A REMARK CONCERNING FEYNMAN KAC FORMULAS FOR THE PERTURBED HARMONIC OSCILLATOR

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Abstract. We give the solution of certain parabolic evolution problems (time-depending perturbations of the heat equation for the harmonic oscillator) as explicit integrals on a space of continuous functions, called the Wiener space. The methods are based on the Mehler formula giving the solution of the unperturbed problem and on the use of discretization to split the difficulties.

1. Introduction.

The aim of this article is to give an *explicit* expression for the solutions of the problem

$$(1)\qquad \begin{cases} \dfrac{\partial v}{\partial t}(t,x) - \dfrac{\partial^2 v}{\partial x^2}(t,x) + (x^2 + c(t,x))v(t,x) = 0 \quad \text{on} \quad]0,\infty[\times\mathbb{R} \\ \lim\limits_{t\to 0} v(t,x) = v_0(x) \end{cases},$$

as a Feynman Kac type integral on the Wiener space C_W, which is the space of continuous functions on $[0,1]$, equipped with a special measure m_W called Wiener measure (see Subsection 2.1).
Recall that the theory of semigroups and their perturbations gives the existence and uniqueness of the solution of (1) in $C^0(\mathbb{R}^+, L^2(\mathbb{R}))$, if, for example, c is continuous, bounded and satisfies the following Hölder condition

$$(2)\qquad \exists L > 0, \exists \alpha \in [0,1] \ : \ \forall s,t \in [0,\infty),\ \forall x \in \mathbb{R}, \quad |c(t,x) - c(s,x)| \leq L|t-s|^{\alpha} \ .$$

The problem of giving such explicit solutions has already been studied under more restrictive conditions than in the present case. In [3], [6], the case when the potential $V(t,x) = x^2 + c(t,x)$ is replaced by a general, unbounded but time-independant potential $V(x)$ is studied and in [5], the author deals with the case when $V(t,x)$ is bounded.
The present work is devoted to the study of the mixed case (1). The main result of this paper is the following

Theorem 1. *Let $v_0 \in L^2(\mathbb{R})$. Let c be continuous and bounded on $[0,\infty) \times \mathbb{R}$. Suppose that c belongs to $L^2(]0,T[\times\mathbb{R})$ for any $T > 0$ and satisfies the Hölder condition (2). The* *function v defined on $]0,\infty)\times\mathbb{R}$ by*

$$
\begin{aligned}
v(t,x) = &\int_{C_W} v_0(\sqrt{2t}\ w(1)+x)\ \exp\left(-t\int_0^1 \left(x+\sqrt{2t}\ w(s)\right)^2\ ds\right)\\
&\qquad\exp\left(-t\int_0^1 c\left(t(1-s),\sqrt{2t}\ w(s)+x\right)\ ds\right)\ dm_W(w)
\end{aligned}
\tag{3}
$$

is the solution of (1) in $C^0(\mathbb{R}^+,L^2(\mathbb{R}))$.

This result differs from the usual Feynman-Kac formula, for the function $w\in C_W$ is never evaluated in t. No well known change of measure ([6], [1], [9]) on C_W allows to get one formula from the other one.
One interest would be to get regularity results (or to study the dependency on a parameter) more easily than using the theory of semi-groups, by straightforward derivations.
Problem (1) will be considered as a perturbation of the heat equation related to the harmonic oscillator H. Section 2 contains the notions needed about Wiener integrals, followed by some facts about the heat kernel for the harmonic oscillator, in particular Mehler's formula. The proof of a preliminary version of Theorem 1, under restrictive regularity conditions, is completed in Sections 3 and 4 by constructing a sequence of functions expressed as integrals on C_W and converging to the solution of (1). The regularity conditions are weakened in Section 6, which leads to Theorem 1. In section 5, a slight modification of section 3 gives an alternate demonstration of Feynman-Kac formula, which does not rely on Itô calculus. Eventually we give some explicit expressions of Wiener integrals, which can be deduced from Theorem 1 (annexe A).

2. Preliminaries

2.1. **Wiener integrals.** The construction of the Wiener measure is detailed, for example, in [6] and [10]. The (classical) *Wiener space*, denoted by C_W, is the set of all real-valued continuous functions w on $[0,1]$, with $w(0)=0$. It can be equipped with a probability measure m_W called *Wiener measure*, which is defined on a σ-field $\mathfrak{M}^*$ containing all sets of the type

$$J=\{w\in C_W\ :\ (w(t_1),\dots,w(t_n))\in\mathcal{H}\},$$

where $t_0=0<t_1<\cdots<t_n\le 1$ and $\mathcal{H}$ is a Borel set of $\mathbb{R}^n$. For this kind of set the measure is given by

$$m_W(J)=\int_{\mathcal{H}} f_{t_1,\dots,t_n}(\xi_1,\dots,\xi_n)\ d\xi_1\dots d\xi_n,$$

where $f_{t_1,\dots,t_n}$ is the *normal density*

$$f_{t_1,\dots,t_n}(\xi_1,\dots,\xi_n)=\left((2\pi)^n\prod_{i=1}^n(t_i-t_{i-1})\right)^{-1/2}\exp\left(-\frac{1}{2}\sum_{i=1}^n\frac{(\xi_i-\xi_{i-1})^2}{t_i-t_{i-1}}\right)$$

(with $\xi_0=0$).
The integral of an integrable, real-valued function F defined on C_W will be denoted by $E^W[F]$ or $\int_{C_W} F(w)\ dm_W(w)$. When $F(w)$ depends on the values of w at finitely many

fixed points $0 \leq t_1 < t_2 < \dots t_n$ of $[0,1]$, $E^W[F]$ can be expressed as follows. Let φ be a measurable real-valued function on $\mathbb{R}^n$ and let $\mathcal{R}^1$ denote the Borel sets on $\mathbb{R}$. The mapping

$$\begin{array}{rccl} F: & (C_W, \mathfrak{M}^*) & \to & (\mathbb{R}, \mathcal{R}^1) \\ & w & \mapsto & \varphi(w(t_1), \dots, w(t_n)) \end{array}$$

is measurable and

$$\int_{C_W} F(w)\, dm_W(w) = \int_{\mathbb{R}^n} \varphi(\xi_1, \dots, \xi_n) f_{t_1,\dots,t_n}(\xi_1, \dots, \xi_n)\, d\xi_1 \dots d\xi_n, \tag{4}$$

in the sense that the existence of one side implies the existence of the other and the equality. When F depends on the value of w at infinitely many values of t, it is useful to recall that, if the topology on C_W is defined by the uniform norm, any open set is $\mathfrak{M}^*$-measurable and any real-valued, continuous function is $\mathfrak{M}^*$-measurable.
One important propertie is that the so-called coordinate process, defined on C_W by

$$\forall t \in [0,1],\ \forall w \in C_W,\ X_t(w) = w(t)$$

is a Brownian motion. A consequence is Fernique's Theorem, which states that there exists a positive α such that the integral $\int_{C_W} e^{\alpha ||w||^2}\, dm_W(w)$ converges, $||w||$ being the uniform norm of w ([6]).

2.2. **Heat equation for the harmonic oscillator.**

Classically the problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) - \dfrac{\partial^2 v}{\partial x^2}(t,x) + x^2 v(t,x) = 0 & (t,x) \in]0,\infty[\times\mathbb{R}, \\ v(0) = v_0 \end{cases} \tag{5}$$

(where $v_0 \in L^2(\mathbb{R})$) has a unique solution v belonging to $C^0([0,\infty[, L^2(\mathbb{R}))$ ([8]). More precisely, there exists a semigroup $(U_t)_{t\geq 0}$ of $L^2(\mathbb{R})$-contractions such that, for all $t > 0$, $v(t,\cdot) = U_t v_0$. In the case of the harmonic oscillator this semigroup is explicitely given by Mehler's formula ([4])

$$(U_t v_0)(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \mathrm{sh}(2t)}} \exp\left(-\frac{1}{2}\left((x^2+z^2)\frac{\mathrm{ch}(2t)}{\mathrm{sh}(2t)} - \frac{2xz}{\mathrm{sh}(2t)}\right)\right) v_0(z)\, dz. \tag{6}$$

One can check directly that $v : (t,x) \mapsto (U_t v_0)(x)$ is infinitely derivable on $]0,\infty[\times\mathbb{R}$. The change of variable $z - \dfrac{x}{\mathrm{ch}(2t)} = y$ gives another expression,

$$v(t,x) = (U_t v_0)(x) = \int_{\mathbb{R}} q(t,x,y)\, v_0\left(y + \frac{x}{\mathrm{ch}(2t)}\right)\, dy \tag{7}$$

where

$$q(t,x,y) = (2\pi\ \mathrm{sh}(2t))^{-1/2} \exp\left(-\frac{1}{2}\frac{\mathrm{ch}(2t)}{\mathrm{sh}(2t)} y^2 - \frac{1}{2}\frac{\mathrm{sh}(2t)}{\mathrm{ch}(2t)} x^2\right).$$

Both formulae give solutions of (5) for initial conditions which do not necessarily belong to L^2. For example, if v_0 is continuous and vanishes at infinity, $U_t v_0$ is C^∞ on $]0,\infty[\times\mathbb{R}$ and continuous on $[0,+\infty) \times \mathbb{R}$. The property

$$\forall t, \tau \geq 0\ , \qquad U_t U_\tau v_0 = U_{t+\tau} v_0$$

remains valid as well.

3. Approximation sequence

This Section is the first step in the proof of Theorem 1. It is devoted to the construction of a sequence $(v_n^{(t)})_{n\in\mathbb{N}}$ of functions designed to approximate the solution of (1). For a given positive integer n and a fixed positive t, n factors $e^{\frac{t}{n}H}$ coming from (6) alternate with n factors $e^{\frac{t}{n}c(t_k,x)}$ (as in Trotter's formula), where the t_k s are the bounds of a suitable discretization of $[0,t]$. The expression thus found can be written as an integral on the Wiener space which converges to the function v in Theorem 1. The exponent (t) stresses the fact that the sequence depends on the upper bound of the time interval.

Suppose that the function $c = c(t,x)$ is continuous and bounded on $[0,\infty)\times\mathbb{R}$ and that $v_0 = v_0(x)$ is continuous on $\mathbb{R}$ and vanishes at infinity. Let us split the interval $[0,t]$ into $2n$ subintervals bounded by the $\tau_k = \frac{kt}{2n}$, $k = 0,\dots,2n$ and assign $v_n^{(t)}$ to satisfy, on each subinterval, an incomplete version of

$$\frac{\partial v}{\partial t}(t,x) - \frac{\partial^2 v}{\partial x^2}(t,x) + x^2 v(t,x) + c(t,x)v(t,x) = 0.$$

The sequence $v_n^{(t)}$ is constructed by induction in the following way

- on the even interval $[\tau_{2k},\tau_{2k+1}]$, $v_n^{(t)}$ is a solution of the heat equation for the harmonic oscillator

$$\frac{\partial v}{\partial \tau}(\tau,x) - 2\left(\frac{\partial^2 v}{\partial x^2}(\tau,x) - x^2 v(\tau,x)\right) = 0 \tag{8}$$

  and satisfies the initial condition $\lim_{\tau\to\tau_{2k}} v(\tau,x) = v_n^{(t)}(\tau_{2k},x)$, where $v_n^{(t)}(\tau_{2k},\cdot)$ was built in the preceding step ;
- on the odd interval $[\tau_{2k+1},\tau_{2k+2}]$, $v_n^{(t)}$ satisfies the ordinary differential equation

$$\frac{\partial v}{\partial \tau}(\tau,x) + 2c(\tau_{2k+2},x)v(\tau,x) = 0, \tag{9}$$

  with the initial condition $\lim_{\tau\to\tau_{2k+1}} v(\tau,x) = v_n^{(t)}(\tau_{2k+1},x)$, where $v_n^{(t)}(\tau_{2k+1},\cdot)$ was built in the preceding step.

Both equations have of course an explicit solution

$$v_n^{(t)}(\tau,x) = \left(U_{2(\tau-\tau_{2k})} v_n^{(t)}(\tau_{2k},\cdot)\right)(x) \qquad \text{on} \quad [\tau_{2k},\tau_{2k+1}],$$

$$v_n^{(t)}(\tau,x) = e^{-2c(\tau_{2k+2},x)(\tau-\tau_{2k+1})} v_n^{(t)}(\tau_{2k+1},x) \quad \text{on} \quad [\tau_{2k+1},\tau_{2k+2}].$$

The equation (9) is a constant coefficient linear differential equation since c depends on the fixed time τ_{2k+2}. The initial conditions ensure the continuity with respect to τ. The factor 2 in (8) and (9) compensates the fact that each equation is solved on half of the interval (see Lemma 9 below and [5], where this discretization was introduced, to treat the usual heat equation.)

We first write $v_n^{(t)}(t,x)$ as iterated integrals.

Proposition 2. *For all real x we have*

$$v_n^{(t)}(t,x) = \int_{\mathbb{R}^n} (2\pi \mathrm{sh}(2t/n))^{-n/2}\, v_0\left(\sigma_n + \frac{x}{\mathrm{ch}(2t/n)^n}\right)$$
$$\exp\left(-\frac{1}{2}\frac{\mathrm{ch}(2t/n)}{\mathrm{sh}(2t/n)}\sum_{j=1}^{n}\left(\sigma_{n-j+1} - \frac{\sigma_{n-j}}{\mathrm{ch}(2t/n)}\right)^2\right)$$
$$\exp\left(-\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)}\sum_{j=1}^{n}\left(\sigma_{n-j} + \frac{x}{\mathrm{ch}(2t/n)^{n-j}}\right)^2\right)$$
$$\exp\left(-\frac{t}{n}\sum_{j=1}^{n} c\left(\frac{jt}{n}, \sigma_{n-j} + \frac{x}{\mathrm{ch}(2t/n)^{n-j}}\right)\right) \quad d\sigma_1 \dots d\sigma_n$$

with the convention $\sigma_0 = 0$.

Proof. By induction on k we get that, for all $\tau \in [\tau_{2k}, \tau_{2k+1}]$,

$$v_n^{(t)}(\tau,x) = \int_{\mathbb{R}^{k+1}} q(2(\tau - \tau_{2k}), x, y_{k+1})$$
$$\prod_{j=1}^{k} q\left(t/n, \sum_{l=j+1}^{k+1}\frac{y_l}{\mathrm{ch}(2t/n)^{l-1-j}} + \frac{x}{\mathrm{ch}(4(\tau-\tau_{2k}))\mathrm{ch}(2t/n)^{k-j}}, y_j\right)$$
$$\prod_{j=1}^{k}\exp\left(-\frac{t}{n}c\left(\tau_{2j}, \sum_{l=j+1}^{k+1}\frac{y_l}{\mathrm{ch}(2t/n)^{l-1-j}} + \frac{x}{\mathrm{ch}(4(\tau-\tau_{2k}))\mathrm{ch}(2t/n)^{k-j}}\right)\right) \tag{10}$$
$$v_0\left(\sum_{l=1}^{k+1}\frac{y_l}{\mathrm{ch}(2t/n)^{l-1}} + \frac{x}{\mathrm{ch}(4(\tau-\tau_{2k}))\mathrm{ch}(2t/n)^{k}}\right) \quad dy_1 \dots dy_{k+1}.$$

For $\tau = \tau_{2n} = t$ and $k+1 = n$, this gives

$$v_n^{(t)}(t,x) = \int_{\mathbb{R}^n} v_0\left(\sum_{l=1}^{n}\frac{y_l}{\mathrm{ch}(2t/n)^{l-1}} + \frac{x}{\mathrm{ch}(2t/n)^n}\right)$$
$$\prod_{j=1}^{n} q\left(t/n, \sum_{l=j+1}^{n}\frac{y_l}{\mathrm{ch}(2t/n)^{l-1-j}} + \frac{x}{\mathrm{ch}(2t/n)^{n-j}}, y_j\right)$$
$$\prod_{j=1}^{n}\exp\left(-\frac{t}{n}c\left(\tau_{2j}, \sum_{l=j+1}^{n}\frac{y_l}{\mathrm{ch}(2t/n)^{l-1-j}} + \frac{x}{\mathrm{ch}(2t/n)^{n-j}}\right)\right) \quad dy_1 \dots dy_n,$$

with the convention that a sum is equal to zero if its lower index is strictly superior to its upper index (this is useful for $j = n$). Let

$$\sigma_k = \sum_{l=n+1-k}^{n} \frac{y_l}{\mathrm{ch}(2t/n)^{l-1-n+k}} \ , \ 1 \leq k \leq n \tag{11}$$

(recall $\sigma_0 = 0$). Reciprocally,

$$y_j = \sigma_{n+1-j} - \frac{\sigma_{n-j}}{\mathrm{ch}(2t/n)} \ , j = 1, \dots, n.$$

and this change of variables $y \to \sigma$ has Jacobian equal to 1. Hence

$$v_n^{(t)}(t,x) = \int_{\mathbb{R}^n} v_0 \left(\sigma_n + \frac{x}{(\mathrm{ch}(2t/n))^n} \right)$$

$$\prod_{j=1}^{n} q \left(\frac{t}{n}, \sigma_{n-j} + \frac{x}{(\mathrm{ch}(2t/n))^{n-j}}, \sigma_{n+1-j} - \frac{\sigma_{n-j}}{\mathrm{ch}(2t/n)}, \right)$$

$$\prod_{j=1}^{n} \exp \left(-\frac{t}{n} c \left(jt/n, \sigma_{n-j} + \frac{x}{(\mathrm{ch}(2t/n))^{n-j}} \right) \right) \quad d\sigma_1 \dots d\sigma_n.$$

Replacing q by its expression gives the Proposition. ☐

The second step is the transformation of this integral on $\mathbb{R}^n$ into an integral on the Wiener space. Since C_W does not depend on n we then shall be able to let n converge to infinity. These computations lead to the following Proposition.

Proposition 3. *Let v_0 be continuous and bounded on $\mathbb{R}$. Suppose c is continuous and has a lower bound on $[0, \infty) \times \mathbb{R}$. Moreover let c have an x-derivative c_x at each point of $]0, \infty) \times \mathbb{R}$ and suppose c_x has an extension to $[0, \infty) \times \mathbb{R}$ which is continuous and bounded.*
When n goes to infinity the numerical sequence $v_n^{(t)}(t,x)$ converges to a limit $v(t,x)$ defined by

$$v(t,x) = \int_{C_W} v_0(\sqrt{2t}\ w(1) + x) \ \exp \left(-t \int_0^1 \left(x + \sqrt{2t}\ w(s) \right)^2 \ ds \right)$$

$$\exp \left(-t \int_0^1 c \left(t(1-s), \sqrt{2t} w(s) + x \right) \ ds \right) \ dm_W(w).$$

Proof. To bring out the normal density one uses the change of variables

$$\xi_j = \sigma_j (n \ \mathrm{sh}(2t/n))^{-1/2}, \ 1 \leq j \leq n \tag{12}$$

and the convention $\xi_0 = \sigma_0 = 0$. It follows

$$\begin{aligned} v_n^{(t)}(t,x) &= \int_{\mathbb{R}^n} (2\pi/n)^{-n/2} v_0\left((n\ \mathrm{sh}(2t/n))^{1/2}\xi_n + \frac{x}{\mathrm{ch}(2t/n)^n} \right) \\ & \exp\left(-\frac{1}{2}(n\ \mathrm{ch}(2t/n) \sum_{\ell=1}^{n} \left(\xi_{n-\ell+1} - \frac{\xi_{n-\ell}}{\mathrm{ch}(2t/n)} \right)^2 \right) \\ & \exp\left(-\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)} \sum_{\ell=1}^{n} \left((n\ \mathrm{sh}(2t/n))^{1/2}\xi_{n-\ell} + \frac{x}{\mathrm{ch}(2t/n)^{n-\ell}} \right)^2 \right) \\ & \exp\left(-\frac{t}{n}\sum_{\ell=1}^{n} c\left(\frac{\ell t}{n}, (n\ \mathrm{sh}(2t/n))^{1/2}\xi_{n-\ell} + \frac{x}{\mathrm{ch}(2t/n)^{n-\ell}} \right) \right) \quad d\xi_1 \ldots d\xi_n \ . \end{aligned}$$

We introduce the normal density for equidistant points $0 \le t_i = i/n \le 1$,

$$f_{t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) = (2\pi/n)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} n(\xi_i - \xi_{i-1})^2 \right),$$

to make the transition to Wiener space more natural. We first obtain

$$\begin{aligned} v_n^{(t)}(t,x) = \int_{\mathbb{R}^n} & v_0\left((n\ \mathrm{sh}(2t/n))^{1/2}\xi_n + \frac{x}{\mathrm{ch}(2t/n)^n} \right) f_{t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) \\ \exp & \left(\frac{n}{2}\sum_{i=1}^{n}(\xi_i - \xi_{i-1})^2 - \frac{1}{2}(n\ \mathrm{ch}(2t/n)\sum_{j=1}^{n}\left(\xi_j - \frac{\xi_{j-1}}{\mathrm{ch}(2t/n)} \right)^2 \right) \\ \exp & \left(-\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)}\sum_{j=0}^{n-1}\left((n\ \mathrm{sh}(2t/n))^{1/2}\xi_j + \frac{x}{\mathrm{ch}(2t/n)^j} \right)^2 \right) \\ \exp & \left(-\frac{t}{n}\sum_{j=0}^{n-1} c\left(\frac{(n-j)t}{n}, (n\ \mathrm{sh}(2t/n))^{1/2}\xi_j + \frac{x}{\mathrm{ch}(2t/n)^j} \right) \right) \quad d\xi_1 \ldots d\xi_n \ , \end{aligned}$$

and then, by formula (4),

$$v_n^{(t)}(t,x) = \int_{C_W} F_n(w)G_n(w)H_n(w)\ v_0\left((n\ \mathrm{sh}(2t/n))^{1/2}w(1) + \frac{x}{\mathrm{ch}(2t/n)^n} \right) \quad dm_W(w),$$

with

$$F_n(w) = \exp\left(\frac{n}{2}\sum_{i=1}^{n}\left[(1-\mathrm{ch}(2t/n))w(t_i)^2 + (1-\mathrm{ch}(2t/n)^{-1})w(t_{i-1})^2\right]\right)$$

$$G_n(w) = \exp\left(-\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)}\sum_{j=0}^{n-1}\left((n\ \mathrm{sh}(2t/n))^{1/2}w(t_j) + \frac{x}{\mathrm{ch}(2t/n)^j}\right)^2\right)$$

$$H_n(w) = \exp\left(-\frac{t}{n}\sum_{j=0}^{n-1} c\left(\frac{(n-j)t}{n}, (n\ \mathrm{sh}(2t/n))^{1/2}w(j/n) + \frac{x}{\mathrm{ch}(2t/n)^j}\right)\right).$$

The Proposition is now a consequence of the dominated convergence Theorem. The convergences and estimations concerning the four factors are treated in the Lemma just below

Lemma 4. *Denote by m a lower bound of c. For all $w \in C_W$,*

- $0 \le F_n(w) \le 1$ *and* $\lim_{n\to\infty} F_n(w) = 1$,
- $0 \le G_n(w) \le 1$ *and* $\lim_{n\to\infty} G_n(w) = \exp\left(-t\int_0^1 \left(x + \sqrt{2t}\ w(s)\right)^2\ ds\right)$,
- $0 \le H_n(w) \le e^{-mt}$ *and* $\lim_{n\to\infty} H_n(w) = \exp\left(-t\int_0^1 c\left(t(1-s), \sqrt{2t}\ w(s) + x\right)\ ds\right)$,
- $\left|v_0\left((n\ \mathrm{sh}(2t/n))^{1/2}w(1) + \frac{x}{\mathrm{ch}(2t/n)^n}\right)\right| \le ||v_0||_\infty$ *and* $\lim_{n\to\infty} v_0\left((n\ \mathrm{sh}(2t/n))^{1/2}w(1) + \frac{x}{\mathrm{ch}(2t/n)^n}\right) = v_0(\sqrt{2t}\ w(1) + x)$.

Proof of Lemma 4
We shall use more than once the continuity of $w \in C_W$ to compute limits of Riemann sums like $\lim_{n\to\infty}\sum_{j=0}^{n-1} w(t_j)$. The positivity and the estimates proposed are obvious, except for F_n which we shall treat first.
Let us develop the argument of the exponential

$$A_n = \frac{n}{2}\sum_{i=1}^{n}\left[(1-\mathrm{ch}(2t/n))w(t_i)^2 + (1-\mathrm{ch}(2t/n)^{-1})w(t_{i-1})^2\right].$$

Since $t_n = 1$ and $w(t_0) = w(0) = 0$,

$$A_n = \frac{n}{2}(1-\mathrm{ch}(2t/n))w(1)^2 - \frac{n}{2}\frac{(\mathrm{ch}(2t/n)-1)^2}{\mathrm{ch}(2t/n)}\sum_{i=1}^{n-1} w(t_i)^2 \le 0\ ,$$

and then $F_n \le 1$.
Now we claim that A_n converges to 0. An asymptotic expansion is enough for the term

containing $w(1)$. The second term can be written as

$$\frac{n^2}{2}\frac{(\mathrm{ch}(2t/n)-1)^2}{\mathrm{ch}(2t/n)} \quad\times\quad \frac{1}{n}\sum_{i=1}^{n-1} w(t_i)^2.$$

In this product the second factor converges to $\int_0^1 w^2\, ds$ and the first one to 0, using an asymptotic expansion.

To compute the limit of G_n we split the argument of the exponential into three terms :

$$A_n = -\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)}\sum_{j=0}^{n-1}(n\ \mathrm{sh}(2t/n))w(t_j)^2$$

$$B_n = -\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)}\sum_{j=0}^{n-1}(n\ \mathrm{sh}(2t/n))^{1/2}w(t_j)\frac{x}{\mathrm{ch}(2t/n)^j}$$

$$C_n = -\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)}\sum_{j=0}^{n-1}\frac{x^2}{\mathrm{ch}(2t/n)^{2j}}\ .$$

A direct computation gives

$$A_n = -\frac{n^2}{2}\ \frac{\mathrm{sh}(2t/n)^2}{\mathrm{ch}(2t/n)}\ \frac{1}{n}\sum_{j=0}^{n-1} w(t_j)^2 \quad\longrightarrow\quad -\frac{(2t)^2}{2}\int_0^1 w^2(s)\ ds\ .$$

The third term is a geometric sum. An asymptotic expansion leads to

$$C_n \longrightarrow -tx^2.$$

The second term is decomposed as

$$B_n \ = \underbrace{-x\,\frac{n^{3/2}\mathrm{sh}(\frac{2t}{n})^{3/2}}{\mathrm{ch}(\frac{2t}{n})}\ \frac{1}{n}\sum_{j=0}^{n-1} w(t_j)}_{D_n}\ \underbrace{-x\,\frac{n^{3/2}\mathrm{sh}(\frac{2t}{n})^{3/2}}{\mathrm{ch}(\frac{2t}{n})}\ \frac{1}{n}\sum_{j=0}^{n-1} w(t_j)(\mathrm{ch}(\frac{2t}{n})^{-j}-1))}_{E_n}\ .$$

Since

$$\frac{n^{3/2}\mathrm{sh}(2t/n)^{3/2}}{\mathrm{ch}(2t/n)} = (2t)^{3/2}(1+o(1/n)),$$

one has

$$D_n \longrightarrow -x(2t)^{3/2}\int_0^1 w(s)\ ds.$$

Moreover

$$0 \le 1-\frac{1}{\mathrm{ch}(2t/n)^j} \le 1-\frac{1}{\mathrm{ch}(2t/n)^n} \sim 2t^2n^{-1}$$

then

$$|E_n| \leq |x| \frac{n^{3/2}\mathrm{sh}(2t/n)^{3/2}}{\mathrm{ch}(2t/n)} \quad (1 - \mathrm{ch}(2t/n)^{-n})) \quad \frac{1}{n}\sum_{j=0}^{n-1} |w(t_j)| \quad \longrightarrow \quad 0.$$

To sum up, $(\mathrm{ch}(2t/n))^{-j}$ can be replaced by 1 in all the $w(t_j)(\mathrm{ch}(2t/n))^{-j}$. We conclude that

$$\begin{aligned} A_n + B_n + C_n \quad &\longrightarrow \quad -2t^2 \int_0^1 w^2(s)\ ds \quad - x(2t)^{3/2} \int_0^1 w(s)\ ds \quad - tx^2 \\ &= \quad -t \int_0^1 \left(x + (2t)^{1/2} w(s)\right)^2\ ds. \end{aligned}$$

Let us turn to H_n. For all j and all w we can write

$$-c\left(\frac{(n-j)t}{n}, (n\ \mathrm{sh}(2t/n))^{1/2} w(j/n) + \frac{x}{\mathrm{ch}(2t/n)^j}\right) \leq -m,$$

hence the estimate $H_n(w) \leq e^{-mt}$. Denote by $||c_x||_\infty$ the uniform norm of c_x on $\in [0,\infty) \times \mathbb{R}$. The mean value theorem implies that

$$\begin{aligned} &\left| c\left(t - \frac{j}{n}t, (n\ \mathrm{sh}(2t/n))^{1/2} w(j/n) + \frac{x}{\mathrm{ch}(2t/n)^j}\right) - c\left(t - \frac{j}{n}t, \sqrt{2t}\ w(j/n) + x\right)\right| \\ &\quad \leq ||c_x||_\infty \left|(n\ \mathrm{sh}(2t/n))^{1/2} w(j/n) - \sqrt{2t}\ w(j/n)\right| + ||c_x||_\infty \left|\frac{x}{\mathrm{ch}(2t/n)^j} - x\right| \\ &\quad \leq ||c_x||_\infty ||w||_\infty \left|(n\ \mathrm{sh}(2t/n))^{1/2} - \sqrt{2t}\right| + ||c_x||_\infty |x| \left|\frac{1}{\mathrm{ch}(2t/n)^n} - 1\right|, \end{aligned}$$

in which the last term is independant of j. Therefore

$$\begin{aligned} &\frac{t}{n}\left|\sum_{j=0}^{n-1} c\big(t - \frac{j}{n}t, (n\ \mathrm{sh}(\frac{2t}{n}))^{1/2} w(j/n) + \frac{x}{\mathrm{ch}(\frac{2t}{n})^j}\big) - \sum_{j=0}^{n-1} c\big(t - \frac{j}{n}t, \sqrt{2t}\ w(j/n) + x\big)\right| \\ &\quad \leq t||c_x||_\infty \big(||w||_\infty \left|(n\ \mathrm{sh}(2t/n))^{1/2} - \sqrt{2t}\right| + |x| \left|\frac{1}{\mathrm{ch}(2t/n)^n} - 1\right|\big)\ , \end{aligned}$$

which converges to 0 when n goes to infinity. We deduce that $H_n(w)$ and

$$\exp\left(-\frac{t}{n}\sum_{j=0}^{n-1} c\left(t - \frac{j}{n}t, \sqrt{2t}\ w(j/n) + x\right)\right)$$

converge to the same limit. Since the function $u \mapsto c\left(t - ut, \sqrt{2t}\ w(u) + x\right)$ is continuous on $[0,1]$, this limit is

$$\exp\left(-t\int_0^1 c\left(t(1-s), \sqrt{2t}\ w(s) + x\right)\ ds\right).$$

Let us treat the last point of the Lemma. Recall that v_0 is continuous and bounded. Its argument goes to $(2t)^{1/2}w(1)+x$ because

$$\mathrm{ch}(2t/n)^n = \exp\left(\frac{2t^2}{n} + o(\frac{1}{n^2})\right) \to 1,$$

which completes the proof. □

4. Preliminary version of Theorem 1

We still need to show that the function v constructed above as the limit, at time t, of the sequence $(v_n^{(t)})_{n\in\mathbb{N}}$ is a solution of (1). The demonstration requires stronger regularity conditions on v_0 and c than the ones used to compute the limit. Here is the (weaker) version of Theorem 1 which will be proved in this section.

Theorem 5. *Let v_0 be a C^4 function over $\mathbb{R}$, which has bounded derivatives of order up to 4. Suppose $v_0(x)$ converges to 0 when x goes to infinity.*
Let c be a function which

- *is continuous and bounded on $[0,\infty)\times\mathbb{R}$,*
- *is C^1 on $]0,\infty)\times\mathbb{R}$,*
- *has bounded space derivatives, up to order 4, these derivatives being continuous and bounded on $]0,\infty)\times\mathbb{R}$.*

The function v defined on $]0,\infty)\times\mathbb{R}$ by

$$v(t,x) = \int_{C_W} v_0(\sqrt{2t}\ w(1)+x)\ \exp\left(-t\int_0^1 \left(x+\sqrt{2t}\ w(s)\right)^2\ ds\right)$$

$$\exp\left(-t\int_0^1 c\left(t(1-s),\sqrt{2t}\ w(s)+x\right)\ ds\right)\ dm_W(w)$$

is a solution of (1).

We need preliminary results. The first Lemma shows that the sequence $(v_n^{(t)})$ and one of its derivatives converge on a dyadic subset of $[0,t]$. The second one gives uniform estimates concerning some of the derivatives of $v_n^{(t)}$. The last result proves that a subsequence of $(v_n^{(t)})$ converges *uniformly*, and on $[0,t]$ itself.

Lemma 6. *Let $t\in]0,\infty[$, let $\mathcal{D}$ be the set*

$$\mathcal{D} = \left\{\tau\in[0,t]\ :\ \exists n_0\in\mathbb{N}, \exists k_0\in\{0,\dots,2^{n_0}-1\}, \tau = \frac{k_0}{2^{n_0}}t.\right\}$$

For all $\tau\in\mathcal{D}$ and all $x\in\mathbb{R}$,

$$\lim_{n\to\infty} v_{2^n}^{(t)}(\tau,x) = v(\tau,x) \quad \text{and} \quad \lim_{n\to\infty} \frac{\partial^2 v_{2^n}^{(t)}}{\partial x^2}(\tau,x) = \frac{\partial^2 v}{\partial x^2}(\tau,x),$$

where v is the function defined in Theorem 1.

The rest of this Section is devoted to the proof of the Lemmas and of Theorem 5.

Proof of Lemma 6
It is easier to express $v_n^{(t)}(\tau,x)$ when $\tau = kt/n$ is a bound of the subdivision. Therefore we are led to consider nested subdivisions. A dyadic point $\tau = \frac{k}{2^n}t$, which is already a bound of the subdivision with $2{\cdot}2^n$ intervals, is a bound of all following subdivisions. The set $\mathcal{D}$ is the unions of all such points. Formula (10) gives $v_n^{(t)}(\tau,x)$ for $\tau \in [\frac{2k}{2n}t, \frac{2k+1}{2n}t]$. Adapting the proof of Proposition (3) we deduce the expression of $v_{2^{n+p}}^{(t)}(\tau,x)$ when $\tau = \frac{k}{2^n}t = \frac{2^pk}{2^{n+p}}t$ and obtain

(13)

$$
\begin{aligned}
v_{2^{n+p}}^{(t)}(\tau,x) = \int_{C_W} & v_0\left(\sqrt{2^pk\ \mathrm{sh}(2t/2^{n+p})}w(1) + \frac{x}{\mathrm{ch}(2t/2^{n+p})^{2^pk}}\right) \\
&\exp\left(-\frac{1}{2}\frac{\mathrm{sh}(2t/2^{n+p})}{\mathrm{ch}(2t/2^{n+p})}\sum_{l=0}^{2^pk-1}\left(\sqrt{2^pk\ \mathrm{sh}(2t/2^{n+p})}w\left(\frac{l}{2^pk}\right) + \frac{x}{\mathrm{ch}(2t/2^{n+p})^l}\right)^2\right) \\
&\exp\left(\frac{2^pk}{2}(1-\mathrm{ch}(2t/2^{n+p})w(1)^2 - \frac{2^pk}{2}\frac{(\mathrm{ch}(2t/2^{n+p})-1)^2}{\mathrm{ch}(2t/2^{n+p})}\sum_{l=1}^{2^pk-1} w\left(\frac{l}{2^pk}\right)^2\right) \\
&\exp\left(-\frac{t}{2^{n+p}}\sum_{l=0}^{2^pk-1} c(\frac{2^pk-l}{2^{n+p}}t, \sqrt{2^pk\ \mathrm{sh}(2t/2^{n+p})}w(\frac{l}{2^pk}) + \frac{x}{\mathrm{ch}(2t/2^{n+p})^l})\right)\ dm_W(w).
\end{aligned}
$$

The integrated terms are similar to the F_n, G_n, H_n treated in Lemma 4. We get the limit of $v_{2^{n+p}}^{(t)}(\tau,x)$ by letting p go to infinity in the integral and do not need the additional hypotheses. The only difference is that

$$\lim_{p\to\infty} 2^pk\ \mathrm{sh}(2t/2^{n+p}) = \tau.$$

To find the limit of $\partial_x^2 v_{2^n}^{(t)}(\tau,x)$, one has to derivate (13) twice with respect to x. The derivatives of the first exponential term contain the expression

$$M = -\frac{\mathrm{sh}(2t/2^{n+p})}{\mathrm{ch}(2t/2^{n+p})}\sum_{l=0}^{2^pk-1}\left(\sqrt{2^pk\ \mathrm{sh}(2t/2^{n+p})}w\left(\frac{l}{2^pk}\right) + \frac{x}{\mathrm{ch}(2t/2^{n+p})^l}\right)\frac{1}{\mathrm{ch}(2t/2^{n+p})^l},$$

as well as its square and its derivative (with respect to x). We estimate M by $K(||w||_\infty + |x|)$ with a constant K depending only on t. The other exponential terms and their derivatives are bounded with respect to w. Therefore we can apply the dominated convergence Theorem, since $\int_{C_W} ||w||^2\ dm_W(w)$ is bounded (by Fernique's Theorem, see Section 2). As for the limits themselves, the same techniques can be applied as in the proof of Lemma 4. Note that the estimate of $\partial_x^3 c$ is needed to treat terms containing $\partial_x^2 c$ for we use the mean value theorem. □

Lemma 7. *There exists $C > 0$, depending on t and x, such that*

$$\forall j \in \{0, \dots, 4\},\ \forall n \in \mathbb{N}^*,\ \forall k \in \{0, \dots, n-1\},\ \forall \tau \in \left[\frac{kt}{n}, \frac{(k+1)t}{n}\right],$$

$$\left|\frac{\partial^j v_n^{(t)}(\tau, x)}{\partial x^j}\right| \leq C.$$

Proof of Lemma 7
We first treat the interval $[\frac{2kt}{2n}, \frac{(2k+1)t}{2n}]$. Formula (10) shows that

$$v_n(\tau, x) = \int_{\mathbb{R}^{k+1}} (2\pi \mathrm{sh}(4(\tau - kt/n)))^{-1/2} (2\pi \mathrm{sh}(2t/n))^{-k/2}$$

$$\exp\left(-\frac{1}{2}\frac{\mathrm{sh}(4(\tau - kt/n))}{\mathrm{ch}(4(\tau - kt/n))} x^2\right) \exp\left(-\frac{1}{2}\frac{\mathrm{ch}(4(\tau - kt/n))}{\mathrm{sh}(4(\tau - kt/n))}\sigma_1^2\right)$$

$$\exp\left(-\frac{1}{2}\frac{\mathrm{ch}(2t/n)}{\mathrm{sh}(2t/n)} \sum_{j=1}^{k} \left(\sigma_{k+2-j} - \frac{\sigma_{k+1-j}}{\mathrm{ch}(2t/n)}\right)^2\right)$$

$$\exp\left(-\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)} \sum_{j=1}^{k} \left(\sigma_{k+1-j} + \frac{x}{\mathrm{ch}(4(\tau - kt/n))\mathrm{ch}(2t/n)^{k-j}}\right)^2\right)$$

$$\exp\left(-\frac{t}{n}\sum_{j=1}^{k} c\left(\frac{j}{n}t, \sigma_{k+1-j} + \frac{x}{\mathrm{ch}(4(\tau - kt/n))\mathrm{ch}(2t/n)^{k-j}}\right)\right)$$

$$v_0\left(\sigma_{k+1} + \frac{x}{\mathrm{ch}(4(\tau - kt/n))\mathrm{ch}(2t/n)^{k}}\right) \quad d\sigma_1 \dots d\sigma_{k+1}.$$

The space-derivatives of order at most 4 of all terms but one are bounded by constants depending on t, $||v_0^{(j)}||$ and $||\partial_x^j c||$ ($j \leq 4$), but not on k, n, τ or x. The derivative of

$$\exp\left(-\frac{1}{2}\frac{\mathrm{sh}}{\mathrm{ch}}(4(\tau - \frac{kt}{n}))x^2\right) \exp\left(-\frac{1}{2}\frac{\mathrm{sh}}{\mathrm{ch}}(\frac{2t}{n}) \sum_{j=1}^{k}\left(\sigma_{k+1-j} + \frac{x}{\mathrm{ch}(4(\tau - \frac{kt}{n}))\mathrm{ch}(\frac{2t}{n})^{k-j}}\right)^2\right)$$

is less easy to treat. It is the product of the exponential term and of

$$-\frac{\mathrm{sh}}{\mathrm{ch}}(4(\tau - \frac{kt}{n}))x - \frac{\mathrm{sh}}{\mathrm{ch}}(\frac{2t}{n})\sum_{j=1}^{k}\left(\sigma_{k+1-j} + \frac{x}{\mathrm{ch}(4(\tau - \frac{kt}{n}))\mathrm{ch}(\frac{2t}{n})^{k-j}}\right)\frac{1}{\mathrm{ch}(4(\tau - \frac{kt}{n}))\mathrm{ch}(\frac{2t}{n})^{k-j}}.$$

We then have to estimate expressions such as

$$\left(\sum a_s b_s c_s\right)^j \exp\left(-\frac{1}{2}\sum a_s b_s^2\right),$$

where the a_s are quotients sh/ch and the c_s are the $(\mathrm{ch}(4(\tau-\frac{kt}{n}))^{-1}\mathrm{ch}(\frac{2t}{n})^{-s}$. Applying Cauchy-Schwartz inequality to the sum outside of the exponential term shows it is smaller than $\sqrt{\sum a_s b_s^2 \sum a_s c_s^2}$. The first factor is absorbed by the exponential and the second one is a geometric sum, which is bounded.
It is eventually possible to estimate the derivatives of order $0 \le j \le 4$ by

$$C\int_{\mathbb{R}^{k+1}}(2\pi\mathrm{sh}(4(\tau-kt/n)))^{-1/2}(2\pi\mathrm{sh}(2t/n))^{-k/2}\exp\left(-\frac{1}{2}\frac{\mathrm{ch}(4(\tau-kt/n))}{\mathrm{sh}(4(\tau-kt/n))}\sigma_1^2\right)$$

$$\exp\left(-\frac{1}{2}\frac{\mathrm{ch}(2t/n)}{\mathrm{sh}(2t/n)}\sum_{j=1}^{k}\left(\sigma_{k+2-j}-\frac{\sigma_{k+1-j}}{\mathrm{ch}(2t/n)}\right)^2\right)\ d\sigma_1\ldots d\sigma_{k+1}\ ,$$

where the constant C depends on t, $||v_0^{(j)}||_\infty$ et $||\partial^j c||_\infty$ $(0\le j\le 4)$ only. The integral is equal to

$$\mathrm{ch}\left(4(\tau-\frac{kt}{n})\right)^{-1/2}\mathrm{ch}(2t/n)^{-k/2}\le 1,$$

which establishes our claim for the interval $[\frac{2k}{2n}t, \frac{2k+1}{2n}]$.
On the following interval, $[\frac{2k+1}{2n}t, \frac{2k+2}{2n}]$,

$$v_n^{(t)}(\tau,x)=\exp\left(-2(\tau-\frac{2k+1}{2n}t)c(\frac{k+1}{n}t),x)\right)v_n^{(t)}(\frac{2k+1}{2n}t,x).$$

It is the product of two functions having bounded space-derivatives (of order at most 4). This proves the estimations on $[0,t]$. □

Lemma 8. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions continuous on $[0,T]$, piecewise C^1 on $[0,T]$ and satisfying*

$$\exists C>0\ :\ \forall n\in\mathbb{N},\ \forall\tau\in[0,T]\ \textit{such that}\ u_n'(\tau)\ \textit{exists},\quad |u_n'(\tau)|\le C.$$

Suppose D is a dense subset of $[0,T]$ and u, a continuous function on $[0,T]$, such that

$$\forall\tau\in D,\ \lim_n u_n(\tau)=u(\tau).$$

Then the sequence (u_n) has a subsequence which converges to u uniformly on $[0,T]$.

Proof. The bounds on the derivatives show that $(u_n)_{n\in\mathbb{N}}$ is bounded and equicontinuous on $[0,T]$. Ascoli's Theorem yields the existence of a uniformly converging subsequence. Its limit $\tilde{u}$ is continuous on $[0,T]$ and equal to u on the dense subset D, which justifies the Lemma. □

Proof of Theorem 5.
Let x be fixed. We have proved that the (numerical) sequences $(v_{2^n}^{(t)}(\tau,x))_{n\in\mathbb{N}}$ and $((\partial_x)^2 v_{2^n}^{(t)}(\tau,x))_{n\in\mathbb{N}}$ converge (respectively) to $v(\tau,x)$ and $\partial_x^2 v(\tau,x)$ for all τ belonging to the dense subset $\mathcal{D}$ defined in Lemma 6. Now to prove the uniform convergence on $[0,t]$ we apply Lemma 8. The continuity of $v_{2^n}^{(t)}(\cdot,x)$ is a consequence of its definition. The continuity of $\partial_x^2 v_{2^n}^{(t)}(\cdot,x)$ can be proved by studying the expressions appearing in the proof of Lemma 7 and so is the derivability with respect to τ.

To establish the bounds on the τ-derivatives let us recall that

(1) On the even intervals $[2kt/2.2^n, (2k+1)t/2.2^n]$,

$$\frac{\partial v_{2^n}^{(t)}(\tau,x)}{\partial\tau} = 2\frac{\partial^2 v_{2^n}^{(t)}(\tau,x)}{\partial x^2} - 2x^2 v_{2^n}^{(t)}(\tau,x),$$

(2) on the odd intervals $[(2k+1)t/2.2^n, (2k+2)t/2.2^n]$,

$$\frac{\partial v_{2^n}^{(t)}(\tau,x)}{\partial\tau} = -2c((2k+2)t/2.2^n, x)v_{2^n}^{(t)}(\tau,x).$$

The bounds on the time derivatives come from the estimates on the space derivatives : to show that $\partial_\tau v_{2^n}^{(t)}$ is bounded one needs the x-derivatives up to order 2, to treat $\partial_\tau(\partial_x)^2 v_{2^n}^{(t)}(\tau,x)$ one needs the x- derivatives up to order 4. The bounds concerning the space-derivatives have been established in Lemma 7. It follows that a subsequence of $(v_{2^n}^{(t)})$ (resp. of $(\partial_\tau v_{2^n}^{(t)})$) converges uniformly on $[0,t]$. Let us denote its indexes by $\varphi(n)$.
Next we write more concisely the system of equations defining $v_n^{(t)}$ (according to the interval). On $[0,t]\setminus\{kt/2^n\}, 0\le k\le 2^n$, $v_n^{(t)}$ satisfies

$$\frac{\partial v_n^{(t)}(\tau,x)}{\partial\tau} = 2\beta_n(\tau)\left(\frac{\partial^2 v_n^{(t)}(\tau,x)}{\partial x^2} - x^2 v_n^{(t)}(\tau,x)\right) - 2(1-\beta_n(\tau))c_n(\tau,x)v_n^{(t)}(\tau,x),$$

with

$$\begin{aligned} \beta_n(\tau) &= 1 && \text{on} \quad \left]\frac{2k}{2n}t, \frac{2k+1}{2n}t\right] \\ &= 0 && \text{on} \quad \left]\frac{2k+1}{2n}t, \frac{2k+2}{2n}t\right] \\ \text{and } c_n(\tau,x) &= c(\tfrac{2k+2}{2n}t, x) && \text{on} \quad \left]\frac{2k}{2n}t, \frac{2k+2}{2n}t\right]. \end{aligned}$$

This equation still holds for the subsequence indexed by $\varphi(n)$. The uniform convergence allows us to integrate on any subinterval $[0,s]$ of $[0,t]$:

$$v_{\varphi(n)}^{(t)}(s,x) - v_{\varphi(n)}^{(t)}(0,x) =$$
$$2\int_0^s \beta_{\varphi(n)}(\tau)\left(\frac{\partial^2 v_{\varphi(n)}^{(t)}(\tau,x)}{\partial x^2} - x^2 v_{\varphi(n)}^{(t)}(\tau,x)\right) - (1-\beta_{\varphi(n)}(\tau))c_{\varphi(n)}(\tau,x)v_{\varphi(n)}^{(t)}(\tau,x)\, d\tau.$$

Now let n tend to infinity. The following result ([5]) shows what become of β_n and of the factor 2 :

Lemma 9. *Let β_n be the function defined above. Let (ψ_n) be a sequence of functions belonging to $\in L^1([0,t])$ and suppose it converges uniformly on $[0,t]$ to a limit ψ. Then, for all $0\le\tau\le\sigma\le t$,*

$$\lim_{n\to\infty}\int_\tau^\sigma \beta_n\psi_n\, ds = \lim_{n\to\infty}\int_\tau^\sigma (1-\beta_n)\psi_n\, ds = \frac{1}{2}\int_\tau^\sigma \psi\, ds.$$

Eventually, for all $s \leq t$, we obtain

$$v(s,x) - v_0(x) = \int_0^s \left(\frac{\partial^2 v(\tau,x)}{\partial x^2} - x^2 v\tau, x) \right) - c(\tau,x)v(\tau,x) \; d\tau \; .$$

As $v(s,x)$ does not depend on t (the intermediates $v_n^{(t)}$ depend on t but not the limit), the function v is a solution of (1). □

5. Another proof of Feynman Kac formula

Before completing the proof of Theorem 1 we shall see that a small modification of the method developped in Section3 yields the following expression for v. For sufficiently small t,

$$v(t,x) = \int_{C_W} v_0(w(2t)+x) \; \exp\left(-\frac{1}{2} \int_0^{2t} (x+w(s))^2 \; ds \right)$$

$$\exp\left(-\frac{1}{2} \int_0^{2t} c\,(t-s/2, w(s)+x) \; ds \right) \; dm_W(w) \; .$$

With $u(t,x) = v(t/2,x)$, u satisfies

$$\frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \frac{1}{2}(x^2 + c(t/2,x))u(t,x) = 0,$$

which explains the differences with the usual expression.

The first point is that the demonstration does not use the Itô integral at all. What is, perhaps, more significant is that both expressions of v are not linked by a "classical" change of variable on Wiener space.

Proof. It essentially follows the same steps as in Section 3. Starting from Proposition 2, we consider the time sequence $(t_k)_{0 \leq k \leq n}$ with $t_k = k\frac{2t}{n}$ instead of $t_k = k/n$. This yields

$$v_n^{(t)}(t,x) = \int_{C_W} \left(\frac{2t}{n\mathrm{sh}(2t/n)} \right)^{n/2} v_0\left(w(2t) + \frac{x}{\mathrm{ch}(2t/n)^n} \right)$$

$$\exp\left(-\frac{1}{2}\frac{\mathrm{ch}(2t/n)}{\mathrm{sh}(2t/n)} \sum_{j=1}^{n} \left[w(2jt/n) - \frac{w(2(j-1)t/n)}{\mathrm{ch}(2t/n)} \right]^2 \right)$$

$$\exp\left(\frac{n}{4t} \sum_{j=1}^{n} \left[w(2jt/n) - w(2(j-1)t/n) \right]^2 \right)$$

$$\exp\left(-\frac{1}{2}\frac{\mathrm{sh}(2t/n)}{\mathrm{ch}(2t/n)} \sum_{j=0}^{n-1} \left(w(2jt/n) + \frac{x}{\mathrm{ch}(2t/n)^j} \right)^2 \right)$$

$$\exp\left(-\frac{t}{n} \sum_{j=0}^{n-1} c\left(\frac{(n-j)t}{n}, w(2jt/n) + \frac{x}{\mathrm{ch}(2t/n)^j} \right) \right) \; dm_W(w).$$

Most of the terms are bounded and converge as in Section 3 or even more easily. It just remains to treat

$$-\frac{1}{2}\frac{\mathrm{ch}(2t/n)}{\mathrm{sh}(2t/n)}\sum_{j=1}^{n}\left[w(2jt/n)-\frac{w(2(j-1)t/n)}{\mathrm{ch}(2t/n)}\right]^2+\frac{n}{4t}\sum_{j=1}^{n}\left[w(2jt/n)-w(2(j-1)t/n)\right]^2.$$

This expression splits into A_n+B_n where

$$A_n=\left(\frac{n}{4t}-\frac{1}{2}\frac{\mathrm{ch}(2t/n)}{\mathrm{sh}(2t/n)}\right)w(2t)^2-\sum_{j=1}^{n-1}w(2jt/n)^2\frac{(\mathrm{ch}(2t/n)-1)^2}{2\mathrm{ch}(2t/n)\mathrm{sh}(2t/n)}$$

and

$$B_n=\left(\frac{n}{2t}-\frac{1}{\mathrm{sh}(2t/n)}\right)\sum_{j=1}^{n-1}w(2jt/n)\left(w(2jt/n)-w(2(j-1)t/n)\right).$$

The first term is negative and converges to 0. The second one can be estimated as follows

$$\begin{aligned}|B_n| &\leq \left|\frac{n}{2t}-\frac{1}{\mathrm{sh}(2t/n)}\right|\frac{n}{2t}\sqrt{\frac{2t}{n}\sum_{j=1}^{n-1}w(2jt/n)^2}\sqrt{\frac{2t}{n}\sum_{j=1}^{n-1}\left(w(2jt/n)-w(2(j-1)t/n)\right)^2}\\ &\leq M\sqrt{2t}||w||\sqrt{\frac{2t}{n}\sum_{j=1}^{n-1}\left(w(2jt/n)-w(2(j-1)t/n)\right)^2}.\end{aligned}$$

As $\sum_{j=1}^{n-1}\left(w(2jt/n)-w(2(j-1)t/n)\right)^2$ is the quadratic variation of a Brownian motion, the subsequence for $n=2^p$ converges to $\sqrt{2t}$ and it is smaller than $4n||w||^2$. To sum up, $|B_n|\leq 4tM||w||^2$ and converges to 0. This estimation and Fernique's theorem allow us to use Lebesgue dominated convergence theorem, provided t is small enough. □

6. Proof of Theorem 1

To get the optimal form of the Theorem, it remains to prove that formula (3) gives a solution of Problem (1) even if v_0 and c satisfy much weaker assumptions. This will be done by approximating general v_0 and c by regular functions and showing that the solution of the approximating problem converges to that of the real problem.

Proposition 10. *For $v_0\in L^2(\mathbb{R})$ and c measurable and inferiorly bounded on $]0,\infty)\times\mathbb{R}$ we define, following formula (3),*

$$\begin{aligned}S(v_0,c)(t,x)= &\int_{C_W}v_0(\sqrt{2t}\ w(1)+x)\ \exp\left(-t\int_0^1\left(x+\sqrt{2t}\ w(s)\right)^2\ ds\right)\\ &\exp\left(-t\int_0^1 c\left(t(1-s),\sqrt{2t}\ w(s)+x\right)\ ds\right)\ dm_W(w)\ .\end{aligned}$$

For all $0 \le \alpha < \beta < \infty$, $S(v_0, c)$ *belongs to* $L^2([\alpha,\beta]\times\mathbb{R})$. *Moreover,*

$$\int_{[\alpha,\beta]\times\mathbb{R}} |S(v_0,c)(t,x)|^2 \; dtdx \le ||v_0||^2 \int_\alpha^\beta e^{-2t\inf(c)} \; dt \; . \tag{14}$$

Proof .
Let us consider

$$\begin{aligned} I \;&= \int_\alpha^\beta \int_{\mathbb{R}} \int_{C_W} |v_0(\sqrt{2t}\; w(1)+x)|^2 \exp\left(-2t\int_0^1 \left(x+\sqrt{2t}\; w(s)\right)^2 \; ds\right) \\ &\times \exp\left(-2t\int_0^1 c\left(t(1-s), \sqrt{2t}\; w(s)+x\right) \; ds\right) \; dm_W(w) \; dxdt \quad . \end{aligned}$$

The first exponential factor is smaller than 1 and the second one, than $\exp(-2t\inf(c))$. By Fubini's Theorem,

$$\begin{aligned} I \;&\le \int_\alpha^\beta e^{-2t\inf(c)} \int_{C_W} \left(\int_{\mathbb{R}} |v_0(\sqrt{2t}\; w(1)+x)|^2 \; dx\right) \; dm_W(w)dt \\ &\le \int_\alpha^\beta e^{-2t\inf(c)} \int_{C_W} \left(\int_{\mathbb{R}} |v_0(\xi)|^2 \; d\xi\right) \; dm_W(w)dt \; , \end{aligned}$$

with $\xi = \sqrt{2t}\; w(1)+x$. Eventually,

$$I \le ||v_0||^2 \int_\alpha^\beta e^{-2t\inf(c)} \; dt.$$

This shows that the integral

$$\int_{C_W} |v_0(\sqrt{2t}\; w(1)+x)|^2 e^{-2t\int_0^1 \left(x+\sqrt{2t}\; w(s)\right)^2 \; ds} e^{-2t\int_0^1 c\left(t(1-s),\sqrt{2t}\; w(s)+x\right)\; ds} \; dm_W(w)$$

converges for almost all $(t,x) \in [\alpha,\beta]\times\mathbb{R}$. By Hölder's inequality it follows that $S(v_0,c)(t,x)$ is defined for the same (t,x) and that $S(v_0,c)$ satisfies the inequality (14). $\square$

Proposition 11. *Let* $(v_0^{(n)}) \in L^2(\mathbb{R})^{\mathbb{N}}$ *converge to* v_0 *in* $L^2(\mathbb{R})$ *and* $(c^{(n)}) \in L^2(\mathbb{R}^+\times\mathbb{R})^{\mathbb{N}}$ *converge to* $c \in L^2(\mathbb{R}^+\times\mathbb{R})$. *Assume that the* $c^{(n)}$ *and* c *have a common lower bound* $\mu \in \mathbb{R}$.
Then, for all $\alpha, \beta \in \mathbb{R}^+$ *satisfying* $0 \le \alpha \le \beta < \infty$, $S(v_0^{(n)}, c^{(n)})$ *converges to* $S(v_0,c)$ *in* $L^2([\alpha,\beta]\times\mathbb{R})$.

Proof . Let us introduce the intermediate $S(v_0, c^{(n)})$. Then by the preceding Proposition

$$||S(v_0,c^{(n)}) - S(v_0^{(n)},c^{(n)})||_{L^2([\alpha,\beta]\times\mathbb{R})} \le ||v_0 - v_0^{(n)}||_{L^2(\mathbb{R})} \left(\int_\alpha^\beta e^{-2t\mu} \; dt\right)^{1/2}$$

and this term converges to 0.
The second term $S(v_0, c^{(n)}) - S(v_0, c)$ is more delicate. Let

$$I_n(t,x) = \int_{C_W} v_0(\sqrt{2t}w(1)+x)e^{-t\int_0^1 \left(x+\sqrt{2t}w(s)\right)^2 ds}$$
$$\left(e^{-t\int_0^1 c^{(n)}\left(t(1-s),\sqrt{2t}w(s)+x\right)ds} - e^{-t\int_0^1 c\left(t(1-s),\sqrt{2t}w(s)+x\right)ds}\right) dm_W \ ,$$

so that we may write

$$||S(v_0,c^{(n)}) - S(v_0,c)||^2_{L^2([\alpha,\beta]\times\mathbb{R})} = \int_\alpha^\beta \int_{\mathbb{R}} I_n(t,x)^2 \ dt \ dx.$$

We shall prove that the sequence $(I_n(t,x))_n$ converges to 0 when n goes to infinity and that it is uniformly bounded by a function $g(t,x)$ belonging to $L^2([\alpha,\beta]\times\mathbb{R})$. Then, thanks to the dominated convergence Theorem, $||S(v_0,c^{(n)}) - S(v_0,c)||^2_{L^2([\alpha,\beta]\times\mathbb{R})}$ will converge to 0. As usual, the first exponential term of $I_n(t,x)$ is smaller than 1. Both $-t\int_0^1 c^{(n)}(t(1-s),\sqrt{2t}w(s)+x)ds$ and $-t\int_0^1 c(t(1-s),\sqrt{2t}w(s)+x)ds$ being smaller than $-t\mu$,

$$\left|e^{-t\int_0^1 c^{(n)}\left(t(1-s),\sqrt{2t}w(s)+x\right)ds} - e^{-t\int_0^1 c\left(t(1-s),\sqrt{2t}w(s)+x\right)ds}\right|$$
$$\leq \left|t\int_0^1 (c^{(n)}-c)(t(1-s),\sqrt{2t}\ w(s)+x)ds\right| e^{-t\mu}$$

and we can estimate $|I_n|$ by

$$|I_n(t,x)| \leq te^{-t\mu} \underbrace{\int_{C_W} |v_0(\sqrt{2t}w(1)+x)| \int_0^1 \left|(c^{(n)}-c)(t(1-s),\sqrt{2t}\ w(s)+x)\right| ds dm_W}_{M_n(t,x)} \ .$$

Thanks to (4), $M_n(t,x)$ can be written as an integral on $\mathbb{R}$:

$$M_n(t,x) \ = \int_0^1 \int_{\mathbb{R}^2} |v_0(\sqrt{2t}\xi_2+x)| \left|(c^{(n)}-c)(t(1-s),\sqrt{2t}\ \xi_1+x)\right| f_{s,1}(\xi_1,\xi_2)d\xi_1 d\xi_2\ ds,$$

where $f_{s,1}$ is the gaussian density. The change of variables

$$u=\sqrt{2t}\xi_2+x,\ v=t(1-s),\ w=\sqrt{2t}\ \xi_1+x$$

gives

$$M_n(t,x) \ = \frac{1}{4\pi t}\int_{\mathbb{R}^3} \frac{\mathbf{1}_{[0,t]}(v)}{\sqrt{v(t-v)}} \left|(c^{(n)}-c)(v,w)\right| e^{-\frac{(w-x)^2}{4(t-v)}} e^{-\frac{(u-w)^2}{4v}} |v_0(u)|\ dudvdw.$$

As

$$\int_{\mathbb{R}} e^{-\frac{(u-w)^2}{4v}} |v_0(u)|\ du \leq ||v_0||_{L^2(\mathbb{R})}(2\pi v)^{1/4},$$

$M_n(t,x)$ is smaller than

$$\frac{1}{4\pi t}||v_0||_{L^2(\mathbb{R})}(2\pi)^{1/4}||c-c^{(n)}||_{L^2(\mathbb{R}^+\times\mathbb{R})}\sqrt{\int_{\mathbb{R}^2}\frac{\mathbf{1}_{[0,t]}(v)}{v(t-v)}v^{1/2}\exp\left(-\frac{(w-x)^2}{2(t-v)}\right)\,dvdw.}$$

The integral appearing in the square root converges and does not depend on n, which shows that $M_n(t,x)$ and $I_n(t,x)$ go to 0.
Now for the uniform estimate. Clearly

$$|I_n(t,x)|\le\int_{C_W}|v_0(\sqrt{2t}w(1)+x)|(e^{-t\mu}+e^{-t\mu})dm_W:=g(t,x).$$

The function g is in $L^2([\alpha,\beta]\times\mathbb{R})$ since

$$\begin{aligned}\int_\alpha^\beta\int_{\mathbb{R}}g(t,x)^2\,dxdt &\le\int_\alpha^\beta 4e^{-2t\mu}\int_{\mathbb{R}}\left(\int_{C_W}|v_0(\sqrt{2t}w(1)+x)|dm_W\right)^2\,dxdt\\ &\le\int_\alpha^\beta 4e^{-2t\mu}\int_{\mathbb{R}}\int_{C_W}|v_0(\sqrt{2t}w(1)+x)|^2dm_W\,dxdt\\ &\le\int_\alpha^\beta 4e^{-2t\mu}\int_{C_W}\int_{\mathbb{R}}|v_0(\sqrt{2t}w(1)+x)|^2\,dx\,dm_Wdt\,.\end{aligned}$$

The change of variables $\xi=\sqrt{2t}w(1)+x$ allows to write

$$\int_\alpha^\beta\int_{\mathbb{R}}g(t,x)^2\,dxdt\le||v_0||^2_{L^2(\mathbb{R})}\le\int_\alpha^\beta 4e^{-2t\mu}dt<\infty,$$

which concludes the proof. □

Any $v_0\in L^2(\mathbb{R})$ can be approximated (in $L^2(\mathbb{R})$) by a sequence $(v_0^{(n)})$ of functions satisfying the hypotheses of Theorem 5. Similarly, any function c belonging to $L^2(]0,\infty)\times\mathbb{R})\cap L^\infty(]0,\infty)\times\mathbb{R})$ is the limit (in $L^2(]0,\infty)\times\mathbb{R})$) of a sequence $(c^{(n)})_n$ of functions satisfying the hypotheses of Theorem 5. Moreover we can suppose this sequence to be bounded in $L^\infty(]0,\infty)\times\mathbb{R})$. Then, according to Theorem 5, $v_n:=S(v_0^{(n)},c^{(n)})$ is a solution of

$$\begin{cases}\dfrac{\partial v_n}{\partial t}(t,x)-\dfrac{\partial^2 v_n}{\partial x^2}(t,x)+(x^2+c^{(n)}(t,x))v_n(t,x)=0\quad\text{on}\quad]0,\infty[\times\mathbb{R}\\[2ex] v_n(0,x)=v_0^{(n)}(x)\;.\end{cases}$$

Let φ be a smooth test function on $]0,\infty[\times\mathbb{R}$. Integrations by part give

$$<v_n,\frac{\partial\varphi}{\partial t}>-<v_n,\frac{\partial^2\varphi}{\partial x^2}>+<x^2v_n,\varphi>+<c^{(n)}v_n,\varphi>=0,$$

where the brackets stand for $L^2(]0,\infty[\times\mathbb{R})$ products. As v_n converges to $S(v_0,c)$ in any $L^2([\alpha,\beta]\times\mathbb{R})$, we obtain

$$< S(v_0,c), \frac{\partial\varphi}{\partial t} > - < S(v_0,c), \frac{\partial^2\varphi}{\partial x^2} > + < x^2 S(v_0,c), \varphi > + < cS(v_0,c), \varphi >= 0.$$

Moreover, $v_n(0,\cdot) = v^{(n)}$ converges to v_0, which gives the equality $S(v_0,c)(0,\cdot) = v_0$. Hence $S(v_0,c)$ is a solution of (1) in a weak sense.

Appendix A. Computation of some Wiener integrals

Another consequence of Theorem 1 is the following Proposition :

Proposition 12. *Suppose v_0 satisfies, for all positive t and real x*

$$\int_{\mathbb{R}} (2\pi \text{ sh}(2t))^{-1/2} \exp\left(-\frac{1}{2}\frac{\text{ch}(2t)}{\text{sh}(2t)}y^2 - \frac{1}{2}\frac{\text{sh}(2t)}{\text{ch}(2t)}x^2\right) \left|v_0\left(y + \frac{x}{\text{ch}(2t)}\right)\right| \, dy < \infty.$$

Then for all $t > 0$ and $x \in \mathbb{R}$ we can write

$$\int_{C_W} v_0(\sqrt{2t}\; w(1) + x) \; \exp\left(-t\int_0^1 \left(x + \sqrt{2t}\; w(s)\right)^2 \, ds\right) \, dm_W(w)$$

$$= \int_{\mathbb{R}} (2\pi \text{ sh}(2t))^{-1/2} \exp\left(-\frac{1}{2}\frac{\text{ch}(2t)}{\text{sh}(2t)}y^2 - \frac{1}{2}\frac{\text{sh}(2t)}{\text{ch}(2t)}x^2\right) \left|v_0\left(y + \frac{x}{\text{ch}(2t)}\right)\right| \, dy \;.$$

Proof. Suppose v_0 belongs to $L^2(\mathbb{R})$. Then Theorem 1 shows that, as the perturbation c is equal to 0, the left hand side is the solution of the heat equation for the harmonic oscillator, with initial condition v_0. The right hand side is the solution $U_t v_0$ of the same problem, given by Mehler's formula. The equality follows.
When v_0 is not in $L^2(\mathbb{R})$, the equality holds for $v_0\varphi_n$ where φ_n is a convenient truncature. Then the assumption on v_0 allows to use the Theorems of dominated and of monotone convergence. □

When $v_0(x) = 1, x$ or x^2, it is easy to compute $U_t v_0$ and to deduce the following equalities from these computations :

Corollary 13. *For $v_0 = 1$ we obtain*

$$k(t,x) := \int_{C_W} e^{-t\int_0^1 \left(x+\sqrt{2t}\; w(s)\right)^2 \, ds} \, dm_W(w) = \frac{1}{\sqrt{\text{ch}(2t)}} \exp\left(-\frac{1}{2}\frac{\text{sh}(2t)}{\text{ch}(2t)}x^2\right).$$

The case $v_0(x) = x$ gives

$$\int_{C_W} w(1) \exp\left(-t\int_0^1 \left(x + \sqrt{2t}\; w(s)\right)^2 \, ds\right) \, dm_W(w) = \frac{x(1-\text{ch}(2t))}{\sqrt{2t}\; \text{ch}(2t)} \; k(t,x)$$

and for $v_0(x) = x^2$ we get

$$\int_{C_W} w(1)^2 \exp\left(-t\int_0^1 (x+\sqrt{2t}\, w(s))^2 \, ds\right) dm_W(w) = \frac{1}{2t}\left(\frac{(1-\text{ch}(2t))^2 x^2}{\text{ch}^2(2t)} + \frac{\text{sh}(2t)}{\text{ch}(2t)}\right) \; k(t,x) \;.$$

As the perturbation c is equal to 0, [3] and [6] prove that, under certain conditions, the l.h.s. is the solution of the heat equation problem for H. Nevertheless these integrals are not mentioned explicitly in the literature, to the author's knowledge.

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