(v_1, \dots, v_r)$, and does not belong to $\mathbb{Z}_{\geqslant 0}(v_1, \dots, v_r)$. However all the coefficients of the new $\mathbb{Q}_{\geqslant 0}$-combination belong to the semiopen interval $[0, 1)$. This means that $\{v_0'; v_1, \dots, v_r\}$ is an ENSS. □

Let $v_0, v_1, \dots, v_r$ be some vectors in a rational vector space L, and let f be a linear function on L. We call f a *discriminating linear function* for the collection $\{v_0; v_1, \dots, v_r\}$ if the value $f(v_0)$ cannot be represented as a linear combination of values $f(v_1), \dots, f(v_r)$ with nonnegative integer coefficients. If v_0 belongs to $\mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geqslant 0}(v_1, v_2, \dots, v_r)$ and it can be represented as a $\mathbb{Q}_{\geqslant 0}$-combination of linearly independent vectors $v_1, \dots, v_r$ with coefficients from the semiopen interval $[0, 1)$, then the existence of a discriminating function guarantees that $\{v_0; v_1, \dots, v_r\}$ is an ENSS.

Assume that the set of vectors $M \subset \mathbb{Q}^n$ has rank d, $d \leqslant n$, and $L = \langle v \mid v \in M \rangle$ is the linear span of vectors from M. The set M is called *unimodular* if for any linearly independent vectors $v_1, \dots, v_d \in M$ the value of the d-dimensional volume $\mathrm{vol}_d(v_1, v_2, \dots, v_d)$ has constant absolute value. If one fixes a basis in L, then the condition above is equivalent to the fact that absolute values of all nonzero determinants $|\det(v_1, v_2, \dots, v_d)|$, $v_1, v_2, \dots, v_d \in M$, computed in this basis are equal.

If the set M is unimodular, then its intersection with any subspace $L_1 \subset L$ is also unimodular. This is clear if we choose in L any basis compatible with L_1.

Lemma 4. *Any unimodular set of vectors M is hereditary normal.*

Proof. On the contrary, suppose that there exists an ENSS $\{v_0; v_1, \dots, v_r\}$ in M. Denote by d' the dimension of the subspace $L_1 = \langle v_1, \dots, v_r \rangle$, $d' \leqslant d$. The set $\{v_1, \dots, v_r\}$ is unimodular. According to the definition of an ENSS, vectors in the corresponding $\mathbb{Q}_{\geqslant 0}$-combination for v_0 are linearly independent. Complete them with elements of M to a basis $v_1, v_2, \dots, v_{d'}$ of the space L_1. Due to unimodularity and Cramer's formulae for the solution of a system of linear equations, the coordinates of each v_i, $i > d'$, in the basis $v_1, v_2, \dots, v_{d'}$ are equal to 0, 1, and -1. Then, using the initial $\mathbb{Z}$-combination for v_0 and substituting the decompositions for v_i in it, we obtain that v_0 has integer coordinates in the basis $v_1, \dots, v_{d'}$. Since v_0 has a unique decomposition in any fixed basis, all the coefficients of the initial $\mathbb{Q}_{\geqslant 0}$-combination are integers. □

We say that a subset $M \subset \mathbb{Q}^n$ of rank d is *almost unimodular* if we can choose a subset $\{v_1, v_2, \dots, v_d\} \subseteq M$ such that $\mathrm{vol}_d(v_1, v_2, \dots, v_d) = m$, and for any other vector $w \in M$ and for each i the value

$$\mathrm{vol}_d(v_1, v_2, \dots, \widehat{v_i}, \dots, v_d, w)$$

is divisible by m. If one fixes a basis in the linear space $\langle M \rangle$, then this property can be checked by comparing the values of the corresponding determinants in the given basis instead of computing vol_d's. The value $m = \det(v_1, v_2, \dots, v_d)$ is called the *volume* of an almost unimodular subset.

Lemma 5. *Let M be an almost unimodular set of volume m and of rank d. Then all the determinants in M are divisible by m.*

Proof. Expand each $w \in M$ in the basis $(v_1, v_2, \dots, v_d)$. By Cramer's formulae, they all have integer coordinates:

$$\text{if } w = a_1 v_1 + \ldots + a_d v_d, \quad \text{then } a_i = \frac{\det(v_1, v_2, \dots, \widehat{v_i}, w, \dots, v_d)}{\det(v_1, v_2, \dots, v_d)}, \quad \text{where all } a_i \in \mathbb{Z},$$

because of almost unimodularity. Furthermore, for any vectors $w_1, \dots, w_d \in M$ we have $\det(w_1, \dots, w_d) = \det A \cdot \det(v_1, \dots, v_d)$, where A is an integer matrix expressing the set of vectors $w_1, \dots, w_d$ in the basis $(v_1, v_2, \dots, v_d)$. Since $\det A \in \mathbb{Z}$, the value $\det(w_1, \dots, w_d)$ is divisible by m, and the proof is completed. □

Lemma 6. *Consider an almost unimodular set M such that all determinants in M are contained in the set $m \cdot \{1, a_1, \dots, a_k\}$ and for some vectors $w_1, \dots, w_d \in M$ we have that $\det(w_1, \dots, w_d)$ equals am. Then, if we decompose any vector $w \in M$ in the basis $w_1, \dots, w_d$, the coefficients belong to the set $\{\pm 1/a, \pm a_1/a, \dots, \pm a_k/a\}$.*

Proof. The proof follows directly from Cramer's formulae, see above. □

By a *primitive subset* $\{v_1, \dots, v_d\}$ in an almost unimodular set of volume m we mean any d-element subset such that its determinant equals $\pm m$. In fact, Lemma 5 says that for any primitive subset $\{v_1, \dots, v_d\} \subseteq M$ the set M belongs to $\mathbb{Z}(v_1, \dots, v_d)$.

Example. Consider the set M containing 16 points $\{(\pm 1, \pm 1, \pm 1, \pm 1)\}$. It is easy to see that determinants of all 4-tuples equal 0, 8, or 16. This means that M is almost unimodular.

Lemma 7. *Suppose that an almost unimodular set M of rank d and of volume m is not hereditary normal, and $\{v_0; v_1, \dots, v_r\}$ is an ENSS. Assume that the corresponding $\mathbb{Q}_{\geqslant 0}$-combination for v_0 involves only the linearly independent vectors $v_1, \dots, v_{d'}$.*

(i) *If $d' = \operatorname{rk}\langle v_1, \dots, v_{d'}\rangle = d$, then $|\operatorname{vol}_d(v_1, \dots, v_d)| \neq m$.*
(ii) *If $d' < d$, then for any vectors $w_{d'+1}, \dots, w_d \in M$ linearly independent with $v_1, \dots, v_{d'}$ one has $|\operatorname{vol}_d(v_1, \dots, v_{d'}, w_{d'+1}, \dots, w_d)| \neq m$.*

Proof. (i) If $|\operatorname{vol}_d(v_1, \dots, v_{d'})| = m$, then by Lemma 5 the vector v_0 decomposes with integer coefficients in the basis $v_1, \dots, v_d$. Since $v_1, \dots, v_d$ are linearly independent, this $\mathbb{Z}$-combination coincides with the initial $\mathbb{Q}_{\geqslant 0}$-combination, contradiction with the fact that it is an ENSS.

(ii) We may suppose that vectors $w_{d'+1}, \dots, w_d$ enter in the initial $\mathbb{Q}_{\geqslant 0}$-combination for v_0 with zero coefficients, and then use the reasoning of the previous part. □

Lemma 8. *Consider an almost unimodular set M of rank d which is not hereditary normal and such that all values of determinants in it equal 0, $\pm m$, or $\pm 2m$. Let $\{v_0; v_1, \dots, v_r\}$ be an ENSS such that the corresponding $\mathbb{Q}_{\geqslant 0}$-combination for v_0 contains exactly vectors $v_1, \dots, v_l$. Then there exists a vector v_0' and a subset $\{v_{i_1}, \dots, v_{i_s}\} \subseteq \{v_1, \dots, v_l\}$ such that $\{v_0'; v_1, \dots, v_r\}$ is an ENSS and all the coefficients in the corresponding $\mathbb{Q}_{\geqslant 0}$-combination $v_0' = q_{i_1}' v_{i_1} + \dots q_{i_s}' v_{i_s}$ equal 0 and $\frac{1}{2}$.*

Proof. Assume that the set $\{v_1, \dots, v_r\}$ has rank d'. First we show that if $d' < d$, then in $\langle v_1, \dots, v_r\rangle$ all $\operatorname{vol}_{d'}(w_1, \dots, w_{d'}) \in \{0, \pm m', \pm 2m'\}$, $w_i \in \{v_1, \dots, v_r\}$ for some m' (if $d = d'$, this is a tautology). For this choose a basis from vectors $w_i \in M$ in $L = \langle M\rangle$ compatible with $L_1 = \langle v_1, \dots, v_r\rangle$. Let $w_{d'+1}, \dots, w_d$ be vectors from this basis belonging to $L \setminus L_1$. Consider all possible values

$$\{\operatorname{vol}_d(u_1, \dots, u_{d'}, w_{d'+1}, \dots, w_d) \mid u_1, \dots, u_{d'} \in L_1\}.$$

In the basis constructed above the corresponding matrices have the form $\begin{pmatrix} A & B \\ 0 & E \end{pmatrix}$, where E stands for the identity matrix, which gives $\det A \in \{\pm m, \pm 2m\}$.

Now we can omit the points from $M \setminus L_1$ and suppose that $d' = d$, $m' = m$. Moreover, vectors $v_1, \dots, v_l$ are linearly independent. If $l < d$, then augment $\{v_1, \dots, v_l\}$ by $(d-l)$ vectors $v_{l+1}, \dots, v_d$, linearly independent with $\{v_1, \dots, v_l\}$, and assume that $v_{l+1}, \dots, v_d$ appear in the initial $\mathbb{Q}_{\geqslant 0}$-combination for v_0 with zero coefficients. If $\operatorname{vol}(v_1, \dots, v_d) = m$, then by Lemma 7 the set $\{v_0; v_1, \dots, v_r\}$ is not an ENSS. If $\operatorname{vol}(v_1, \dots, v_d) = 2m$, then by Lemma 6 all the vectors v_i, $i > d$, can be decomposed in the basis $v_1, v_2, \dots, v_d$ with integer or half-integer coefficients. Substituting these decompositions in the initial $\mathbb{Z}$-combination for v_0, we obtain that the initial $\mathbb{Q}_{\geqslant 0}$-combination has integer or half-integer coefficients (last $d - l$ of them being zero). If some of them are $\geqslant 1$, then, as in Lemma 3, we replace the vector v_0 by v_0' and obtain that all coefficients equal 0 or $\frac{1}{2}$. □

2. The root system B_n

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Q}^n$. The root system B_n, where $n \geqslant 2$, is formed by vectors $\{\pm e_i \pm e_j\,, \pm e_i \,|\, 1 \leqslant i, j \leqslant n,\, i \neq j\}$. With respect to the system of simple roots

$$\alpha_1 = e_1 - e_2,\, \alpha_2 = e_2 - e_3, \ldots,\, \alpha_{n-1} = e_{n-1} - e_n,\, \alpha_n = e_n$$

the fundamental weights have the form

$$\pi_1 = e_1,\, \pi_2 = e_1 + e_2,\, \ldots,\, \pi_{n-1} = e_1 + \ldots + e_{n-1},\, \pi_n = \frac{1}{2}(e_1 + \ldots + e_n).$$

The root lattice Ξ coincides with the lattice of all integer points in $\mathbb{Q}^n$. Next, the weight lattice Λ has the form

$$\Lambda = \{(\ell_1, \ell_2, \ldots, \ell_n) \mid 2\ell_i \in \mathbb{Z}, \ell_i - \ell_j \in \mathbb{Z}, i, j = 1, \ldots, n\}.$$

The Weyl group W acts by permutations on the set of coordinates and by changing signs of an arbitrary set of coordinates. A weight $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ is dominant if and only if $\ell_1 \geqslant \ldots \geqslant \ell_n \geqslant 0$. If all coordinates of λ are integers (or all together half-integers but not integers), then the set $M(\lambda)$ consists of all integer (or half-integer but not integer, respectively) points in the polytope $P(\lambda)$.

2.1. **Positive results.**

Case 1. $\lambda = \pi_1 = (1, 0, \ldots, 0)$. Then $M(\lambda) = \{\pm e_i \,|\, 1 \leqslant i \leqslant n\}$. Obviously, this subset is unimodular, and by Lemma 4 it is hereditary normal.

Case 2. $\lambda = 2\pi_2 = (1, 1)$, $n = 2$. It is easy to check case-by-case that any subset in the set of vectors $\{\pm e_1 \pm e_2, \pm e_1, \pm e_2\}$ is saturated.

Case 3. $\lambda = \pi_n = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, $2 \leqslant n \leqslant 4$. In this case we have

$$M(\lambda) = \left\{ \underbrace{\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)}_{n \text{ coordinates}} \right\}.$$

One has to check that this set is hereditary normal.

Multiply all coordinates of all vectors by 2. The problem does not change but now all the coordinates are integers: $M'(\lambda) = \{\underbrace{(\pm 1, \pm 1, \ldots, \pm 1)}_{n \text{ coordinates}}\}$.

For $n = 2, 3$ $M'(\lambda)$ is unimodular, so by Lemma 4 it is hereditary normal.

Consider $n = 4$. The values of all nonzero determinants in $M'(\lambda)$ equal ± 8 and ± 16. This means that $M'(\lambda)$ is almost unimodular. Find all 4-tuples of vectors such that their determinant equals 16. For any vector v from M' the vector $-v$ also belongs to M', hence we will look for such 4-tuples up to sign: we will check only one vector from each pair of opposite vectors, namely, the one with the first coordinate equalling 1. Also we may assume that the first vector in this 4-tuple is $(1, 1, 1, 1)$. Using case-by-case consideration, we see that it can be only the following set (given by rows of the following matrix):

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Next, suppose that there exists an ENSS $\{v_0; v_1, \ldots, v_r\}$. Using Lemmas 8 and 7, we obtain that the corresponding $\mathbb{Q}_{\geqslant 0}$-combination for v_0 is a sum of some $\pm w_1$, $\pm w_2$, $\pm w_3$, $\pm w_4$ with coefficients 0 and 1/2, i.e. $q_1 v_1 + q_2 v_2 + q_3 v_3 + q_4 v_4$, where each $q_i \in \{0, \frac{1}{2}\}$ and $v_i = \pm w_i$. Since every two coordinates of v_0 differ by an even integer, $2(\pm q_3 \pm q_4)$ is even, hence $(\pm q_3 \pm q_4)$ is integer, hence q_3 and q_4 are both 0 or $\frac{1}{2}$. For other pairs of coefficients q_i one can proceed analogously. Since $v_0 \neq 0$, all the coefficients equal $\frac{1}{2}$. If we augment $\{v_1, v_2, v_3, v_4\}$ by any new vector v_5 from $M'(\lambda)$ (due to Lemma 2(ii) it is not $\pm w_i$, $i=1,\dots,4$) we will express v_0 as a $\mathbb{Z}_{\geqslant 0}$-combination. Indeed, any vector v_5 from $M(\lambda)$ which is not equal to $\pm w_i$, $i=1,\dots,4$ can be represented as $\pm\frac{1}{2}w_1\pm\frac{1}{2}w_2\pm\frac{1}{2}w_3\pm\frac{1}{2}w_4$ (for example, $(1,1,1,-1)=1/2(w_1+w_2+w_3-w_4)$), which gives a similar representation in vectors v_1,v_2,v_3,v_4. But, if we have any vector v_5 of the form $\pm\frac{1}{2}v_1\pm\frac{1}{2}v_2\pm\frac{1}{2}v_3\pm\frac{1}{2}v_4$, the vector $\frac{1}{2}(v_1+v_2+v_3+v_4)$ can be obtained for any choice of signs: we add to v_5 those vectors v_i at which v_5 has coefficient $-\frac{1}{2}$. This shows that v_0 can be expressed in v_i with $\mathbb{Z}_{\geqslant 0}$-coefficients. Hence, it is not an ENSS.

2.2. **Some negative results.**

Counterexample 1. $\lambda=2\pi_1=2e_1$, $n=2$. Consider the following subset in $M(\lambda)$: $v_1=2e_1$, $v_2=e_1+e_2$, $v_3=e_2$, $v=e_1=v_1/2=v_2-v_3$. Take a discriminating linear function: $f=3x_1+4x_2$ (see Section 1.2), then $f(v_1)=6$, $f(v_2)=7$, $f(v_3)=4$, $f(v)=3$. It is clear that 3 cannot be represented as a sum of integers 4, 6, and 7.

Counterexample 2. $\lambda=\pi_2=e_1+e_2$, $n\geqslant 3$. Let $v_1=e_1+e_2$, $v_2=e_1-e_2$, $v_3=e_2-e_3$, $v_4=-e_3$. Then $v=e_1=\frac{1}{2}((e_1+e_2)+(e_1-e_2))=(e_1-e_2)+(e_2-e_3)-(-e_3)$, but $e_1\notin\mathbb{Z}_{\geqslant 0}(v_1,v_2,v_3,v_4)$. Indeed, let $f=3x_1+x_2-5x_3$. Then $f(v_1)=4$, $f(v_2)=2$, $f(v_3)=6$, $f(v_4)=5$, $f(v)=3$, but 3 cannot be represented as a sum of integers 2, 4, 5, and 6.

Counterexample 3. $\lambda=\pi_1+\pi_n=(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$, $n\geqslant 2$. Let

$$v_1=\left(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2}\right),\ v_2=\left(\frac{3}{2},-\frac{1}{2},\dots,-\frac{1}{2}\right),\ v_3=\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right).$$

Then $v=(1,0,\dots,0)=1/3(v_1+v_2)=v_1-v_3$, and if one considers the first coordinate, it is clear that

$$v\notin\mathbb{Z}_{\geqslant 0}(v_1,v_2,v_3).$$

Counterexample 4. $\lambda=\pi_n=\left(\frac{1}{2},\dots,\frac{1}{2}\right)$, $n=5$. To simplify the notation, multiply all the coordinates by 2. Let

$$\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\end{pmatrix}=\begin{pmatrix} 1&1&1&1&-1\\ 1&1&1&-1&1\\ 1&1&-1&1&1\\ 1&-1&1&1&1\\ -1&1&1&1&1\\ 1&1&1&-1&-1\end{pmatrix},$$

$$v=\frac{1}{3}(v_1+v_2+v_3+v_4+v_5)=(1,1,1,1,1)=v_1+v_2-v_6,$$

$$f=3x_1+3x_2+3x_3+2x_4+2x_5,$$

$$f(v_1)=f(v_2)=9,\ f(v_3)=f(v_4)=f(v_5)=7,\ f(v_6)=5,\ f(v)=13.$$

Note that 13 cannot be decomposed as a sum of integers 9, 7, and 5. Hence, it is an NSS.

2.3. **Reduction to the already examined cases.** By a *shift for B_n* we call the procedure of replacing the vector $\lambda=(\ell_1,\dots,\ell_n)$ with the vector $\lambda'=(\ell_1,\dots,\ell_i-1,\dots,\ell_n)$, if $\ell_i\geqslant 1$. Notice that λ' always belongs to $M(\lambda)$ because $\lambda-\lambda'\in\Xi$ and λ' is a convex linear combination of vectors λ and $(\ell_1,\dots,-\ell_i,\dots,\ell_n)$ with suitable coefficients (these vectors both belong to $M(\lambda)$).

Lemma 9. *Let $n\geqslant 3$. If $\lambda\in\Xi\setminus\Phi$, then the vector e_1+e_2 belongs to $M(\lambda)$.*

Proof. Let $\lambda=(\ell_1,\dots,\ell_n)$, $\ell_1,\dots,\ell_n\in\mathbb{Z}$. Since λ is a dominant weight, we have $\sum_1^n\ell_i\geqslant 2$. If $\sum_1^n\ell_i>2$ and $\ell_i>0$, then the point $(\ell_1,\dots,\ell_{i-1},\ell_i-1,\ell_{i+1},\dots,\ell_n)$ belongs to $M(\lambda)$ (apply the shift). Repeating this procedure, we show that there is a point $\lambda'\in M(\lambda)$ with $\sum_1^n\ell'_i=2$. It is either a root e_i+e_j, or $2e_i$, in the second case we can obtain $\lambda''=2e_j$ by acting with the Weyl group, and the midpoint of $\lambda'\lambda''$ is the point $e_i+e_j\in M(\lambda')\subseteq M(\lambda)$, hence $e_1+e_2\in M(\lambda)$, as well. □

Now we show how all cases from B_n, which do not appear in Theorem 1, can be reduced to Examples 1 – 4, using Corollary 2. If all coordinates of λ are integers and $n \geqslant 3$, then any weight λ which does not belong to Ξ can be reduced to e_1+e_2 by Lemma 9, i.e. Counterexample 2 can be applied. If all coordinates of λ are integers and $n=2$, then $\lambda=(\ell_1,\ell_2)\neq(2,0)$ but it is not a root, which gives $\ell_1 \geqslant 2$, hence $(2,0)\in M(\lambda)$, and we can apply Corollary 2 to Counterexample 1.

If all coordinates of $\lambda=(\ell_1,\dots,\ell_n)$ are not integers, i.e. for all i the value $2\ell_i$ is odd, and if in addition there exists a number i such that $2\ell_i \geqslant 3$, then the point $(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$ belongs to $M(\lambda)$ (apply several shifts), and one can apply Corollary 2 to Counterexample 3. Finally, if $\lambda=\left(\frac{1}{2},\dots,\frac{1}{2}\right)$, then we have $M(\lambda)=\{(\underbrace{\pm1/2,\pm1/2,\dots,\pm1/2}_{n\text{ coordinates}})\}$. Multiply all the coordinates of all the vectors by 2, this does not change the problem, but now all the coordinates are integers: $M'(\lambda)=\{(\underbrace{\pm1,\pm1,\dots,\pm1}_{n\text{ coordinates}})\}$. For $n=5$ see Counterexample 4, for $n>5$ an NSS can be constructed in the following way: take Counterexample 4 for $n=5$ and append $n-5$ coordinates equal to the 5th coordinate to each vector.

3. The root system C_n

Again let $e_1,\dots,e_n$ be the standard basis of $\mathbb{Q}^n$. The root system C_n, $n\geqslant 3$, is formed by vectors $\{\pm e_i \pm e_j\,,\pm 2e_i \mid 1\leqslant i,j\leqslant n,\ i\neq j\}$. With respect to the system of simple roots

$$\alpha_1=e_1-e_2,\ \alpha_2=e_2-e_3,\dots,\ \alpha_{n-1}=e_{n-1}-e_n,\ \alpha_n=2e_n$$

the fundamental weights have the form

$$\pi_1=e_1,\ \pi_2=e_1+e_2,\ \dots,\ \pi_n=e_1+\ldots+e_n.$$

The root lattice Ξ coincides with the lattice of all integer points in $\mathbb{Q}^n$ with the even sum of coordinates. The weight lattice Λ consists of all integer points in $\mathbb{Q}^n$. The Weyl group W acts by permutations on the set of coordinates and by sign changes on an arbitrary subset of coordinates. A weight $\lambda=(\ell_1,\ell_2,\dots,\ell_n)$ is dominant if and only if $\ell_1\geqslant\ldots\geqslant\ell_n\geqslant 0$. The set $M(\lambda)$ coincides with the set of integer points in $P(\lambda)$ such that their sum of coordinates has the same parity as λ.

3.1. Positive results.

Case 4. $\lambda=\pi_1=e_1$. Then M is unimodular, hence hereditary normal (cf. Case B_n).

Case 5. $\lambda=\pi_2=e_1+e_2$, $n=3,4$. For $n=3$ this set is unimodular, hence it is hereditary normal. For $n=4$ it is almost unimodular. All nonzero determinants are equal to ±2 or ±4, and a 4-tuple of vectors with the determinant ±4 without loss of generality coincides with the set of rows of the matrix

$$\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}=\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}.$$

Now the proof is analogous to Case 3. Suppose that there exists an ENSS $\{v_0;v_1,v_2,v_3,v_4\}$. Without loss of generality the vectors $v_1,\dots,v_4$ are as above, otherwise simultaneously change the sing of some coordinate in all the vectors. Assume that the corresponding $\mathbb{Q}_{\geqslant0}$-combination is $v_0=q_1v_1+q_2v_2+q_3v_3+q_4v_4$. Then all $q_i\in\{0,1/2\}$, $q_1=q_2$, and $q_3=q_4$. But the sum of coordinates of v_0 is even, consequently, all q_i equal $\frac{1}{2}$ simultaneously. To obtain v_0 as a $\mathbb{Z}$-combination, it is necessary to append a vector which is not $\pm v_i$. However, it is easy to see that all the other vectors in $M(\lambda)$ equal $\pm\frac{1}{2}v_1\pm\frac{1}{2}v_2\pm\frac{1}{2}v_3\pm\frac{1}{2}v_4$. If we augment $\{v_1,\dots,v_4\}$ by any vector of this form, then the vector $\frac{1}{2}v_1+\frac{1}{2}v_2+\frac{1}{2}v_3+\frac{1}{2}v_4$ will be easily obtained as a $\mathbb{Z}_{\geqslant0}$-combination (cf. Section 2.1).

3.2. **Some negative results.**

Counterexample 5. $\lambda = \pi_1 + \pi_2 = (2,1,0)$. Let

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Take $v = (1,1,1)$. We have $v = 1/3(v_1 + v_2 + v_3) = v_1 + v_2 - v_4$. To check that v is not a $\mathbb{Z}_{\geqslant 0}$-combination of v_i, consider the discriminating function $f = 100x_1 + 10x_2 + x_3$. Then $f(v_1) = 210$, $f(v_2) = 21$, $f(v_3) = 102$, $f(v_4) = 120$, but $f(v) = 111$.

Counterexample 6. Let $\lambda = 2\pi_1 = 2e_1$, $n = 3$. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $v = e_1 + e_2 = 1/2(v_1 + v_2) = v_3 - v_4$. To check that v is not a $\mathbb{Z}_{\geqslant 0}$-combination of vectors v_i, consider the discriminating function $f = 5x_1 + 3x_2 + 9x_3$. Then $f(v_1) = 10$, $f(v_2) = f(v_4) = 6$, $f(v_3) = 14$, but $f(v) = 8$.

Counterexample 7. Take $\lambda = \pi_3 = e_1 + e_2 + e_3$, $n = 3$. Consider the following vectors:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $v = e_1 = 1/2(v_1 + v_2) = v_1 - v_3 + v_4$. To verify that v is not a $\mathbb{Z}_{\geqslant 0}$-combination of vectors v_i, consider the discriminating function $f = 11x_1 + 6x_2 - 14x_3$. Then $f(v_1) = 3$, $f(v_2) = 19$, $f(v_3) = 6$, $f(v_4) = 14$, but $f(v) = 11$.

Counterexample 8. $\lambda = \pi_4 = e_1 + e_2 + e_3 + e_4$, $n = 4$. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Take $v = (1,1,0,0) = \frac{1}{2}(v_1 + v_2) = v_3 - v_4$. Consider the discriminating function $f = 5x_1 + 5x_2 + 8x_3 - x_4$. Then $f(v_1) = 17$, $f(v_2) = f(v_4) = 3$, $f(v_3) = 13$, $f(v) = 10$. It is clear that v_1 and v_3 cannot be used in a $\mathbb{Z}_{\geqslant 0}$-combination. But 10 is not divisible by 3, and we cannot obtain v, using only v_2 and v_4.

Counterexample 9. Take $\lambda = \pi_2 = e_1 + e_2$, $n = 5$. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $v = e_1 + e_2 = 1/2(v_1 + v_2 + v_3 + v_4) = v_2 + v_3 + v_5 - v_6$. Take $f = 5x_1 + 6x_2 + x_3 + 2x_4 + 20x_5$. We have $f(v_1) = 6$, $f(v_2) = 4$, $f(v_3) = 8$, $f(v_4) = 4$, $f(v_5) = 21$, $f(v_6) = 22$, but $f(v) = 11$.

Remark 1. Counterexamples $5 - 7$ work for all $n \geqslant 3$, Counterexample 8 works for all $n \geqslant 4$, and Counterexample 9 works for all $n \geqslant 5$. Indeed, we can append $n - 3$ zero coordinates ($n - 4$ and $n - 5$, respectively) to each vector.

3.3. **Reduction to the already examined cases.** Consider two cases: (i) all $\ell_i \in \{0,1\}$; (ii) there is at least one ℓ_i with $|\ell_i| \geqslant 2$.

First consider case (i): all $\ell_i \in \{0,1\}$, which means that $\lambda = \pi_k = e_1 + e_2 + \ldots + e_k$, $k \leqslant n$.

Lemma 10. *An NSS for the pair (k, n_0) is at the same time an NSS for all the pairs (k, n), where $n \geqslant n_0$.*

Proof. Append $n - n_0$ zero coordinates to each vector. □

Lemma 11. *An NSS for the pair (k, n), where $k+2 \leqslant n$, is also an NSS for the pair $(k+2, n)$.*

Proof. Use Corollary 2: if $\lambda = e_1 + \ldots + e_{k+2}$, then

$$e_1 + e_2 + \ldots + e_k = \lambda - (e_{k+1} + e_{k+2}) = \frac{1}{2}(\lambda + (e_1 + \ldots + e_k - e_{k+1} - e_{k+2})),$$

hence it belongs to $M(\lambda)$. This means that an NSS for (k, n) is also an NSS for $(k+2, n)$. □

Now take any pair (k, n), not equal to $(1, n)$, $(2, 2)$, $(2, 3)$, and $(2, 4)$, where $k \leqslant n$.

If k is even and $n \leqslant 4$, then it is the pair $(4, 4)$, i.e. we get Counterexample 8. If k is even and $n \geqslant 5$, then we can modify Counterexample 9 to get the required NSS: first apply Lemma 10, and then apply Lemma 11. If k is odd and $k \geqslant 3$, then we can modify Counterexample 7 to get the required NSS in the same way.

Now consider case (ii).

Definition 1. By a *shift for C_n* we denote the procedure of replacing the point

$$\lambda = (\ldots, l, \ldots, l', \ldots)$$

with the point $\lambda' = (\ldots, l-1, \ldots, l'+1, \ldots)$ (at the same places) when $l - l' \geqslant 2$.

The point λ' belongs to $M(\lambda)$. Indeed, the point $(\ldots, l', \ldots, l, \ldots)$ belongs to $M(\lambda)$. Its convex hull with λ with a suitable coefficient equals λ'. Notice that for $l - l' \geqslant 2$ the sum of squares of coordinates of the point λ decreases after a shift by a positive integer:

$$(l-1)^2 + (l'+1)^2 = l^2 - 2l + 1 + l'^2 + 2l' + 1 = l^2 + l'^2 - 2(l - l' - 1) < l^2 + l'^2.$$

Hence, we can consequently apply only finitely many shifts.

Lemma 12. *Let $\lambda = (\ell_1, \ldots, \ell_n)$, such that $\ell_i \geqslant 2$ for some i. Then either $(2, 0, \ldots, 0)$ or $(2, 1, 0, \ldots, 0)$ belongs to $M(\lambda)$.*

Proof. Since λ is dominant, we have $\ell_1 \geqslant 2$. Now change λ, during this process it can be nondominant. Change sign at any coordinate, e.g. at ℓ_n, in such a way that $\ell_n \leqslant 0$, and shift it with ℓ_1 several times till the moment when ℓ_1 becomes 2. If meanwhile ℓ_n becomes positive, then change its sign to make it negative, and so on. Then fix $\ell_1 = 2$ and shift other coordinates in any possible way, changing signs at some coordinates, if needed. This process is finite, and if no further shift is possible, then it is either the point $(2, 0, \ldots, 0)$, or the point $(2, 1, 0, \ldots, 0)$. □

In case (ii) we can apply Lemma 12 and then Corollary 2: the required NSS'es for all highest weights will be produced either from Counterexample 5 or from Counterexample 6.

4. The root system D_n

As before, let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Q}^n$. The root system D_n, $n \geqslant 4$, consists of vectors $\{\pm e_i \pm e_j \mid 1 \leqslant i, j \leqslant n,\ i \neq j\}$. With respect to the system of simple roots

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \ldots,\ \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = e_{n-1} + e_n$$

the fundamental weights have the form

$$\pi_1 = e_1,\ \pi_2 = e_1 + e_2,\ \ldots,\ \pi_{n-2} = e_1 + \ldots + e_{n-2},$$
$$\pi_{n-1} = \frac{1}{2}(e_1 + \ldots + e_{n-1} - e_n),\ \pi_n = \frac{1}{2}(e_1 + \ldots + e_{n-1} + e_n).$$

The root lattice Ξ coincides with the lattice of all integer points in $\mathbb{Q}^n$ with even sum of coordinates. Next, the weight lattice Λ has the form

$$\Lambda = \{(\ell_1, \ell_2, \ldots, \ell_n) \mid 2\ell_i \in \mathbb{Z}, \ell_i - \ell_j \in \mathbb{Z}, i, j = 1, \ldots, n\}.$$

The Weyl group W acts by permutations of the set of coordinates and by changing signs on any set of coordinates of even cardinality. The weight $\lambda = (\ell_1, \ell_2, \ldots, \ell_n)$ is dominant if and only if $\ell_1 \geqslant \ldots \geqslant \ell_n$, $\ell_{n-1} + \ell_n \geqslant 0$. If all the coordinates of λ are integers (strictly half-integers), then the set $M(\lambda)$ consists of all integer (strictly half-integer) points in the polytope $P(\lambda)$, such that their sum of coordinates differs with the sum of coordinates of λ by an even number.

The reasoning for D_n has another structure than in the preceding cases. The cases of integer and noninteger coordinates of a fundamental weight are considered separately. Many NSS'es are taken from Section 3. *Shift for D_n* is the same as Shift for C_n.

4.1. **Coordinates of all weights are integers.**

Case 6. $\lambda = \pi_1 = e_1$. The set $M(\lambda)$ is hereditary normal, the proof is analogous to Case 1 of B_n.

Case 7. $\lambda = \pi_2 = e_1 + e_2$, $n = 4$. The set $M(\lambda)$ coincides with the analogous set from Case 5. That set is hereditary normal.

In all other cases we construct NSS'es. We often use NSS'es constructed for C_n, it is only necessary to check that for D_n the weights under consideration indeed belong to $M(\lambda)$. If a point v has a zero coordinate, then its orbits under the Weyl groups in cases C_n and D_n coincide, because the coordinate equal to 0 can be, if needed, multiplied by -1.

Counterexample 10. $\lambda = \pi_1 + \pi_2 = (2, 1, 0, 0)$. We can use Counterexample 5.

Counterexample 11. $\lambda = 2\pi_1 = (2, 0, 0, 0)$, $n = 4$. Counterexample 6 with the appended column of zeroes works.

Counterexample 12. $\lambda = \pi_3 + \pi_4 = e_1 + e_2 + e_3$, $n = 4$. Counterexample 7 with the appended column of zeroes works.

Counterexample 13. $\lambda = \pi_2 = e_1 + e_2$, $n = 5$. Counterexample 9 can be applied.

Counterexample 14. $\lambda = 2\pi_4 = e_1 + e_2 + e_3 + e_4$, $n = 4$. Counterexample 8 can be applied.

Now, using Corollary 2 applied to Counterexamples 10–14, we show that NSS'es exist in all the remaining cases: a) all coordinates of the highest weight equal ± 1, and their sum is odd; b) all coordinates of the highest weight equal ± 1, and their sum is even; c) λ has a coordinate such that its absolute value is 2 or more.

In this subsection all coordinates are integer, consequently, any set of weights for any n can be considered as a set of weights for a greater n, filling new coordinates with zeroes. Hence, Counterexamples 10, 11, 12, and 14 provide us with NSS'es for highest weights of the same form for all $n \geqslant 4$, and Counterexample 13 — for all $n \geqslant 5$.

In case a), if there are at least 5 nonzero coordinates, make two last of them zero. For this take λ', which differs from λ by the signs of two last coordinates, and take the midpoint of the interval $\lambda\lambda'$ instead of λ. Then make two more coordinates zero, etc., finally we reduce this case to Counterexample 12.

In case b), if we have only two nonzero coordinates, we can obtain an NSS from Counterexample 13: just append the required number of zeroes. If there are 4 nonzero coordinates, then an NSS can be obtained from Counterexample 14 by appending the required number of zeroes. If there are more than 4 nonzero coordinates (recall that their number is even and each equals ± 1), then make two last of them zero, then two more, and repeat this procedure up to the moment when their number equals 4.

In case c), depending on the parity of $\sum_1^n \ell_i$, one has to show that either the point $(2,0\dots,0)$ or the point $(2,1,0,\dots,0)$ belongs to $M(\lambda)$. The proof is almost the same as in the case of C_n. On the first step change λ to a point having at least one zero coordinate. Actually, take λ' which is obtained from λ by a sign change of ℓ_{n-1} and ℓ_n simultaneously. It is important that λ' has negative coordinates. Now work with λ'. Shift ℓ_2 with any negative coordinate, e.g. with ℓ_{n-1}, till the moment when one of them becomes zero. Permute $n-1$ last coordinates to make $l_n = 0$. Secondly, apply the algorithm from the proof of Lemma 12 to $n-1$ first coordinates of λ'. If in that algorithm it is required to change λ' with a point $w\lambda'$, w belongs to the Weyl group for C_n, then the same procedure can be applied for D_n. Actually, if w changes the sign of an odd number of coordinates, which is allowed for C_n but not for D_n, then to be applicable for D_n, the element w will also change the sign of the nth coordinate, i.e. the resulting point $w\lambda'$ will be the same since $\ell_n = 0$. This modification of w will belong to the Weyl group of D_n. Now, exactly as in the case of C_n, apply Lemma 12 and then Corollary 2, obtaining the required NSS'es for all highest weights either from Counterexample 10 or from Counterexample 11.

4.2. **Coordinates of all weights are noninteger and there exists a coordinate whose absolute value is not less than $\frac{3}{2}$.**

Lemma 13. *Under these conditions $M(\lambda)$ contains a point of the form $\left(\frac{3}{2},\frac{1}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right)$, where l'_i are half-integers, $i=4,\dots,n$.*

Proof. Since $\lambda = (\ell_1,\dots,\ell_n)$ is dominant, ℓ_1 is one of the coordinates with the maximal absolute value. Now change λ, letting it be nondominant. If $\ell_1 > 3/2$, we take λ' which is obtained from λ by a sign change of two last coordinates. It is important for us that λ' has negative coordinates. Now work with λ'. Shift ℓ_1 with any negative coordinates till the moment when ℓ_1 becomes equal to $3/2$. If needed, during this process make a sign change of two last coordinates again. Now fix ℓ_1 and perform the same procedure with ℓ_2 till the moment when $\ell_2 = 1/2$. If now ℓ_3 and ℓ_4 have the same sign, then change signs at ℓ_2 and ℓ_4 and shift ℓ_3 and ℓ_4 till the moment when one of them becomes $\pm 1/2$. If at some step they have the same sign, change the sign at ℓ_2 and ℓ_4. Permuting the coordinates, if needed, we may suppose that we obtained the point $(3/2,\pm 1/2,\pm 1/2,\dots)$. Now, if needed, change the signs at the pairs of coordinates $2,4$ and $3,4$. ☐

Now consider the following NSS:

$$v_1 = \left(\frac{3}{2},\frac{1}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right),\ v_2 = \left(-\frac{1}{2},-\frac{3}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right),\ v_3 = \left(\frac{1}{2},\frac{3}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right),$$
$$v_4 = \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right) = \frac{1}{2}\left(\left(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right) + \left(\frac{1}{2},\frac{3}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right)\right).$$

Then $v_0 = \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},l'_4,l'_5,\dots,l'_n\right) = \frac{1}{2}(v_1+v_2) = v_1 + v_4 - v_3$. If one looks at the third coordinate, it is clear that it is indeed an NSS.

4.3. **All coordinates of weights are noninteger and all coordinates of the highest weight are less than 1.**

Under these conditions $\lambda = (1/2,\dots,1/2,\pm 1/2) \in \{\pi_{n-1},\pi_n\}$. Since π_{n-1} and π_n are dual, it suffices to consider only the case $\lambda = \pi_n$.

Case 8. For $n=4$ the set $M(\lambda)$ is a subset of $M(\pi_4)$ for B_4 (see Case 3). Since in the case of C_n all the subsets were saturated, here it is also true.

Now the aim is to show that for $n = 5$, 6 the answer is positive, and for $n \geqslant 7$ it is negative.

Case 9. $\lambda = \pi_5 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $n = 5$. Here

$$M(\lambda) = \left\{ \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \mid \text{even number of minuses} \right\}.$$

To simplify the notation, multiply all the coordinates by 2. We obtain the set

$$M'(\lambda) = \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \mid \text{even number of minuses}\}.$$

Let us show that $M'(\lambda)$ is almost unimodular of volume 16. To compute a determinant of five arbitrary vectors, write them as a matrix and add the first row of this matrix to all the other rows. Now rows 2–5 are even, hence the volume of the determinant is divisible by 16. For the following vectors

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

the determinant equals 16, hence $M'(\lambda)$ is almost unimodular. Notice that each vector has length $\sqrt{5}$. The value of the determinant is at the same time the volume of the parallelepiped generated by these vectors, and the absolute value of the last number does not exceed $(\sqrt{5})^5 < 64$, hence equals 16, 32, or 48.

Letting $m = 16$, we obtain that all possible nonzero values of determinants are $\pm m$, $\pm 2m$, or $\pm 3m$.

Lemma 14. *If for some vectors $v_1, \ldots, v_5 \in M'(\lambda)$ the scalar product $(v_1, v_2) = -3$, then*

$$|\det(v_1, \ldots, v_5)| < 3m.$$

Proof. Each vector from $M'(\lambda)$ has length $\sqrt{5}$. Let S_{12} be the area of the parallelogram generated by vectors v_1 and v_2. Since $(v_1, v_2) = -3$, we have $S_{12} = 4$. From geometrical reasons

$$|\det(v_1, \ldots, v_5)| \leqslant S_{12} \cdot (\sqrt{5})^3 < 48 = 3m.$$

□

Lemma 15. *Take $v_1, \ldots, v_6 \in M'(\lambda)$.*

(i) *If $|\det(v_1, \ldots, v_5)| = 3m$, then one may suppose that*

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

up to the permutation of lines and up to the simultaneous sign change in pairs of columns.

(ii) *The following is impossible: $|\det(v_1, v_2, v_3, v_4, v_5)| = |\det(v_1, v_2, v_3, v_4, v_6)| = 3m$.*

Proof. (i) It follows from Lemma 14 that no two of these vectors differ in four coordinates. Hence, any two of these vectors differ exactly in 2 coordinates. Without loss of generality $v_1 = (1, 1, 1, 1, 1)$ and $v_2 = (-1, -1, 1, 1, 1)$. Then each of the three other vectors has exactly two -1's. Say that two first coordinates are *prefix*. To differ with v_2 exactly in two coordinates, each of the remaining vectors must have exactly one prefix coordinate equal to -1. The pigeonhole principle gives us that two of them (say, v_3 and v_4) have the same prefix coordinate equal to -1, without loss of generality this prefix coordinate is the first coordinate. Then the first coordinate of v_5 also equals -1, otherwise v_5 cannot differ simultaneously with v_2, v_3, and v_4 in two coordinates. Since all the vectors are pairwise distinct, we obtain the same set as in the formulation of the Lemma.

(ii) It follows from the previous part that v_5 and v_6 cannot differ in two coordinates. Hence, if one supposes that first five vectors are as above, then v_6 has four coordinates equal to -1. Without loss of generality this is either vector $(1,-1,-1,-1,-1)$ or vector $(-1,1,-1,-1,-1)$. No one of these vectors works. $\square$

Lemma 16. *Suppose that for some vectors $v_1,\dots,v_6 \in M'(\lambda)$ all the absolute values of their nonzero determinants are greater than m. Then all these determinants equal $\pm 2m$.*

Proof. On the contrary, suppose that there is a determinant equalling $\pm 3m$. Then Lemma 15 shows that all the other nonzero determinants equal $\pm 2m$. But the alternating sum of six determinants of 5-tuples of our vectors equals $\det(v_1-v_2, v_1-v_3,\dots,v_1-v_6)$. In the corresponding matrix all the entries are even, hence the determinant is divisible by $32 = 2m$. Contradiction with the fact that $3m \pm 2m \pm \ldots \pm 2m$ is not divisible by $2m$. $\square$

Consider an ENSS $\{v_0; v_1, v_2, \dots, v_s\}$. If the rank d of this set is less than 5, take $5-d$ vectors from $M'(\lambda)$ to make the rank equal to 5, and assign to them zero coefficients in the corresponding to v_0 $\mathbb{Z}$- and $\mathbb{Q}_{\geqslant 0}$-combinations. Now suppose that this ENSS is $\{v_0; v_1, v_2, v_3, v_4, v_5, \dots, v_s\}$, and only v_1, v_2, v_3, v_4, v_5 appear in the $\mathbb{Q}_{\geqslant 0}$-combination (maybe with zero coefficients). We may count all the determinants of the form $\det(v_1,\dots,\widehat{v_i},\dots,v_5,v_s)$, where one of the first 5 vectors is thrown out and one new vector is taken instead of it. Case a) one of them equals $\pm m$, case b) for all the nonzero determinants their absolute value is greater than m.

In case b) Lemma 16 gives us that we have $s-5$ unimodular six-element subsets

$$\{v_1,\dots,v_5,v_j\}, \quad 6 \leqslant j \leqslant s,$$

with $m' = 2m$. In each of them v_j can be expressed in $v_1,\dots,v_5$ with integer coefficients, hence the determinant of each 5-tuple in the set $\{v_1,\dots,v_s\}$ is divisible by $2m$, hence equals $\pm 2m$. This ENSS is hereditary normal by Lemma 4, a contradiction.

Case a) needs more punctuality. Lemma 7 gives us that the determinant $\pm m$ does not coincide with $\det(v_1, v_2, \dots, v_5)$. Without loss of generality $\det(v_1,\dots,v_4,v_6) = 16$ (if it equals -16, transpose two first vectors, then the determinant will change sign). By our assumption $\det(v_1,\dots,v_5) = \pm 2m$ or $\pm 3m$.

Lemma 17. *There are no vectors $w_1,\dots,w_6$ in $M'(\lambda)$ such that the following is true (simultaneously):*

$$\det(w_1,\dots,w_5) = \pm 2m, \quad \det(w_5,w_2,w_3,w_4,w_6) = \pm 2m,$$

these determinants have different signs, and $\det(w_1,\dots,w_4,w_6) = \pm m$.

Proof. Straightforward check using software Maple 7, [9]. $\square$

Lemma 18. *There are no vectors $w_1,\dots,w_6$ in $M'(\lambda)$ such that*

$$\det(w_1,\dots,w_5) = -2m \text{ and } \det(w_1,\dots,w_4,w_6) = -3m.$$

Proof. Using Lemma 15, we may assume that $w_6 = (1,1,1,1,1)$ and

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

The hyperplane $\langle w_1, w_2, w_3, w_4\rangle$ is defined by the equation $2x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Since $\det(w_1,\dots,w_5) < 0$ and $\det(w_1,\dots,w_6) < 0$, we have that w_5 and w_6 belong to the same half-space with respect to this hyperplane. Hence, exactly two coordinates of w_5 equal -1. Without loss of generality $w_5 = (1,-1,-1,1,1)$, but the corresponding determinant equals $-16 = -m$, a contradiction. $\square$

Recall that we had an ENSS. Let

$$v_0 = q_1 v_1 + \ldots + q_5 v_5 \tag{1}$$

and

$$v_0 = z_1 v_1 + \ldots + z_4 v_4 + z_6 v_6 \tag{2}$$

be the initial $\mathbb{Q}_{\geqslant 0}$- and $\mathbb{Z}$-combinations. Consider the decomposition

$$v_5 = y_1 v_1 + \ldots + y_4 v_4 + y_6 v_6$$

of the vector v_5 in the basis $\{v_1, v_2, v_3, v_4, v_6\}$. It can be re-written as

$$v_6 = -\frac{y_1}{y_6} v_1 - \ldots - \frac{y_4}{y_6} v_4 + \frac{1}{y_6} v_5. \tag{3}$$

Substituting (3) into (2), we obtain

$$v_0 = z_1 v_1 + z_2 v_2 + z_3 v_3 + z_4 v_4 + z_6 \left(-\frac{y_1}{y_6} v_1 - \ldots - \frac{y_4}{y_6} v_4 + \frac{1}{y_6} v_5\right). \tag{4}$$

Compare (1) and (4). From the uniqueness of the decomposition in a basis it follows that

$$q_1 = z_1 - z_6 \frac{y_1}{y_6}, \ldots, q_4 = z_4 - z_6 \frac{y_4}{y_6}, q_5 = z_6 \frac{1}{y_6}, \text{ all } q_i \in [0, 1). \tag{5}$$

If $|y_6| = 3$, i.e. $|\det(v_1, v_2, \ldots, v_5)| = 3m$, then by Lemma 15

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

The linear combination of these vectors with $\mathbb{Q}_{\geqslant 0}$-coefficients $q_1, \ldots, q_5$ must belong to the weight lattice multiplied by two, this means that all coordinates of the resulting vector must have the same parity. Subtracting the third coordinate from the second one, we obtain that $2(q_2 - q_3)$ is even, which implies $q_2 = q_3$, and analogously $q_2 = q_3 = q_4 = q_5$. The first coordinate of v_0 equals $q_1 - 4q_2$, while all the others equal $q_1 + 2q_2$. These numbers must also have the same parity, consequently, $q_2 \in \{0, \frac{1}{3}, \frac{2}{3}\}$. Since $q_1 - 4q_2$ and $q_1 + 2q_2$ are both integers and cannot simultaneously equal 0, we obtain that $q_1 = q_2 \in \{\frac{1}{3}, \frac{2}{3}\}$. Hence, v_0 equals either $(-1, 1, 1, 1, 1)$ or $(-2, 2, 2, 2, 2)$.

Lemma 19. *Take vectors $(v_1, v_2, v_3, v_4, v_5)$ from Lemma 15 and any vector*

$$v_6 \in M'(\lambda) \setminus \{v_1, \ldots, v_5\}.$$

Then the vector $(-1, 1, 1, 1, 1)$ can be represented as a $\mathbb{Z}_{\geqslant 0}$-combination of vectors

$$(v_1, v_2, v_3, v_4, v_5, v_6).$$

Proof. Up to the permutation of the last four coordinates, we can assume that v_6 is either $(1, 1, 1, -1, -1)$, or $(1, -1, -1, -1, -1)$, or $(-1, -1, -1, -1, 1)$. Consider these cases separately.

(i) $v_6 = (1, 1, 1, -1, -1)$. Then

$$(-1, 1, 1, 1, 1) = (1, 1, 1, -1, -1) + (-1, -1, 1, 1, 1) + (-1, 1, -1, 1, 1).$$

(ii) $v_6 = (1, -1, -1, -1, -1)$. Then

$$(-1, 1, 1, 1, 1) = 2(1, -1, -1, -1, -1) + (1, 1, 1, 1, 1) + (-1, -1, 1, 1, 1) + \\ + (-1, 1, -1, 1, 1) + (-1, 1, 1, -1, 1) + (-1, 1, 1, 1, -1).$$

(iii) $v_6 = (-1,-1,-1,-1,1)$. Then
$$(-1,1,1,1,1) = (-1,-1,-1,-1,1) + (1,1,1,1,1) + (-1,1,1,1,-1).$$
□

It remains to consider the case when $|y_6| = 2$. Here all $q_i \in \{0, \frac{1}{2}\}$.

If z_6 is even, then $q_1 = z_1 - z_6\frac{y_1}{y_6} = z_1 - y_1\frac{z_6}{y_6}$ is an integer from the interval $[0,1)$, hence it equals 0. Analogously all the other q_i's, $i = 2,\dots,5$, equal 0, consequently, $v_0 = 0$. A contradiction.

If z_6 is odd, then the saturation property is checked in the following way. We need to construct a $\mathbb{Z}_{\geqslant 0}$-combination for v_0, to do this take the vector v_6 and add several vectors from $v_1,\dots,v_5$ with suitable positive coefficients to obtain v_0. To show that it is possible, it is enough to verify that if we decompose both v_0 and v_6 in the basis $v_1,\dots,v_5$, then any pair of corresponding coordinates differs by an integer and that all coordinates of v_6 are strictly less than 1. This guarantees that they do not exceed the corresponding coordinates of v_0, since we know that the coordinates of v_0 equal q_i and belong to the interval $[0,1)$.

By (5), cases $i = 1,2,3,4$ and $i = 5$ should be considered separately. Since cases $i = 1,2,3,4$ are symmetrical, consider only cases $i = 1$ and $i = 5$. Since $\frac{z_6-1}{y_6}$ is integer, we have that
$$q_1 - \left(-\frac{y_1}{y_6}\right) = z_1 - z_6\frac{y_1}{y_6} + \frac{y_1}{y_6} = z_1 + \frac{(1-z_6)y_1}{y_6}$$
is integer, analogously $q_5 - \frac{1}{y_6} = \frac{z_6-1}{y_6}$ is integer, i.e. all the differences of the corresponding coordinates are integer. We also know that $y_1 = \det(v_5,v_2,v_3,v_4,v_6)$ and
$$y_6 = \det(v_1,v_2,v_3,v_4,v_5),$$
which means that $|y_1| \in \{0,1,2,3\}$ and $|y_6| = 2$. It follows from Lemmas 15, 17, and 18 that the number $-\frac{y_1}{y_6}$ is neither 1 nor $\frac{3}{2}$. In all the other cases the inequality $-\frac{y_i}{y_6} < 1$ is held for all i, $1 \leqslant i \leqslant 4$. It is also clear that $\frac{1}{y_6} < 1$. Hence, adding some v_i's ($1 \leqslant i \leqslant 5$), we can obtain v_0 from v_6, and the ENSS under consideration is not an ENSS. Therefore $M'(\lambda)$ is hereditary normal.

Case 10. $\lambda = \pi_6 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, $n = 6$. Here
$$M(\lambda) = \{(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2) \mid \text{even number of minuses}\}.$$
For convenience let us work with the set
$$M'(\lambda) = 2M(\lambda) = \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \mid \text{even number of minuses}\}.$$

Lemma 20. *The set $M'(\lambda)$ is almost unimodular of volume 64. The values of determinants equal ± 64 and ± 128, or, equivalently, $\pm m$ and $\pm 2m$.*

Proof. Consider a subset $\{v_1, v_2, \dots, v_6\} \subseteq M'$. Without loss of generality $v_1 = (1,1,1,1,1,1)$. Add v_1 to each of the other vectors and write down the obtained 6 vectors as rows of a matrix. The rows from the second till the sixth are even, hence the determinant is divisible by 32, and if we divide the rows from the second till the sixth by 2, the number of 1's in each of the rows of the remaining matrix will be even. Now add to the first column of the new matrix the sum of all other columns. The new first column is even, hence the determinant of the original matrix is divisible by 64. Now find an upper bound for it. Split the vectors in three pairs and generate a parallelogram with each pair, then the volume of the parallelepiped does not exceed the product of areas of these three parallelograms. Each vector in $M'(\lambda)$ has length $\sqrt{6}$, the absolute value of the scalar product of two arbitrary vectors equals 2, hence the area of each parallelogram equals $6^{2/2}\sqrt{1-(1/3)^2} = 2^{5/2}$. Finally, the volume does not exceed $2^{15/2} < 192$, consequently, its absolute value equals 64 or 128. □

Suppose that we have an ENSS $\{v_0; v_1, v_2, v_3, v_4, v_5, v_6\}$ in $M'(\lambda)$. Consider a $\mathbb{Q}_{\geqslant 0}$-combination corresponding to the vector v_0. By Lemma 7 we have that $|\det(v_1, v_2, v_3, v_4, v_5, v_6)|$ equals 128, consequently, by Lemma 6 the coefficients of the initial $\mathbb{Q}_{\geqslant 0}$-combination equal 0 or $\frac{1}{2}$.

Lemma 21. *If the determinant of the set of vectors $\{v_1, v_2, \ldots, v_6\} \subseteq M'$ equals 128, then up to multiplying vectors v_i by -1, multiplying pairs of coordinates simultaneously by -1 and interchanging columns and rows we may assume that it is the set of rows*

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}, \ w_i \in \{v_i, -v_i\}.$$

Proof. The set $M'(\lambda)$ contains a vector $-v$ for each vector v. Hence, to compute the determinants, we may consider only 16 vectors instead of 32. Indeed, take only vectors with the positive sum of coordinates:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Without loss of generality the minor of size six contains two first rows of this matrix. By the direct check in Maple 7 [9], we obtain that if its determinant is 128, then it is either the set of rows (1 2 3 4 5 6) or the set (1 2 7 8 9 10). Now notice that these two minors interchange if we transpose columns 5 and 6. □

Now reasoning is analogous to Case 3: to obtain v_0 as a $\mathbb{Z}$-combination, we have to use at least one more vector v_7. According to Lemma 2(ii), $v_7 \neq -v_i$, $1 \leqslant i \leqslant 6$. Show that if we augment the given six vectors by any other vector v_7, we obtain v_0 as a $\mathbb{Z}_{\geqslant 0}$-combination. For convenience till the end of this proof we suppose that $M'(\lambda)$ consists of points $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ having odd number of -1's. Then we can reformulate the result of the preceding lemma in the following way: if the determinant of the given 6 vectors $v_1, \ldots, v_6$ equals 128, then, acting with the Weyl group, we can map one of the (unordered) sets $(\pm v_1, \pm v_2, \dots, \pm v_6)$ to the set of rows

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}. \tag{6}$$

The vector $v_7 \in M'(\lambda)$ does not equal $\pm v_i$, hence three of its coordinates equal 1, and three other equal -1. It is easy to see that any such vector can be expressed as a $\mathbb{Q}$-linear combination of rows of matrix (6) if one takes three rows with the coefficient $1/2$, and three others with the coefficient $-1/2$. Hence, the decomposition of the vector v_7 in vectors $v_1, \dots, v_6$ has the form $(\pm\frac{1}{2}v_1 \pm \frac{1}{2}v_2 \pm \frac{1}{2}v_3 \pm \frac{1}{2}v_4 \pm \frac{1}{2}v_5 \pm \frac{1}{2}v_6)$.

Now express v_0 as a $\mathbb{Q}$-linear combination of rows of (6). Let r_i be the coefficient of the ith row. By Lemma 6, each r_i equals 0 or $\pm\frac{1}{2}$ (because the rows of (6) may be not the initial v_i's but $-v_i$'s). Show that all r_i are zero. Compare the first and the second coordinates of v_0. Their difference is even. On the other hand, it equals $(-r_1 + r_2 + r_3 + r_4 + r_5 + r_6) - (r_1 - r_2 + r_3 + r_4 + r_5 + r_6) = 2(r_2 - r_1)$. Consequently, either r_1 and r_2 are both zero or both nonzero. Repeating this procedure for other pairs of columns, we obtain that if $v_0 \neq 0$, then all r_i are nonzero. Hence, $v_0 = \frac{1}{2}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6)$. Notice that the vector $\frac{1}{2}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6)$ can be obtained from any vector v_7 of the form $(\pm\frac{1}{2}v_1 \pm \frac{1}{2}v_2 \pm \frac{1}{2}v_3 \pm \frac{1}{2}v_4 \pm \frac{1}{2}v_5 \pm \frac{1}{2}v_6)$ by adding several v_i's (cf. Section 2.1). Consequently, $M'(\lambda)$ is hereditary normal.

For $n \geqslant 7$ multiply all the coordinates by 2. After this all the coordinates of the initial vectors become ± 1. Now construct an NSS.

Counterexample 15. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

Then $v = (2, 0, 0, 0, 0, 0, 0) = \frac{1}{2}(v_1 + v_2 + v_3 + v_4) = v_5 + v_6 + v_7 - v_1$. If we consider the first coordinate, then if v is a $\mathbb{Z}_{\geqslant 0}$-combination of some v_i, it is the sum of two v_i's. But no pairwise sum equals v, and it is indeed an NSS.

Counterexample 15 can be easily modified for the greater values of n. Indeed, append $n-7$ coordinates equalling 1 to each vector. It is easy to see that for $n \geqslant 7$ these vectors belong to $M(\lambda)$ for $\lambda = \pi_{n-1}$. Since 1 is at the same time the first coordinate of all v_i, each linear combination of v_i's will have the same value on each appended coordinate and on the first coordinate.

Theorem 1 is proved.

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