ON QUASITRIANGULAR STRUCTURES IN HOPF ALGEBRAS ARISING FROM EXACT GROUP FACTORIZATIONS

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Abstract. We show that bicrossed product Hopf algebras arising from exact factorizations in almost simple finite groups, so in particular, in simple and symmetric groups, admit no quasitriangular structure.

1. Introduction and main result

We shall work over an algebraically closed field k of characteristic zero. Let H be a finite dimensional Hopf algebra. The category $\operatorname{Rep} H$ of its finite dimensional representations is a *finite tensor category* with tensor product induced by the comultiplication of H and unit object k. When H is semisimple, $\operatorname{Rep} H$ is a fusion category over k, that is, a semisimple finite tensor category over k.

Tensor categories of the form $\operatorname{Rep} H$ are characterized by being endowed with a fiber functor to the category Vec_k of vector spaces over k. The forgetful functor $\operatorname{Rep} H \to \operatorname{Vec}_k$ is a fiber functor and other fiber functors correspond to *twisting* the comultiplication of H.

An important class of Hopf algebras over k is that of quasitriangular Hopf algebras, introduced by Drinfeld in [7]. These are Hopf algebras H endowed with a so called R-matrix $R \in H \otimes H$. When (H, R) is quasitriangular, the category $\operatorname{Rep} H$ is a braided category over k with respect to the braiding $c : U \otimes V \to V \otimes U$ induced by the action of the R-matrix, for representations $U, V \in \operatorname{Rep} H$.

The main feature of quasitriangular Hopf algebras is that they give rise to universal solutions of the Quantum Yang-Baxter Equation. They are also related to the construction of invariants of knots and 3-manifolds. See [9].

In this paper we consider the question of the existence of quasitriangular structures in a class of semisimple Hopf algebras arising from exact factorizations of finite groups. These are amongst the first examples of non-commutative non-cocommutative Hopf algebras. They were first discovered and studied by G. I. Kac in the late 60's.

Suppose that $G = F\Gamma$ is an exact factorization of the finite group G, into its subgroups Γ and F, such that $F \cap \Gamma = 1$. Equivalently, F and Γ form a matched pair of finite groups with the actions $\triangleleft\colon \Gamma \times F \to \Gamma$, $\triangleright\colon \Gamma \times F \to F$, defined by

$$sx = (x \triangleleft s)(x \triangleright s),$$

$x \in F$, $s \in \Gamma$. We shall call an exact factorization *proper* if F and Γ are proper subgroups of G.

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Associated to this exact factorization and appropriate cohomology data σ and τ, there is a semisimple bicrossed product Hopf algebra $H = k^{\Gamma}\,{}^{\tau}\#_{\sigma} kF$. The Hopf algebra H fits canonically into an *abelian* exact sequence of Hopf algebras

$$k \to k^{\Gamma} \to H \to kF \to k. \tag{1.1}$$

Moreover, every Hopf algebra fitting into such exact sequence can be described in this way. This gives a bijective correspondence between the equivalence classes of Hopf algebra extensions (1.1) and a certain abelian group $\operatorname{Opext}(k^{\Gamma}, kF)$ associated to the matched pair (F, Γ). We refer the reader to [18, 19] for the notion of abelian exact sequence and the cohomology theory underlying it.

In the case of bicrossed products $H = k^{\Gamma}\#kF$, the classification of so-called *positive* quasitriangular structures in H appears in the paper [15]. These are related to set-theoretical solutions of Yang-Baxter equation.

The main result of this note is the following.

Theorem 1.1. *Let $G = F\Gamma$ be a proper exact factorization of the finite group G. Let also H be a Hopf algebra as in* (1.1) *associated to this factorization. Assume G is an almost simple group. Then H admits no quasitriangular structure.*

Recall that a finite group G is called *almost simple* if G is isomorphic to a group $\tilde{G}$ such that $N \leq \tilde{G} \leq \operatorname{Aut} N$, for some non-abelian finite simple group N. In particular, the following are almost simple groups:

(a) G a finite simple group, and

(b) $G = \mathbb{S}_n$ is the symmetric group on n symbols, $n \geq 5$.

Hence the theorem implies that any bicrossed product Hopf algebra associated to an exact factorization in any group in one of the classes (a) or (b) admits no quasitriangular structure.

There is a vast literature on the classification of factorizations in finite and, in particular, finite simple and almost simple groups. Group factorizations are related to the problem of determining the finite primitive permutation groups with a regular subgroup. We mention, in particular, the references [25] for the cases of alternating and symmetric groups, [10] for a determination of all exact factorizations in sporadic simple groups, and [13, 14] for all the maximal factorizations of simple and almost simple groups.

Theorem 1.1 will be proved in Sections 4 and 5. See Theorems 4.2 and 5.2. We treat separately the case of simple groups in Section 4. In the case of general almost simple groups, we make use in our proof of the classification of finite simple groups. We point out that this is not needed in the particular case of symmetric groups, which is stated in Proposition 5.3.

Our proof relies on the classification of full fusion subcategories of the categories of representations of (twisted) quantum doubles of a finite group, due to D. Naidu, D. Nikshych and S. Witherspoon [21]; indeed, the results in [21] amount to a description of all group-theoretical braided fusion categories.

The Drinfeld double of a Hopf algebra H as in (1.1) is twist equivalent to a twisted Drinfeld double of G, as introduced by Dijkgraaf, Pasquier and Roche [6], in view of the description of the Drinfeld double of H given in [2, 22]. On the other hand, a quasitriangular structure in H provides a canonical embedding of $\operatorname{Rep} H$ into $\operatorname{Rep} D(H)$; so that the category $\operatorname{Rep} H$ must coincide with one of those appearing in the classification result of [21]. This would also be a fruitful approach in classifying quasitriangular structures in any given abelian extension or, furthermore, in a group-theoretical Hopf algebra in general.

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2. Preliminaries

2.1. **Finite dimensional Hopf algebras and their representations.** Let H be a finite dimensional Hopf algebra over k. Recall that a *twist* in H is an invertible element $J \in H \otimes H$ such that

$$(\Delta \otimes \mathrm{id})(J)(J \otimes 1) = (\mathrm{id} \otimes \Delta)(J)(1 \otimes J), \tag{2.1}$$

$$(\epsilon \otimes \mathrm{id})(J) = (\mathrm{id} \otimes \epsilon)(J) = 1. \tag{2.2}$$

If $J \in H \otimes H$ is a twist, then there is a new Hopf algebra $(H^J, \Delta^J, \mathcal{S}^J)$, where $H^J = H$ as algebras, with comultiplication $\Delta^J(h) = J^{-1}\Delta(h)J$, and antipode $\mathcal{S}^J(h) = v^{-1}\mathcal{S}(h)v$, $\forall h \in H$, with $v = m(\mathcal{S} \otimes \mathrm{id})(J)$.

The Hopf algebras H and H' are called *twist equivalent* if $H' \simeq H^J$. It is known that H and H' are twist equivalent if and only if $\operatorname{Rep} H \simeq \operatorname{Rep} H'$ as tensor categories [23]. Therefore, properties like being semisimple or (quasi)triangular, are preserved under twisting deformations.

Remark 2.1. Full tensor subcategories of $\operatorname{Rep} H$ correspond, via tannaka duality arguments, to quotient Hopf algebras of H. That is, every full tensor subcategory of $\operatorname{Rep} H$ is of the form $\operatorname{Rep} L$ where $\pi : H \to L$ is a quotient Hopf algebra and the inclusion $\operatorname{Rep} L \to \operatorname{Rep} H$ is given by restriction of representations along π.

Suppose H and H' are twist equivalent. Then $\operatorname{Rep} H \simeq \operatorname{Rep} H'$ as tensor categories and therefore there is a bijective correspondence between quotient Hopf algebras of H and H'.

Indeed, if $\pi : H \to L$ is a Hopf algebra map and $J \in H \otimes H$ is a twist, then $(\pi \otimes \pi)(J)$ is a twist for L and $\pi : H^J \to L^{(\pi\otimes\pi)(J)}$ is a Hopf algebra map. This establishes the correspondence mentioned above.

2.2. **Quasitriangular structures.** Recall that a quasitriangular structure in Hopf algebra H consists of the data of an invertible element $R \in H \otimes H$, called an R-*matrix*, satisfying:

(QT1) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$.
(QT2) $(\epsilon \otimes \mathrm{id})(R) = 1$.
(QT3) $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.
(QT4) $(\mathrm{id} \otimes \epsilon)(R) = 1$.
(QT5) $\Delta^{\mathrm{cop}}(h) = R\Delta(h)R^{-1}$, $\forall h \in H$.

In that case, (H, R) is called a quasitriangular Hopf algebra. Suppose $R \in H \otimes H$ is an R-matrix. Then there are Hopf algebra maps $f_R : H^{*\mathrm{cop}} \to H$ and $f_{R_{21}} : H^* \to H^{\mathrm{op}}$ given, respectively, by

$$f_R(p) = \langle p, R^{(1)}\rangle R^{(2)}, \quad f_{R_{21}}(p) = \langle p, R^{(2)}\rangle R^{(1)},$$

for all $p \in H^*$, where we use the notation $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$.

Recall that for a finite dimensional Hopf algebra H, its Drinfeld double, $D(H)$, is a quasitriangular Hopf algebra containing H and $H^{*\,\mathrm{cop}}$ as Hopf subalgebras. We have $D(H) = H^{*\,\mathrm{cop}} \otimes H$ as a coalgebra, with canonical R-matrix $\mathcal{R} = \sum_i h^i \otimes h_i$, where $(h_i)_i$ is a basis of H and $(h^i)_i$ is the dual basis.

Remark 2.2. Suppose (H, R) is a finite dimensional quasitriangular Hopf algebra, and let $D(H)$ be the Drinfeld double of H. Then there is a surjective Hopf algebra map $f : D(H) \to H$, such that $(f \otimes f)\mathcal{R} = R$. The map f is determined by $f(p \otimes h) = f_R(p)h$, for all $p \in H^*$, $h \in H$.

Indeed, we have $\operatorname{Rep} D(H) = \mathcal{Z}(\operatorname{Rep} H)$: the center of the tensor category $\operatorname{Rep} H$, and the map f corresponds to the canonical inclusion of the braided tensor category $\operatorname{Rep} H$ (with braiding determined by the action of the R-matrix) into its center.

2.3. **Abelian extensions and factorizable groups.** Let (F, Γ) be a matched pair of finite groups. That is, F and Γ are endowed with actions by permutations $\Gamma \overset{\triangleleft}{\leftarrow} \Gamma \times F \overset{\triangleright}{\to} F$ such that

$$s \triangleright xy = (s \triangleright x)((s \triangleleft x) \triangleright y), \tag{2.3}$$

$$st \triangleleft x = (s \triangleleft (t \triangleright x))(t \triangleleft x), \tag{2.4}$$

for all $s, t \in \Gamma$, $x, y \in F$.

Given finite groups F and Γ, providing them with a pair of compatible actions is equivalent to giving a group G together with an exact factorization $G = F\Gamma$: the actions $\triangleright$ and $\triangleleft$ are determined by the relations $gx = (g \triangleright x)(g \triangleleft x)$, $x \in F$, $g \in \Gamma$.

Consider the left action of F on k^Γ, $x.\phi(g) = \phi(g \triangleleft x)$, $\phi \in k^\Gamma$, and let $\sigma : F \times F \to (k^\times)^\Gamma$ be a normalized 2-cocycle. Dually, we consider the right action of Γ on k^F, $\psi(x).g = \psi(x \triangleright g)$, $\psi \in k^F$, and let $\tau : F \times F \to (k^\times)^\Gamma$ be a normalized 2-cocycle.

Assume in addition that σ and τ obey the following compatibility conditions:

$$\sigma_{ts}(x, y)\tau_{xy}(t, s) = \tau_x(t, s)\, \tau_y(t \triangleleft (s \triangleright x), s \triangleleft x)\, \sigma_t(s \triangleright x, (s \triangleleft x) \triangleright y)\, \sigma_s(x, y),$$
$$\sigma_1(s, t) = 1, \qquad \tau_1(x, y) = 1,$$

for all $x, y \in F$, $s, t \in \Gamma$, where $\sigma = \sum_{s \in \Gamma} \sigma_s e_s$, $\tau = \sum_{x \in F} \tau_x e_x$, $e_s \in k^\Gamma$ and $e_x \in k^F$ being the canonical idempotents.

The vector space $H = k^\Gamma \otimes kF$ becomes a (semisimple) Hopf algebra with the crossed product algebra structure and the crossed coproduct coalgebra structure. We shall use the notation $H = k^{\Gamma\ \tau}\#_\sigma kF$. The multiplication and comultiplication of H are determined by the formulas

$$(e_g \# x)(e_h \# y) = e_{g \triangleleft x, h}\, \sigma_g(x, y) e_g \# xy,$$
$$\Delta(e_g \# x) = \sum_{st = g} \tau_x(s, t)\, e_s \#(t \triangleright x) \otimes e_t \# x,$$

for all $g, h \in \Gamma$, $x, y \in F$.

There is an exact sequence of Hopf algebras

$$1 \to k^\Gamma \to H \to kF \to 1,$$

also called an *abelian* exact sequence. Conversely, every Hopf algebra H fitting into an exact sequence of this form is isomorphic to $k^\Gamma {}^\tau\#_\sigma kF$ for appropriate actions $\triangleright$, $\triangleleft$, and compatible cocycles σ and τ. See [8, 16, 24, 18, 19].

Remark 2.3. Let $H = k^\Gamma {}^\tau\#_\sigma kF$ as above. For $s \in \Gamma$, let $F^s \subseteq F$ be the stabilizer of s under the action of F. Then the map $\sigma_s : F^s \times F^s \to k^\times$ defines a 2-cocycle on F^s. For each representation ρ of the twisted group algebra $k_{\sigma_s}F^s$, consider the induced representation $V_{s,\rho} = \operatorname{Ind}^H_{k^\Gamma\#_\sigma kF^s} s \otimes \rho$.

The irreducible representations of H are classified by $V_{s,\rho}$, where s runs over a set of representatives of the orbits of F in Γ and ρ runs over the isomorphism classes of irreducible representations of $k_{\sigma_s}F^s$ [20].

Note that $\dim V_{s,\rho} = [F : F^s] \deg \rho$. So, in particular, for all irreducible representation V of H we have that $\dim V$ divides the order of F.

2.3.1. *Kac exact sequence.* Fix a matched pair of groups (F, Γ). The set of equivalence classes of extensions $1 \to k^\Gamma \to H \to kF \to 1$ giving rise to these actions is denoted by $\operatorname{Opext}(k^\Gamma, kF)$: it is a finite group under the Baer product of extensions.

The class of an element of $\operatorname{Opext}(k^\Gamma, kF)$ can be represented by a pair of compatible cocycles (τ, σ). The group $\operatorname{Opext}(k^\Gamma, kF)$ can also be described as the first cohomology group of a certain double complex [18, Proposition 5.2].

An important result of G. I. Kac [8, 19], says that there is a long exact sequence

$$\begin{aligned} 0 &\to H^1(G, k^\times) \xrightarrow{\text{res}} H^1(F, k^\times) \oplus H^1(\Gamma, k^\times) \to \operatorname{Aut}(k^\Gamma \# kF) \\ &\to H^2(G, k^\times) \xrightarrow{\text{res}} H^2(F, k^\times) \oplus H^2(\Gamma, k^\times) \to \operatorname{Opext}(k^\Gamma, kF) \\ &\xrightarrow{\kappa} H^3(G, k^\times) \xrightarrow{\text{res}} H^3(F, k^\times) \oplus H^3(\Gamma, k^\times) \to \dots \end{aligned}$$

This result turns out to be an important tool in calculations related to the Opext group. In this paper we shall use the following result, contained in [22, Theorem 1.3], concerning the map $\kappa : \operatorname{Opext}(k^\Gamma, kF) \to H^3(G, k^\times)$:

Theorem 2.4. *The Drinfeld double $D(H)$ is twist equivalent to the twisted quantum double $D^\omega(G)$, where $\omega = \kappa(\tau, \sigma)$ is the 3-cocycle arising from the Kac exact sequence. In other words, there is an equivalence of fusion categories* $\operatorname{Rep} D(H) \simeq \operatorname{Rep} D^\omega(G)$.

The twisted quantum double $D^\omega(G)$, $\omega \in H^3(G, k^\times)$, is the quasi-Hopf algebra introduced by Dijkgraaf, Pasquier and Roche [6]. For the case of split extensions, that is when $(\tau, \sigma) = 1$ and hence $\omega = 1$, this result was obtained previously in [2].

Corollary 2.5. *Let $G = F\Gamma$ be an exact factorization of the group G. Let also H be a Hopf algebra fitting into an abelian extension $k \to k^\Gamma \to H \to kF \to k$ corresponding to this factorization. Suppose in addition that H admits a quasitriangular structure. Then* $\operatorname{Rep} H$ *is equivalent to a full (braided) fusion subcategory of* $\operatorname{Rep} D^\omega(G)$.

Proof. It follows from Theorem 2.4, in view of Remarks 2.1 and 2.2. □

2.4. **Fusion subcategories of $D^\omega(G)$.** Let G be a finite group and let $\omega \in H^3(G, k^\times)$. Isomorphism classes of simple objects in $\operatorname{Rep} D^\omega(G)$ are parameterized by pairs (g, π), where g runs over a set of representatives of the conjugacy classes of G, and π is an irreducible representation of the twisted group algebra $k_{\beta_g} C_G(g)$, where the 2-cocycle $\beta_g : C_G(g) \times C_G(g) \to k^\times$ is given by

$$\beta_g(x, y) = \frac{\omega(g, x, y)\omega(x, y, g)}{\omega(x, g, y)}.$$

In view of the description of irreducible representations in crossed products given in [20], this can be seen as a consequence of the fact that, as an algebra, $D^\omega(G)$ is a crossed product $D^\omega(G) = k^G \#_\beta kG$ with respect to the adjoint action of G on itself and the 2-cocycle $\beta : kG \otimes kG \to k^G$ determined by

$$\beta_g(x, y) = \frac{\omega(g, x, y)\omega(x, y, y^{-1}x^{-1}gxy)}{\omega(x, x^{-1}gx, y)}.$$

The irreducible representation corresponding to the pair (g, π) may thus be identified with the induced representation $V_{(g,\pi)} = \operatorname{Ind}_{k^G \#_\beta kC_G(g)}^{D^\omega(G)} g \otimes \pi$. Its dimension is therefore

$$\dim V_{(g,\pi)} = [G : C_G(g)] \deg \pi. \tag{2.5}$$

See [1].

Full fusion subcategories of $\operatorname{Rep} D^\omega(G)$ are determined in [21, Theorem 5.11]. They are in bijection with tripes (K_1, K_2, B) where

(a) K_1 and K_2 are normal subgroups of G that centralize each other, and
(b) $B : K_1 \times K_2 \to k^\times$ is a G-invariant ω-bicharacter, that is, it satisfies

$$B(x, yz) = \beta_x^{-1}(y, z)B(x, y)B(x, z), \quad B(tx, y) = \beta_y(t, x)B(t, y)B(x, y),$$

for all $t, x \in K_1$, $y, z \in K_2$, and the invariance condition

$$B(x^{-1}ax, h) = \frac{\beta_a(x, h)\beta_a(xh, x^{-1})}{\beta_a(x, x^{-1})} B(a, xhx^{-1}),$$

for all $x, y \in G$, $h \in K_2$, $a \in K_1$. See [21, Definition 5.4].

The bijection is determined as follows. The triple (K_1, K_2, B) gives rise to the fusion subcategory $\mathcal{C}(K_1, K_2, B)$, defined as the full abelian subcategory of $\operatorname{Rep} D^\omega(G)$ generated by the simple objects $V_{(g,\pi)}$, where g runs over a set of representatives of conjugacy classes of G in K_1, and π is an irreducible β_g-character of $C_G(g)$ such that $\pi(h) = B(g, h) \deg \pi$, for all $h \in K_2$.

In the other direction, a full fusion subcategory $\mathcal{C}$ of $\operatorname{Rep} D^\omega(G)$ determines a triple $(K_1^{\mathcal{C}}, K_2^{\mathcal{C}}, B^{\mathcal{C}})$, where

$$K_1^{\mathcal{C}} = \{gag^{-1} |\, g \in G,\ (a, \pi) \in \mathcal{C} \text{ for some } \pi\},$$
$$K_2^{\mathcal{C}} = \bigcap_{\pi:\, (e,\pi) \in \mathcal{C}} \ker \pi.$$

(An equivalent definition of $K_2^{\mathcal{C}}$ consists in letting $\mathcal{C} \cap \operatorname{Rep} G \simeq \operatorname{Rep} G/K_2^{\mathcal{C}}$.) The ω-bicharacter $B^{\mathcal{C}} : K_1^{\mathcal{C}} \times K_2^{\mathcal{C}} \to k$ is determined by

$$B^{\mathcal{C}}(x^{-1}gx, h) := \frac{\beta_g(x, h)\beta_g(xh, x^{-1})\, \pi(xhx^{-1})}{\beta_g(x, x^{-1}) \deg \pi},$$

where (g, π) is a class of an irreducible representation of $D^\omega(G)$ that belongs to $\mathcal{C}$.

The above assignments are inverse bijections.

By [21, Lemma 5.9] the dimension of the fusion subcategory $\mathcal{C}(K_1, K_2, B)$ corresponding to the triple (K_1, K_2, B) is $|K_1|[G : K_2]$. Hence $\dim \mathcal{C}(K_1, K_2, B) = |G|$ if and only if $|K_1| = |K_2|$.

It follows from the description in [21, (25)] and the definition of G-invariant ω-bicharacter, that $\mathcal{C}(1, 1, 1) \simeq \operatorname{Rep} G$.

As a special case, full fusion subcategories of $\operatorname{Rep} D(G)$ are parameterized by triples (K_1, K_2, B), where

(a') K_2, K_1 are normal subgroups of G centralizing each other;
(b') $B : K_1 \times K_2 \to k^\times$ is a G-invariant bicharacter.

See [21, Theorem 3.12]. Let $\mathcal{C}(K_1, K_2, B)$ be the fusion subcategory of $D(G)$ corresponding to the triple (K_1, K_2, B). The dimension of $\mathcal{C}(K_1, K_2, B)$ is $|K_1||G : K_2|$ [21, Lemma 3.10].

By Corollary 2.5, every quasitriangular structure on a bicrossed product Hopf algebra $H = k^{\Gamma}{}^{\tau}\#_\sigma kF$ as in (1.1) determines a full fusion subcategory of $\operatorname{Rep} D^\omega(G)$, where $G = F\Gamma$ and $\omega = \kappa(\tau, \sigma)$ is the 3-cocycle coming from H in the Kac exact sequence. Hence we get:

Corollary 2.6. *Let $H = k^{\Gamma}{}^{\tau}\#_\sigma kF$ be an abelian extension as in* (1.1). *Then a quasitriangular structure in H determines a triple (K_1, K_2, B) satisfying (a) and (b) with respect to $G = F\Gamma$ and $\omega = \kappa(\tau, \sigma)$, such that $|K_1| = |K_2|$.* □

The following proposition gives some restrictions on the possible triples in terms of the involved factorization.

Proposition 2.7. *Let $G = F\Gamma$ be an exact factorization of the group G. Let also H be a Hopf algebra extension* (1.1) *corresponding to this factorization and assume that H admits a quasitriangular structure.*

Let (K_1, K_2, B) be the triple determined by the quasitriangular structure in H. Then for all $g \in K_1$, and for all irreducible character π of the twisted group algebra $k_{\beta_g} C_G(g)$ such that $\pi(h) = B(g, h) \deg \pi$, for all $h \in K_2$, the product $[G : C_G(g)] \deg \pi$ divides the order of F.

Proof. The group K_1 is determined as the union of the conjugacy classes of elements $g \in G$ such that the (class of the) irreducible representation corresponding to some pair (g, π) belongs to the fusion subcategory $\operatorname{Rep} H = \mathcal{C}(K_1, K_2, B)$ of $D^\omega(G)$ [21, 5.2]. By the description of the irreducible representations belonging to $\mathcal{C}(K_1, K_2, B)$, these are exactly those corresponding to pairs (g, π) where $g \in K_1$ and π is an irreducible character of the twisted group algebra $k_{\beta_g} C_G(g)$, such that $\pi(h) = B(g, h) \deg \pi$, for all $h \in K_2$.

The proposition follows from Remark 2.3 and Formula (2.5) for the dimension of such an irreducible representation. □

3. Some examples associated to symmetric groups

Let C_n be the cyclic group of order n and $\mathbb{S}_n$ the symmetric group on n symbols. Let $H = k^{C_n}\#k\mathbb{S}_{n-1}$ be the bismash product (split abelian extension) associated to the matched pair $(C_n, \mathbb{S}_{n-1})$ arising from the exact factorization $\mathbb{S}_n = C_n\mathbb{S}_{n-1}$.

In this case, the group $\operatorname{Opext}(k^{C_n}, k\mathbb{S}_{n-1})$ corresponding to this matched pair is trivial [17, Theorem 4.1]. Hence H is the unique, up to isomorphisms, Hopf algebra fitting into an abelian exact sequence $k \to k^{C_n} \to H \to k\mathbb{S}_{n-1} \to k$.

The results in this section are a special case of those obtained in Section 5. In particular, Proposition 3.2 follows from Proposition 5.3.

Let $\mathcal{C}(K_1, K_2, B)$ be the fusion subcategory of $D(\mathbb{S}_n)$ corresponding to the triple (K_1, K_2, B) satisfying (a'), (b') and such that $|K_1| = |K_2|$.

Suppose $n \geq 5$. So that the only normal subgroups of $\mathbb{S}_n$ are 1, $\mathbb{A}_n$ and $\mathbb{S}_n$.

Therefore the only fusion subcategory of $D(\mathbb{S}_n)$ of dimension $n!$ is $\mathcal{C}(1;1;1) \simeq k\mathbb{S}_n$.

Combining this with Corollary 2.6 we get the following:

Lemma 3.1. *Suppose $n \geq 5$.*

(1) *If H is quasitriangular, then H is a twisting of $k\mathbb{S}_n$.*
(2) *If H^* is quasitriangular, then H^* is a twisting of $k\mathbb{S}_n$.*

Proof. If H is quasitriangular, then $\operatorname{Rep} H$ is a fusion subcategory of $\operatorname{Rep} D(H) \simeq \operatorname{Rep} D(\mathbb{S}_n)$ of dimension $n!$. Hence $\operatorname{Rep} H \simeq \operatorname{Rep} \mathbb{S}_n$. The other claim is similar, because the same arguments apply as well to $H^* \simeq k^{\mathbb{S}_{n-1}} \# kC_n$, and $D(H) \simeq D(H^{*\,\mathrm{cop}})$. □

Proposition 3.2. *Suppose $n \geq 5$. Then neither H nor H^* admit a quasitriangular structure.*

Proof. Suppose on the contrary that H admits a quasitriangular structure. Then H would be a twisting of the group algebra $k\mathbb{S}_n$, in view of Lemma 3.1. Similarly, if H^* admits a quasitriangular structure, then H^* would be a twisting of $k\mathbb{S}_n$.

Note that twisting preserves Hopf algebra quotients. Consider first the case of the dual Hopf algebra H^*. In this case, there is a quotient Hopf algebra $H^* \to kC_n$, which gives rise to a quotient Hopf algebra $k\mathbb{S}_n \to L$, where L is a certain twisting of kC_n. Thus $L \simeq kC_n$. But this gives n non-equivalent representations of dimension 1 of the group algebra $k\mathbb{S}_n$, which is impossible because $n > 2$. Hence H^* admits no quasitriangular structure.

Similarly, if H admits a quasitriangular structure, then H must be a twisting of $k\mathbb{S}_n$. But H has a quotient Hopf algebra isomorphic to $k\mathbb{S}_{n-1}$. This gives a quotient Hopf algebra $k\mathbb{S}_n \to L$, with $\dim L = (n-1)!$. On the other hand, every such quotient Hopf algebra is of the form kF, where F is a quotient group of $\mathbb{S}_n$. Then necessarily $|F| = 2 = (n-1)!$, which contradicts the assumption that $n \geq 5$. This contradiction shows that H does not admit a quasitriangular structure. The proof of the proposition is now complete. □

Remark 3.3. If $n = 3$, then $H \simeq k\mathbb{S}_3$ is a group algebra. Thus H is quasitriangular in this case, while $J = H^* \simeq k^{\mathbb{S}_3}$ is not.

The arguments fail if $n = 4$ because we could also take (K_1, K_1, B), where $K_1 \subseteq \mathbb{S}_4$ is the Klein subgroup, which is abelian.

4. Hopf algebras associated to simple groups

Let $G = F\Gamma$ be a proper exact factorization of the group G. We assume in this section that the group G is simple non-abelian.

Let H be a semisimple Hopf algebra fitting into an exact sequence

$$k \to k^{\Gamma} \to H \to kF \to k, \tag{4.1}$$

corresponding to the matched pair (F, Γ).

In particular, H is isomophic to a bicrossed product $H \simeq k^{\Gamma}{}^{\tau}\#_{\sigma}kF$, where (σ, τ) represents the element of the group $\operatorname{Opext}(k^{\Gamma}, kF)$ determined by the extension (4.1).

In view of Theorem 2.4 there is an equivalence of fusion categories $\operatorname{Rep} D(H) \simeq \operatorname{Rep} D^{\omega}(G)$, where $\omega \in H^3(G, k^{\times})$ is the 3-cocycle correspondig to the class $[\sigma, \tau] \in \operatorname{Opext}(k^{\Gamma}, kF)$ under the Kac exact sequence 2.3.1.

Lemma 4.1. *Suppose H admits a quasitriangular structure. Then H is twist equivalent to the group algebra kG.*

Proof. Under these assumptions, $\operatorname{Rep} H$ is isomorphic as a fusion category to a full fusion subcategory of $\operatorname{Rep} D(H) \simeq \operatorname{Rep} D^{\omega}(G)$.

In view of the description of such fusion subcategories in 2.4, and since the group G is simple by assumption, the only fusion subcategory of $\operatorname{Rep} D^{\omega}(G)$ of dimension $|G|$ is $\mathcal{C}(1, 1, 1) \simeq \operatorname{Rep} G$. Therefore we must have $\operatorname{Rep} H \simeq \operatorname{Rep} G$, which implies the lemma. □

Theorem 4.2. *Let H be a Hopf algebra fitting into an exact sequence* (4.1)*, where Γ and F are proper subgroups of G. Then H admits no quasitriangular structure.*

Proof. Suppose on the contrary that H admits a quasitriangular structure. Then H is twist equivalent to kG, by Lemma 4.1. On the other hand, H has a Hopf algebra quotient $H \to kF$ that corresponds, because twisting preserves quotients, to a Hopf algebra quotient $kG \to L$, with $1 < \dim L = |F| < |G|$. In particular, $L \simeq k\overline{G}$, where $\overline{G}$ is a quotient of the group G. This is a contradiction, since the group G is simple by assumption. Thus H admits no quasitriangular structure, as claimed. □

Remark 4.3. We have shown, as part of the proof of Proposition 3.2 and Theorem 4.2, that the bicrossed product Hopf algebra H is not twist equivalent to the group algebra of G. For the case where $G = \mathbb{S}_n$, $n = p + 1$ or $p + 2$, $p > 3$ a prime number, and H corresponds to the exact factorization considered in Section 3, this fact follows from the main result of [5] that says that H is not isomorphic as an algebra to any group algebra. The analogous result is true for a bismash product (split extension) Hopf algebra associated to the groups $\mathrm{PGL}_2(q)$, $q \neq 2, 3$, as shown in [4].

5. Hopf algebras associated to almost simple groups

Let G be a finite group. Recall that G is called *almost simple* if it is isomorphic to a group $\tilde{G}$ (that we shall identify with G in what follows) such that $N \leq \tilde{G} \leq \operatorname{Aut} N$, for some non-abelian finite simple group N. In this case, the *socle* of G, that is, the subgroup generated by the minimal normal subgroups of G, coincides with N. Furthermore, a finite group G is almost simple if and only if its socle is a simple non-abelian group.

In particular, we have the following

Lemma 5.1. *Let $N \leq G \leq \operatorname{Aut}(N)$, where N is simple non-abelian. Then for every normal subgroup $1 \neq K \trianglelefteq G$, we have $N \leq K$.* □

Examples of almost simple groups are the non-abelian simple groups. Also, the symmetric groups $\mathbb{S}_n$, are almost simple, for all $n \geq 5$. Indeed, $\mathbb{S}_n \simeq \operatorname{Aut}(\mathbb{A}_n)$, for all $n \neq 6$, and $\operatorname{Aut}(\mathbb{A}_6)/\mathbb{S}_6 \simeq \mathbb{Z}_2$.

In view of the Classification Theorem of finite simple groups, the simple non-abelian group N is isomorphic to exactly one of the following groups:
(i) the alternating group $\mathbb{A}_n$, $n \geq 5$,
(ii) a group of Lie type, or
(iii) one of the twenty-six sporadic simple groups.

These groups, together with their orders, are listed, for instance, in [11]. We shall make use of the order of the outer automorphism groups from [13, 2.1].

Theorem 5.2. *Let G be an almost simple group and let $G = F\Gamma$ be a proper exact factorization of G. Let H be a Hopf algebra fitting into an exact sequence* (4.1). *Then H admits no quasitriangular structure.*

Proof. By Theorem 2.4 we have an equivalence of fusion categories $\operatorname{Rep} D(H) \simeq \operatorname{Rep} D^{\omega}(G)$, where $\omega \in H^3(G, k^{\times})$ is the 3-cocycle correspondig to the class $[\sigma, \tau] \in \operatorname{Opext}(k^{\Gamma}, kF)$ under the Kac exact sequence 2.3.1.

Suppose on the contrary that H admits a quasitriangular structure. Then $\operatorname{Rep} H$ is isomorphic as a fusion category to a full fusion subcategory of $\operatorname{Rep} D(H) \simeq \operatorname{Rep} D^{\omega}(G)$.

In view of the description of such subcategories in 2.4, and since every nontrivial normal subgroup of G contains N, which is not abelian by assumption, the only fusion subcategory of $\operatorname{Rep} D^{\omega}(G)$ of dimension $|G|$ is $\mathcal{C}(1,1,1) \simeq \operatorname{Rep} G$. Therefore we must have $\operatorname{Rep} H \simeq \operatorname{Rep} G$, which implies H is twist equivalent to the group algebra kG. In particular H is isomorphic to kG as an algebra and therefore they have the same irreducible degrees.

The (proper) quotient Hopf algebra $H \to kF$ corresponds to a surjective group homomorphism $f : G \to T$ for some group T such that kT is twist equivalent to kF, so that $|T| = |F|$. Thus, in view of Remark 2.3, we get:

$$\dim V \text{ divides } |F| = |T|, \text{ for all irreducible representation } V \text{ of } G. \tag{5.1}$$

Letting $L \subseteq G$ be the kernel of f, and since L is a nontrivial normal subgroup of G, Lemma 5.1 implies that $N \subseteq L$. In particular, T is isomorphic to a subgroup of $G/N \leq \operatorname{Out}(N)$.

Since the group G is an extension of N, N being a normal subgroup, then the irreducible representations of N should also satisfy Condition (5.1).

We shall show next that such a subgroup T cannot exist, thus proving the theorem. The proof will go case by case, considering the different possibilities for the simple non-abelian group N listed above.

Case (i). $N \simeq \mathbb{A}_n$, $n \geq 5$. If $n \neq 6$, then $G = \mathbb{A}_n$ or $\mathbb{S}_n$ and $\operatorname{Out}(N) = 2$, hence $|T| = 1$ or 2. Combined with Condition (5.1), this implies a contradiction. If $n = 6$, then $\operatorname{Out}(N) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, which gives similarly a contradiction ($\mathbb{A}_6$, hence also G, has irreducible representations of degree > 4).

Case (ii). N is a group of Lie type over a finite field of characteristic p. In this case, the Steinberg representation is an irreducible representation of N of degree equal to the largest power of p dividing the order of the group [3, 12]. That this representation does not satisfy Condition (5.1), follows by inspection of the order of the group $\operatorname{Out}(N)$ in [13, Table 2.1 A and B].

Case (iii). N is a sporadic simple group. In this case, $|\operatorname{Out}(N)| = 1$ or 2. The same argument as in Case (i) discards this possibility. □

As a consequence, we get the following statement, generalizing Proposition 3.2.

Proposition 5.3. *Let H be a Hopf algebra fitting into an exact sequence* (4.1)*, where Γ and F are proper subgroups of $\mathbb{S}_n$, $n \geq 5$. Then H admits no quasitriangular structure.* □

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