Quasi-weak equivalences in complicial exact categories

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Abstract

We introduce a notion of quasi-weak equivalences associated with weak-equivalences in an exact category. It gives us a delooping for (idempotent complete) exact categories and a condition that the negative K-group of an exact category becomes trivial.

1 Introduction

The negative K-theory $\mathbb{K}(\mathcal{E})$ for an exact category $\mathcal{E}$ is introduced in [6] and [7] by M. Schlichting. This generalizes the definition of Bass, Karoubi, Pedersen-Weibel, Thomason, Carter and Yao. The first motivation of our work is to investigate some vanishing conjectures of such negative K-groups:

(a) For any noetherian scheme X of Krull dimension d, $K_{-n}(X)$ is trivial for $n > d$ ([11]).

(b) The negative K-groups of a small abelian category is trivial ([7]).

(c) For a finitely presented group G, $K_{-n}(\mathbb{Z}G) = 0$ for $n > 1$ ([4]).

In [7], it was given a description of $\mathbb{K}_{-1}(\mathcal{E})$ and a condition on vanishing of $\mathbb{K}_{-1}(\mathcal{E})$ for an (essentially small) exact category $\mathcal{E}$ in terms of its *unbounded* derived category $\mathcal{D}(\mathcal{E})$: We have $\mathbb{K}_{-1}(\mathcal{E}) = \mathbb{K}_0(\mathcal{D}(\mathcal{E}))$ and $\mathbb{K}_{-1}(\mathcal{E})$ is trivial if and only if $\mathcal{D}(\mathcal{E})$ is idempotent complete (= Karoubian in the sense of [10], A.6.1). To extend this observation, we shall introduce the notion of *higher derived categories* $\mathcal{D}_n(\mathcal{E})$ and show the following theorem:

Theorem 1.1 (Cor. 4.4)**.** *For an idempotent complete exact category $\mathcal{E}$, we have $\mathbb{K}_{-n}(\mathcal{E}) = \mathbb{K}_0(\mathcal{D}_n(\mathcal{E}))$. Moreover, $\mathbb{K}_{-n}(\mathcal{E})$ is trivial if and only if $\mathcal{D}_n(\mathcal{E})$ is idempotent complete.*

Although we limit our consideration to idempotent complete exact categories to avoid some technical difficulties, the exact categories in the conjectures (a)-(c) above satisfies this condition. Recall that the derived category $\mathcal{D}(\mathcal{E})$ is the triangulated category obtained by formally inverting quasi-isomorphisms in the category of chain complexes $\mathrm{Ch}(\mathcal{E})$. The pair $(\mathrm{Ch}(\mathcal{E}), \mathrm{qis})$ of the category of chain complexes $\mathrm{Ch}(\mathcal{E})$ and qis the class of quasi-isomorphisms forms a *complicial* exact category with weak equivalences (cf. Def. 3.1). More generally, for a complicial exact category with weak equivalences $\mathbf{E} = (\mathcal{E}, w)$ (cf. Def. 3.3), we define a class of weak equivalences qw in the category of chain complexes $\mathrm{Ch}(\mathcal{E})$, which is called *quasi-weak equivalences* associated with w. If w is the class of isomorphisms in $\mathcal{E}$, then qw is just the class of quasi-isomorphisms on $\mathrm{Ch}(\mathcal{E})$. The derived category $\mathcal{D}(\mathbf{E})$ of $\mathbf{E}$ is obtained by formally inverting the quasi-weak equivalences in $\mathrm{Ch}(\mathcal{E})$. Put $\mathrm{Ch}(\mathbf{E}) = (\mathrm{Ch}(\mathcal{E}), qw)$ and one can define the class weak equivalences in $\mathrm{Ch}_n(\mathbf{E}) := \mathrm{Ch}(\mathrm{Ch}_{n-1}(\mathbf{E}))$ inductively. The n-th derived category $\mathcal{D}_n(\mathbf{E})$ associated with $\mathbf{E}$, is the derived category of $\mathrm{Ch}_n(\mathbf{E})$. We also obtain the following theorems on the negative K-theory $\mathbb{K}(\mathbf{E})$ for $\mathbf{E}$ (for definition, see [8]):

Theorem 1.2 (Thm. 4.2)**.** *Assume that $\mathcal{E}$ is idempotent complete. Then we have:*

(i) (Gillet-Waldhausen theorem) $\mathbb{K}(\mathbf{E}) \overset{\sim}{\to} \mathbb{K}(\mathrm{Ch}^b(\mathbf{E}))$,

(ii) (Eilenberg swindle) $\mathbb{K}(\mathrm{Ch}^+(\mathbf{E})) \overset{\sim}{\to} \mathbb{K}(\mathrm{Ch}^-(\mathbf{E})) \overset{\sim}{\to} 0$,

(iii) (Delooping) $\mathbb{K}(\mathrm{Ch}(\mathbf{E})) \overset{\sim}{\to} \Sigma\, \mathbb{K}(\mathbf{E})$*, where Σ is a suspension functor on the stable category of spectra.*

The organization of this note is as follows: In Section 2, we list several axioms about weak equivalences in a category with cofibrations and study their implication. In Section 3, after recalling the definition of complicial exact category with weak equivalences, we consider the notion of *null classes* and investigate the relation to weak equivalences in a complicial exact category. In Section 4, we introduce the quasi-weak equivalences as noted above which is a class of weak equivalences in the exact category of chain complexes $\mathrm{Ch}(\mathcal{E})$ associated with a given weak equivalences w in an exact category $\mathcal{E}$. By using this, we prove the main theorem. Throughout this note, we follow basically the terminologies on algebraic K-theory in [10] and [8].

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2 Weak equivalences in categories with cofibrations

In this section, we list several axioms on weak equivalences in a category with cofibrations and study their implications. Let $\mathcal{C}$ be a category with cofibrations and w be a class of morphisms in $\mathcal{C}$. We denote by $\mathrm{Ar}(\mathcal{C})$ the category of arrows $\mathcal{C} \to \mathcal{C}$. The functors $\mathrm{dom}, \mathrm{ran} : \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ are defined by $\mathrm{dom}(f) = x$ and $\mathrm{ran}(f) = y$ respectively, for $f : x \to y \in \mathrm{Ar}(\mathcal{C})$.

First, we consider the following axioms on w:

(**WE 1**) Every isomorphisms in $\mathcal{C}$ is in w.

(**WE 2**) For composable morphisms f and g in $\mathcal{C}$, if two of f, g and gf are in w, then the other one is also in w.

(**WE 3**) For a commutative diagram in $\mathcal{C}$,

$$(1) \qquad \begin{array}{ccccc} x & \rightarrowtail & y & \twoheadrightarrow & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ x' & \rightarrowtail & y' & \twoheadrightarrow & z' \end{array} \quad ,$$

where the horizontal lines are cofibration sequences, if a and c are in w, then so is b.

(**WE 3**)$'$ For the commutative diagram (1) in $\mathcal{C}$, if a and b are in w, then so is c.

(**WE 4**) For a commutative diagram in $\mathcal{C}$,

$$(2) \qquad \begin{array}{ccccc} y & \overset{i}{\leftarrowtail} & x & \overset{f}{\rightarrow} & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ y' & \underset{i'}{\leftarrowtail} & x' & \underset{f'}{\rightarrow} & z' \end{array} \quad ,$$

where i and i' are cofibrations, if a, b and c are in w, then the induced map $a \sqcup_b c : y \sqcup_x z \to y' \sqcup_{x'} z'$ on pushouts is also cofibration.

(**WE 5**) For any cofibration $x \rightarrowtail y$ in $\mathcal{C}$ and $x \to z$ in w, the induced morphism $y \to y \sqcup_x z$ is in w.

(**WE 6**) (Factorization axiom) There are a functor $\mathrm{Cyl} : \mathrm{Ar}\,\mathcal{C} \to \mathcal{C}$ and, natural transformations $\alpha : \mathrm{dom} \Rightarrow \mathrm{Cyl}$ and $\beta : \mathrm{Cyl} \Rightarrow \mathrm{ran}$ such that for any morphism $f : x \to y$ in $\mathcal{C}$, $\alpha(f) : x \to \mathrm{Cyl}(f)$ is a cofibration, $\beta(f) : \mathrm{Cyl}(f) \to y$ is in w and $\beta(f)\alpha(f) = f$.

(**WE 7**) For a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{i} & y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ x' & \xrightarrow{i'} & y' \end{array} \qquad , \tag{3}$$

with retractions $p : y \to x$ and $p' : y \to x'$ such that $pi = \mathrm{id}_x$ and $p'i' = \mathrm{id}_{x'}$, if b is in w, so is a.

Note that the axiom (**WE 2**) implies that the class w is closed under composition. Next we study logical relations among these axioms as follows:

Proposition 2.1 (cf. [3]; [2], 8.8)**.** (i) (**WE 1**) *and* (**WE 3**) *imply* (**WE 5**).
(ii) (**WE 2**)*,* (**WE 5**) *and* (**WE 6**) *imply* (**WE 4**).
(iii) (**WE 1**) *and* (**WE 4**) *imply* (**WE 3**)$'$.

Proof. (i) Let $i : x \rightarrowtail y$ be a cofibration in $\mathcal{C}$ and $a : x \to x'$ in w. It is easy to see that the sequence $x' \rightarrowtail y' := x' \sqcup_x y \twoheadrightarrow x/y$ is a cofibration sequence. Now applying (**WE 3**) to the following commutative diagram

$$\begin{array}{ccccc} x & \overset{i}{\rightarrowtail} & y & \twoheadrightarrow & x/y \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{x/y}} \\ x' & \rightarrowtail & y' & \twoheadrightarrow & x/y \end{array} \qquad ,$$

we learn that $b : y \to y'$ is also in w.

(iii) Now we consider the following commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccccc} x & \rightarrowtail & y & \twoheadrightarrow & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ x' & \rightarrowtail & y' & \twoheadrightarrow & z' \end{array} \quad ,$$

where the horizontal lines are cofibration sequences, and a and b are in w. Note that the map $0 \to 0$ is in w by (**WE 1**). For the commutative diagram (1) in $\mathcal{C}$, we assume that a and b are in w. The following diagrams are coCartesian:

$$\begin{array}{ccc} x & \rightarrowtail & y \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & z \end{array} \quad , \qquad \begin{array}{ccc} x' & \rightarrowtail & y' \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & z' \end{array} \quad .$$

By (**WE 4**), the map $c = 0 \sqcup_a b$ is in w.

(ii) Consider the commutative diagram (2) in (**WE 4**). First we assume the lemma below we intend to conclude the result.

Lemma 2.2. *Let us assume the axioms* (**WE 2**) *and* (**WE 5**). *In the diagram (2), suppose that both f and f' are cofibrations or in w. Then, $a \sqcup_b c : y \sqcup_x z \to y' \sqcup_{x'} z'$ is in w.*

Applying (**WE 6**) to the diagram (2), we get the following diagrams

$$\begin{array}{ccccccc} y & \xleftarrow{\;i\;} & x & \xrightarrow{\alpha(f)} & \mathrm{Cyl}(f) & \xrightarrow{\beta(f)} & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle c}\downarrow \\ y' & \xleftarrow[\;i'\;]{} & x' & \xrightarrow[\alpha(f')]{} & \mathrm{Cyl}(f') & \xrightarrow[\beta(f')]{} & z' \end{array} \quad ,$$

$$\begin{array}{ccccc} y \sqcup_x \mathrm{Cyl}(f) & \leftarrowtail & \mathrm{Cyl}(f) & \xrightarrow{\beta(f)} & z \\ {\scriptstyle e}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle c}\downarrow \\ y' \sqcup_{x'} \mathrm{Cyl}(f') & \leftarrowtail & \mathrm{Cyl}(f') & \xrightarrow[\beta(f')]{} & z' \end{array} \quad .$$

By Lemma 2.2, we learn that e is in w. Again using Lemma 2.2, we learn that $e \sqcup_d c$ is in w. Thus by Lemma 2.3, (i) below, we notice that $e \sqcup_d c$ is $a \sqcup_b c$. $\square$

Lemma 2.3. *Let us consider a commutative diagram below in a category.*

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \mathrm{I} & \downarrow & \mathrm{II} & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array} .$$

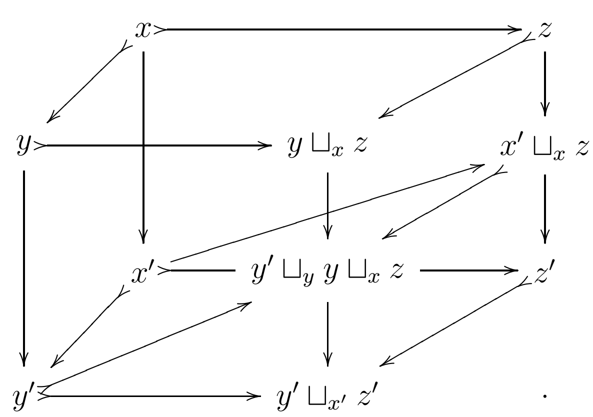
(i) *If the diagrams* I *and* II *are coCartesian squares, then the diagram* I + II *is also.*

(ii) *If the diagram* I + II *and* I *are coCartesian squares, then the diagram* II *is also.*

Proof of Lemma 2.2. First let us assume that both f and f' in the diagram (2) are in w. Then in the diagram below

$$\begin{array}{ccc} y & \xrightarrow{g} & y \sqcup_x z \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a \sqcup_b c} \\ y' & \xrightarrow[g']{} & y' \sqcup_{x'} z' \end{array} ,$$

g and g' are in w by (**WE 5**). Therefore by (**WE 2**), $a \sqcup_b c$ is also. Next let us suppose that f and f' in the diagram (2) are cofibrations in $\mathcal{C}$. Consider the following diagram:



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Claim. The commutative diagram below is coCartesian.

$$\begin{array}{ccc} x' \sqcup_x z & \longrightarrow & z' \\ \downarrow & & \downarrow \\ y' \sqcup_y y \sqcup_x z & \longrightarrow & y' \sqcup_{x'} z' \end{array} .$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} x' & \longrightarrow & x' \sqcup_x z & \longrightarrow & z' \\ \downarrow & \text{I} & \downarrow & \text{II} & \downarrow \\ y' & \longrightarrow & y' \sqcup_y y \sqcup_x z & \longrightarrow & y' \sqcup_{x'} z' \end{array} .$$

The squares $\text{I} + \text{II}$ and I are coCartesian. Therefore by Lemma 2.3 (ii), the diagram II is also coCartesian. □

By (**WE 5**), $y \sqcup_x z \to y' \sqcup_y y \sqcup_x z$ and $z \to x' \sqcup_x z$ are in w. Therefore by (**WE 2**), $x' \sqcup_x z \to z'$ is in w. Hence by (**WE 5**) again, $y' \sqcup_y y \sqcup_x z \to y' \sqcup_{x'} z'$ is in w. Finally by (**WE 2**), the composition $a \sqcup_b c : y \sqcup_x z \to y' \sqcup_y y \sqcup_x z \to y' \sqcup_{x'} z'$ is in w. □

In a category with fibrations $\mathcal{C}$ which is the dual concept of categories with cofibrations, we consider the following axioms:

(**WE 3**)$'^{\text{op}}$ For the commutative diagram (1) in $\mathcal{C}$, if b and c are in w, then so is a.

(**WE 4**)$^{\text{op}}$ For a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} y & \overset{p}{\twoheadrightarrow} & x & \longleftarrow & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ y' & \underset{p'}{\twoheadrightarrow} & x' & \longleftarrow & z' \end{array} ,$$

where p and p' are fibrations, if a, b and c are in w, then $a \times_b c : y \times_x z \to y' \times_{x'} z'$ is also.

(**WE 5**)$^{\text{op}}$ w is stable under base change by fibrations. That is, for any fibration $y \twoheadrightarrow x$ in $\mathcal{C}$ and $z \to x$ in w, the induced morphism $y \times_x z \to y$ is in w.

(**WE 6**)$^{\mathrm{op}}$ There are a functor $M : \mathrm{Ar}\,\mathcal{C} \to \mathcal{C}$ and natural transformations $\gamma : \mathrm{dom} \Rightarrow M$ and $\delta : M \Rightarrow \mathrm{ran}$ such that for any morphism $f : x \to y$ in $\mathcal{C}$, $\gamma(f) : x \to M(f)$ is in w, $\delta(f) : M(f) \to y$ is a fibration and $\delta(f)\gamma(f) = f$.

Thus, one can establish the dual statement of Proposition 2.1. In particular, since an exact category $\mathcal{E}$ is an additive category with bifibrations, we can apply $\mathcal{E}$ to Proposition 2.1 and its dual statement.

3 Complicial exact category

We recall the theory of complicial exact categories with weak equivalences following [8]. Let $\mathrm{Ch}^b(\mathbb{Z})$ be the exact category of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules. Its exact structure is given by the degree-wise split sequences. There is a symmetric monoidal tensor product $\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathrm{Ch}^b(\mathbb{Z}) \to \mathrm{Ch}^b(\mathbb{Z})$ which extends the usual tensor product of free $\mathbb{Z}$-modules defined by $(a \otimes b)^n := \bigoplus_{i+j=n} a^i \otimes b^j$. Its differential $d^n : (a \otimes b)^n \to (a \otimes b)^{n+1}$ is

$$d_a^i \otimes \mathrm{id}_{b^j} + (-1)^i \,\mathrm{id}_{a_i} \otimes d_b^j : a^i \otimes b^j \to (a^{i+1} \otimes b^j) \oplus (a^i \otimes b^{j+1})$$

on $a^i \otimes b^j \subset (a \otimes b)^n$ $(i + j = n)$. The unit is the chain complex $\mathbb{1}$ which is $\mathbb{Z}$ in degree 0 and 0 elsewhere. The complex C is $\mathbb{Z}$ in degrees 0 and -1 and is 0 otherwise. The only non-trivial differential is $d^{-1} = \mathrm{id}_{\mathbb{Z}}$. The complex T is $\mathbb{Z}$ in degree -1 and 0 elsewhere. Note that we have a conflation of chain complexes $\mathbb{1} \rightarrowtail C \twoheadrightarrow T$.

Definition 3.1 ([8], Def. 6.2)**.** An exact category $\mathcal{E}$ is said to be *complicial* if it is equipped with a bi-exact action $\otimes : \mathrm{Ch}^b(\mathbb{Z}) \times \mathcal{E} \to \mathcal{E}$ of the symmetric monoidal category $\mathrm{Ch}^b(\mathbb{Z})$ on $\mathcal{E}$. For an object $x \in \mathcal{E}$, put $Cx := C \otimes x$ and $Tx := T \otimes x$. A sequence $x \rightarrowtail Cx \twoheadrightarrow Tx$ forms a conflation.

For any morphism $f : x \to y$ in a complicial exact category $\mathcal{E}$, the *cone* $\mathrm{Cone}(f)$ is defined by the push-out of f along the inflation $x \rightarrowtail Cx$. We call $\mathrm{Cyl}(f) := y \oplus Cx$ the *cylinder* of f. These make the following conflations: $y \rightarrowtail \mathrm{Cone}(f) \twoheadrightarrow Tx$, and $x \rightarrowtail \mathrm{Cyl}(f) \twoheadrightarrow \mathrm{Cone}(f)$. We associate a triangulated category $\underline{\mathcal{E}}$ called the *stable category* for a complicial exact category $\mathcal{E}$ as follows: An inflation $i : x \rightarrowtail y$ in $\mathcal{E}$ is called a *Frobenius inflation* if for any object u and a morphism $f : x \to Cu$ in $\mathcal{E}$, there is a morphism

$g : y \to Cu$ such that $f = gi$. A deflation $p : x \twoheadrightarrow y$ in $\mathcal{E}$ is called a *Frobenius deflation* if for any object u and a morphism $f : Cu \to y$ in $\mathcal{E}$, there is a morphism $g : Cu \to Cx$ such that $f = pg$. The category $\mathcal{E}$ endowed with the class of Frobenius inflations and Frobenius deflations is a Frobenius exact category. That is, it has enough projective and injective objects and the class of projective objects and the injective objects coincide. An object x is a projective-injective object if and only if it is a direct summand of Cu for some object u in $\mathcal{E}$ ([8], Lem. B.16). For any morphism $f : x \to y$ in $\mathcal{E}$, $x \rightarrowtail Cx$ is a Frobenius inflation, thus the conflations $x \rightarrowtail Cx \twoheadrightarrow Tx$ and $x \rightarrowtail \mathrm{Cyl}(f) \twoheadrightarrow \mathrm{Cone}(f)$ are Frobenius conflations. From now on, we always consider the complicial exact category $\mathcal{E}$ as the Frobenius category. Recall that two maps $f, g : x \to y$ in $\mathcal{E}$ are called *homotopic*, if their difference factors through a projective-injective object.

Lemma 3.2. *For any object x in $\mathcal{E}$, $Cx = C \otimes x$ is contractible.*

Proof. In $\mathrm{Ch}^b(\mathbb{Z})$, the identity map $C \to C$ factor through CC. Hence Cx is contractible. □

The *stable category* $\underline{\mathcal{E}}$ of the Frobenius category $\mathcal{E}$ is the category whose objects are the objects of $\mathcal{E}$ and whose morphisms are the homotopy classes of maps in $\mathcal{E}$. It is known that the stable category $\underline{\mathcal{E}}$ is a triangulated category. Distinguished triangles in $\underline{\mathcal{E}}$ are those triangles which are isomorphic in $\underline{\mathcal{E}}$ to sequences of the form

$$x \xrightarrow{f} y \to \mathrm{Cone}(f) \to Tx \tag{4}$$

for $f : x \to y$ in $\mathcal{E}$.

Definition 3.3 ([8], Def. 6.9)**.** A class of morphisms w in a complicial exact category $\mathcal{E}$ is called a *class of weak equivalences* if w satisfies the following conditions:

(**WE 0**) The tensor product preserves weak equivalences in both variables, that is, if f is a homotopy equivalence in $\mathrm{Ch}^b(\mathbb{Z})$ and g is in w, then $f \otimes g$ is in w,

(**WE 1**)′ Every homotopy equivalence is in w,

(**WE 2**), (**WE 3**) and (**WE 7**) in the last section, namely, w satisfies the 2 out of 3 and and is closed under extensions and retracts.

We denote the class of all classes of weak equivalences by $\mathbf{WE}(\mathcal{E})$. For a class of weak equivalences w in $\mathcal{E}$, we say that the pair $\mathbf{E} := (\mathcal{E}, w)$ is a *complicial exact category with weak equivalences.*

Every class of weak equivalences w in a complicial exact category $\mathcal{E}$ satisfies (**WE 6**). If fact, for a morphism $f : x \to y$ in $\mathcal{E}$, we have a conflation $x \rightarrowtail \mathrm{Cyl}(f) \twoheadrightarrow \mathrm{Cone}(f)$. By Lemma 3.2, $\mathrm{Cyl}(f) = Cx \oplus y$ and y are homotopy equivalence. Hence (**WE 1**)$'$ implies (**WE 6**). Proposition 2.1 says that w satisfies (**WE 4**) and (**WE 5**). Furthermore, it is easy to verify that w satisfies (**WE 4**)$^{\mathrm{op}}$ - (**WE 6**)$^{\mathrm{op}}$. An object x in $\mathcal{E}$ is said to be *w-trivial* if the canonical map $0 \to x$ is in w. Note that by the axioms (**WE 1**)$'$ and (**WE 2**), this condition is equivalent to the condition that the canonical map $x \to 0$ is in w. We denote the class of w-trivial objects by $\mathcal{E}^w$ and sometimes consider $\mathcal{E}^w$ as the full subcategory of $\mathcal{E}$ of the w-trivial objects.

Lemma 3.4. *A morphism $f : x \to y$ in $\mathcal{E}$ is in w if and only if its mapping cone* $\mathrm{Cone}(f)$ *is in $\mathcal{E}^w$.*

Proof. Let us consider the following diagram:

$$\begin{array}{ccccc}
x & \overset{\alpha(f)}{\rightarrowtail} & \mathrm{Cyl}(f) & \twoheadrightarrow & \mathrm{Cone}(f) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta(f)} & & \downarrow \\
y & \underset{\mathrm{id}_y}{\longrightarrow} & y & \longrightarrow & 0
\end{array} ,$$

where $\beta(f)$ is in w by (**WE 6**). Assume that f is in w. Then applying (**WE 4**) to the diagram above, we learn that $\mathrm{Cone}(f)$ is w-trivial. Next suppose that $\mathrm{Cone}(f)$ is w-trivial. Then applying (**WE 4**)$^{\mathrm{op}}$ to the diagram above, we learn that f is in w. □

Next we introduce a *null class* associated with a class of weak equivalences in a complicial exact categories and establish a bijective correspondence between null classes and classes of weak equivalences in the complicial exact category. This is an analogue of a bijective correspondence between thick subcategories and localizing systems in a triangulated category.

Definition 3.5. An additive full subcategory $\mathcal{N}$ of $\mathcal{E}$ is called a *null class* if it satisfies the following axioms:

(**NC 1**) If x is an object in $\mathcal{N}$ and y is an object in $\mathcal{E}$ which is homotopy equivalent to x, then y is also in $\mathcal{N}$.

(**NC 2**) For $a \in \mathrm{Ch}^b(\mathbb{Z})$ and $x \in \mathcal{N}$, we have $a \otimes x \in \mathcal{N}$.

(**NC 3**) For any conflation $x \rightarrowtail y \twoheadrightarrow z$ in $\mathcal{E}$, if x, z are in $\mathcal{N}$, so is y.

(**NC 7**) If there are maps $i : x \to y$ and $p : y \to x$ with $pi = \mathrm{id}_x$ and $y \in \mathcal{N}$, then $x \in \mathcal{N}$.

We denote the class of all null classes on $\mathcal{E}$ by $\mathbf{NC}(\mathcal{E})$.

For any null class $\mathcal{N}$ on $\mathcal{E}$, by (**NC 3**), it becomes an exact category in the natural way and by (**NC 2**), it is complicial. As in the last section, we can consider the stable category $\underline{\mathcal{N}}$ of $\mathcal{N}$ which is a full triangulated subcategory of $\underline{\mathcal{E}}$

Lemma 3.6. *Let $\mathcal{E}$ be a complicial exact category.*

(i) *For any class of weak equivalences w on $\mathcal{E}$, the class of w-trivial objects $\mathcal{E}^w$ is a null class.*

(ii) *Conversely, for any null class $\mathcal{N}$ on $\mathcal{E}$, let us define the class of morphisms $w_{\mathcal{N}}$ by*

$$w_{\mathcal{N}} := \{ f \in \mathrm{Mor}\,\mathcal{E} \mid \mathrm{Cone}(f) \in \mathcal{N} \}.$$

Then $w_{\mathcal{N}}$ is a class of weak equivalences.

(iii) *For the associations $w \mapsto \mathcal{E}^w$ and $\mathcal{N} \mapsto w_{\mathcal{N}}$ gives a bijective correspondence between $\mathbf{WE}(\mathcal{E})$ and $\mathbf{NC}(\mathcal{E})$.*

Proof. (i) Since 0 is w-trivial, it is in $\mathcal{E}^w$. In particular $\mathcal{E}^w$ is non-empty.

(**NC 1**) Let x be a w-trivial object in $\mathcal{E}$ and $f : x \to y$ is a homotopy equivalent. Then by (**WE 1**)$'$, f is in w. Therefore by (**WE 2**), $y \to 0$ is in w.

(**NC 2**) Let x be a w-trivial object in $\mathcal{E}$ and a an object in $\mathrm{Ch}^b(\mathbb{Z})$. Since the end-functor $a \otimes ?$ on $\mathrm{Ch}^b(\mathbb{Z})$ is additive, there is a canonical isomorphism $0 \xrightarrow{\sim} a \otimes 0$. Since $\otimes$ preserves w (**WE 0**), the canonical morphism $a \otimes x \to a \otimes 0$ is in w. Therefore by (**WE 2**), we learn that $a \otimes x \to 0$ is in w.

(**NC 3**) In the diagram below

$$\begin{array}{ccccc} x & \rightarrowtail & y & \twoheadrightarrow & z \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 \end{array},$$

assume that x and z are in $\mathcal{E}^w$. Then by (**WE 3**), y is also in $\mathcal{E}^w$.

(**NC 7**) Assume that there are maps $i : x \to y$ and $p : y \to x$ with $pi = \mathrm{id}_x$ and $y \in \mathcal{E}^w$. By (**WE 7**), we have $x \in \mathcal{E}^w$.

(ii) (**WE 0**) Let $f : a \to b$ be a homotopy equivalence in $\mathrm{Ch}^b(\mathbb{Z})$ and $g : x \to y$ in $w_{\mathcal{N}}$. Note that $f \otimes \mathrm{id}_y$ is a homotopy equivalence in $\mathcal{E}$. By (**NC 1**), we have $\mathrm{Cone}(f \otimes \mathrm{id}_y)$ in $\mathcal{N}$. From the isomorphism $\mathrm{Cone}(\mathrm{id}_a \otimes g) \overset{\sim}{\to} a \otimes \mathrm{Cone}(g)$ and (**NC 2**), we have $\mathrm{Cone}(\mathrm{id}_a \otimes g) \in \mathcal{N}$. From the equality $f \otimes g = (\mathrm{id}_a \otimes g)(f \otimes \mathrm{id}_y)$, there is a distinguished triangle in $\underline{\mathcal{E}}$

$$\mathrm{Cone}(\mathrm{id}_a \otimes g) \to \mathrm{Cone}(f \otimes g) \to \mathrm{Cone}(f \otimes \mathrm{id}_y) \overset{+1}{\to}$$

by the octahedral axiom. Since $\underline{\mathcal{N}}$ is triangulated, $\mathrm{Cone}(f \otimes g)$ in $\underline{\mathcal{N}}$. Then by (**NC 1**), it is in $\underline{\mathcal{N}}$.

(**WE 1**)$'$ Let $f : x \to y$ be a homotopy equivalence in $\mathcal{E}$. Then $\mathrm{Cone}(f)$ is contractible (that is, $\mathrm{Cone}(f) \to 0$ is a homotopy equivalence). Therefore by (**NC 1**), $\mathrm{Cone}(f)$ is in $\mathcal{N}$.

(**WE 2**) Let us consider morphisms $x \overset{f}{\to} y \overset{g}{\to} z$ in $\mathcal{E}$. Then by the octahedral axiom, we have a distinguished triangle in $\underline{\mathcal{E}}$

$$\mathrm{Cone}(f) \to \mathrm{Cone}(gf) \to \mathrm{Cone}(g) \overset{+1}{\to}.$$

If we assume two of f, g and gf are in $w_{\mathcal{N}}$, then their mapping cones are in $\underline{\mathcal{N}}$. Since $\underline{\mathcal{N}}$ is triangulated, the mapping cone of the third one is also in $\underline{\mathcal{N}}$. Then the assertion follows from (**NC 1**).

(**WE 3**) For the commutative diagram (1)

$$\begin{array}{ccccc} x & \rightarrowtail & y & \twoheadrightarrow & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ x' & \rightarrowtail & y' & \twoheadrightarrow & z' \end{array},$$

in $\mathcal{E}$. We have a conflation in $\mathcal{E}$ $\mathrm{Cone}(a) \rightarrowtail \mathrm{Cone}(b) \twoheadrightarrow \mathrm{Cone}(c)$. If we assume $\mathrm{Cone}(a)$ and $\mathrm{Cone}(c)$ are in $\mathcal{N}$, then by (**NC 3**), $\mathrm{Cone}(b)$ is also in $\mathcal{N}$.

(**WE 7**) For the commutative diagram (3), with $b \in w_{\mathcal{N}}$. We have a retraction $\mathrm{Cone}(a) \to \mathrm{Cone}(b)$ with $\mathrm{Cone}(b) \in \mathcal{N}$. Therefore $\mathrm{Cone}(a) \in \mathcal{N}$ by (**NC 7**).

(iii) For any class of weak equivalences w on $\mathcal{E}$, by Lemma 3.4 we have

$$w_{\mathcal{E}^w} = \{f \in \operatorname{Mor}\mathcal{E} \mid \operatorname{Cone}(f) \in \mathcal{E}^w\} = w.$$

For any null class $\mathcal{N}$ on $\mathcal{E}$, we have $\mathcal{E}^{w_\mathcal{N}} = \{x \in \operatorname{Ob}\mathcal{E} \mid Tx = \operatorname{Cone}(x \to 0) \in \mathcal{N}\} = \mathcal{N}$, where the last equality follows from (**NC 2**). □

Let $\mathbf{E} = (\mathcal{E}, w)$ be a complicial exact category with weak equivalences. Since the w-trivial objects $\mathcal{E}^w$ is also a complicial in $\mathcal{E}$, we have its stable category $\underline{\mathcal{E}}^w$ which is a full triangulated subcategory of $\underline{\mathcal{E}}$. The two Frobenius categories $\mathcal{E}$ and $\mathcal{E}^w$ have the same injective-projective objects. The inclusion $\mathcal{E}^w \subset \mathcal{E}$ induces a fully faithful triangulated functor $\underline{\mathcal{E}}^w \to \underline{\mathcal{E}}$ ([8], 6.15). The triangulated category $\mathcal{T}(\mathbf{E})$ associated with $\mathbf{E}$ is the Verdier quotient $\mathcal{T}(\mathbf{E}) = \underline{\mathcal{E}}/\underline{\mathcal{E}}^w$. The distinguished triangles in $\mathcal{T}(\mathbf{E})$ are triangles which are isomorphic to triangles of the form (4). The canonical projection $\pi : \mathcal{E} \to \mathcal{T}(\mathbf{E})$ from the Frobenius category $\mathcal{E}$ induces an isomorphism $w^{-1}\mathcal{E} \overset{\simeq}{\to} \mathcal{T}(\mathbf{E})$, where $w^{-1}\mathcal{E}$ is obtained from $\mathcal{E}$ by formally inverting the weak equivalences (cf. [8], Def. 6.17). Note that π sends a (Frobenius) conflation to a distinguished triangle in $\mathcal{T}(\mathbf{E})$.

Lemma 3.7 ([8], Exerc. 6.16, 6.18)**.** (i) *$\underline{\mathcal{E}}^w$ is closed under retract in $\underline{\mathcal{E}}$. In particular, objects of $\underline{\mathcal{E}}$ which are isomorphic to object of $\underline{\mathcal{E}}^w$ in $\underline{\mathcal{E}}$ are already in $\underline{\mathcal{E}}^w$.*
(ii) *A morphism $f : x \to y$ in $\mathcal{E}$ is a weak equivalence if and only if $\pi(f)$ is an isomorphism in $\mathcal{T}(\mathbf{E})$.*

Proof. (i) Let a be an object in $\underline{\mathcal{E}}^w$ and x an object in $\underline{\mathcal{E}}$. Assume that there are morphisms $p : a \to x$ and $i : x \to a$ such that $pi = \mathrm{id}_x$ in $\underline{\mathcal{E}}$. Hence we have a morphism $H : Cx \to x$ such that $H\iota_x = pi - \mathrm{id}_x$ in $\mathcal{E}$, where $\iota_x : x \rightarrowtail Cx$ is the inflation. Then there are morphisms

$$x \overset{\left(\begin{smallmatrix} i \\ -\iota_x \end{smallmatrix}\right)}{\to} a \oplus Cx \overset{\left(\begin{smallmatrix} p & H \end{smallmatrix}\right)}{\to} x$$

such that $\left(\begin{smallmatrix} p & H \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ -\iota_x \end{smallmatrix}\right) = pi - H\iota_x = pi - (pi - \mathrm{id}_x) = \mathrm{id}_x$. Since $\mathcal{E}^w$ is closed under homotopy equivalences, we have $a \oplus Cx$ is in $\mathcal{E}^w$ and therefore x is in $\mathcal{E}^w$.
(ii) By Lemma 3.4, f is in w if and only if $\operatorname{Cone}(f)$ is in $\mathcal{E}^w$. Since $\mathcal{E}^w$ is closed under homotopy equivalences, this condition is equivalent to that in $\underline{\mathcal{E}}$, $\operatorname{Cone}(f)$ is in $\underline{\mathcal{E}}^w$. By (i), $\underline{\mathcal{E}}^w$ is thick in $\underline{\mathcal{E}}$. Therefore this condition is

equivalent to that $\pi(\mathrm{Cone}(f))$ is isomorphic to 0 in $\mathcal{T}(\mathbf{E})$. Since there is a distinguished triangle $x \overset{\pi(f)}{\to} y \to \mathrm{Cone}(f) \overset{+1}{\to}$ in $\mathcal{T}(\mathbf{E})$, the final condition is equivalent to that $\pi(f)$ is an isomorphism in $\mathcal{T}(\mathbf{E})$. □

If a full subcategory $\mathcal{N}$ in $\mathcal{E}$ satisfies (**NC 2**) and (**NC 3**), $\mathcal{N}$ is also complicial, and the image of $\mathcal{N}$ in the derived category $\mathcal{T}(\mathbf{E})$ by the canonical projection $\pi : \mathcal{E} \to \mathcal{T}(\mathbf{E})$ is a triangulated subcategory. We define the *w-closure* $\mathcal{N}_w$ of $\mathcal{N}$ by the kernel of the composition $\mathcal{E} \overset{\pi}{\to} \mathcal{T}(\mathbf{E}) \to \mathcal{T}(\mathbf{E})/\pi(\mathcal{N})$; the full subcategory of $\mathcal{E}$ whose objects are isomorphic to 0 in $\mathcal{T}(\mathbf{E})/\pi(\mathcal{N})$.

4 Quasi-weak equivalences

Let $\mathbf{E} = (\mathcal{E}, w)$ be an *idempotent complete* exact category with weak equivalences. We shall define a class of weak equivalences qw in the category of chain complexes $\mathrm{Ch}^{\#}(\mathcal{E})$ ($\# \in \{b, +, -, \emptyset\}$) as follows: The quasi-isomorphisms closure of $\mathrm{Ch}^b(\mathcal{E}^w)$ in $\mathrm{Ch}^{\#}(\mathcal{E})$ is denoted by $\mathrm{Ac}^{\#}_w(\mathcal{E})$ which is a null class in $\mathrm{Ch}^{\#}(\mathcal{E})$. The objects in $\mathrm{Ac}^{\#}_w(\mathcal{E})$ are called *w-acyclic complexes*. The class of weak equivalences associated with the null class $\mathrm{Ac}^{\#}_w(\mathcal{E})$ is denoted by qw (Lem. 3.6) which is called *the class of quasi-weak equivalences*. The category of chain complexes with quasi-weak equivalences $\mathrm{Ch}^{\#}(\mathbf{E}) = (\mathrm{Ch}^{\#}(\mathcal{E}), qw)$ forms a complicial exact category with weak equivalences. The null class $\mathrm{Ac}^{\#}_w(\mathcal{E})$ is the qis-closure of $\mathrm{Ch}^b(\mathcal{E}^w)$. Hence qw contains qis the class of quasi-isomorphisms in $\mathrm{Ch}^{\#}(\mathcal{E})$.

Lemma 4.1. *Assume that $\mathcal{E}$ is complicial.*

(i) *The inclusion functor $i : \mathcal{E} \to \mathrm{Ch}^{\#}(\mathcal{E})$ sends a weak equivalence to a quasi-weak equivalence.*

(ii) *The class qw contains chain maps $f : x \to y$ in $\mathrm{Ch}^b(\mathcal{E})$ which are degree-wise weak equivalences.*

Proof. (i) Let $f : x \to y$ be a weak-equivalence in w. Note that the mapping cone $\mathrm{Cone}(f)$ in $\mathcal{E}$ is in $\mathcal{E}^w$ (Lem. 3.4). Now we have a biCartesian square

(5)
$$\begin{array}{ccc} x & \xrightarrow{a} & Cx \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ y & \xrightarrow{b} & \mathrm{Cone}(f) \end{array} .$$

The mapping cone of $i(f)$ in $\mathrm{Ch}^{\#}(\mathcal{E})$ is $\mathrm{Cone}(i(f)) = \cdots \to 0 \to x \xrightarrow{f} y \to 0 \to \cdots$. On the other hand, we have a complex $z := \cdots \to 0 \to Cx \xrightarrow{p} \mathrm{Cone}\, f \to 0 \to \cdots$ in $\mathrm{Ch}^b(\mathcal{E}^w)$. The above diagram (5) gives a chain map $\phi : \mathrm{Cone}(i(f)) \to z$. Its mapping cone in $\mathrm{Ch}^{\#}(\mathcal{E})$ is an acyclic complex

$$\cdots \to 0 \to x \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to 0 \to \cdots .$$

Hence the chain map ϕ is a quasi-isomorphism and thus we have $\mathrm{Cone}(i(f)) \in \mathrm{Ch}^b(\mathcal{E}^w)_{\mathrm{qis}} = \mathrm{Ch}^{\#}(\mathcal{E})^{qw}$.

(ii) We show the assertion by induction on the length of the complexes x and y. The brutal truncation $\sigma^{\geq k}x$ is $\sigma^{\geq k}x := \cdots \to 0 \to 0 \to x^k \to x^{k+1} \to \cdots$ and put $\sigma^{<k}x := x/\sigma^{\geq k}x$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\sigma^{\geq k}x & \rightarrowtail & x & \twoheadrightarrow & \sigma^{<k}x \\
\downarrow{\scriptstyle \sigma^{\geq k}f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \sigma^{<k}f} \\
\sigma^{\geq k}y & \rightarrowtail & y & \twoheadrightarrow & \sigma^{<k}y
\end{array} .$$

Here, the horizontal sequences are conflations. By the assumption on the induction, we have $\sigma^{\geq k}f$ and $\sigma^{<k}f$ are in qw. Thus the assertion follows from (**WE 3**). $\square$

We have a triangulated category $\mathcal{D}^{\#}(\mathcal{E}) = \mathcal{T}(\mathrm{Ch}^{\#}(\mathcal{E}), \mathrm{qis})$ for $\mathcal{E}$, ($\# \in \{b, +, -, \emptyset\}$) that is the derived category associated with the complicial exact category $\mathrm{Ch}^{\#}(\mathcal{E})$ with quasi-isomorphisms as its weak equivalences. Similarly, when $\mathbf{E} = (\mathcal{E}, w)$ is complicial, we denote by $\mathcal{D}^{\#}(\mathbf{E}) = \mathcal{T}(\mathrm{Ch}^{\#}(\mathcal{E}), qw)$ the *derived category* of $\mathbf{E}$, which is defined by the derived category of the complicial exact category with weak equivalence $\mathrm{Ch}^{\#}(\mathbf{E}) := (\mathrm{Ch}^{\#}(\mathcal{E}), qw)$. The non-connective K-theory spectrum $\mathbb{K}(\mathbf{E})$ is defined in [8]. Recall that the associated K-groups in positive degree are agree with Waldhausen K-groups $K_i(\mathbf{E})$ and $\mathbb{K}_0(\mathbf{E}) = K_0(\mathcal{T}(\mathbf{E})^{\sim})$, where $\mathcal{T}(\mathbf{E})^{\sim}$ is the idempotent completion of $\mathcal{T}(\mathbf{E})$. Note also $\mathcal{T}(\mathbf{E})^{\sim}$ becomes a triangulated category ([1]).

Theorem 4.2. *Assume that $\mathbf{E} = (\mathcal{E}, w)$ is complicial. Then we have the following isomorphisms for each $i \in \mathbb{Z}$:*
(i) $\mathbb{K}_i(\mathbf{E}) \stackrel{\sim}{\to} \mathbb{K}_i(\mathrm{Ch}^b(\mathbf{E}))$.
(ii) $\mathbb{K}_i(\mathrm{Ch}^-(\mathbf{E})) = \mathbb{K}_i(\mathrm{Ch}^+(\mathbf{E})) = 0$,
(iii) $\mathbb{K}_i(\mathbf{E}) \stackrel{\sim}{\to} \mathbb{K}_{i+1}(\mathrm{Ch}(\mathbf{E}))$.

Proof. (i) The complicial exact categories $\mathbf{E}$ and $\mathrm{Ch}^b(\mathbf{E})$ admit (**WE 6**). Hence they satisfy the *factorization axiom*, namely, any morphism is a composition of a cofibration followed by a weak equivalence ([7], Appendix). By the very definition of the quasi-weak equivalences, we have $\mathrm{Ac}^b_w(\mathcal{E}) = \mathrm{Ch}^b(\mathcal{E})^{qw}$ the w-acyclic objects in $\mathrm{Ch}^b(\mathcal{E})$ as null classes. Hence we have the following diagram:

$$\begin{array}{ccccc}
\mathbb{K}(\mathcal{E}^w) & \longrightarrow & \mathbb{K}(\mathcal{E}) & \longrightarrow & \mathbb{K}(\mathbf{E}) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
\mathbb{K}(\mathrm{Ac}^b_w(\mathcal{E}), \mathrm{qis}) & \longrightarrow & \mathbb{K}(\mathrm{Ch}^b(\mathcal{E}), \mathrm{qis}) & \longrightarrow & \mathbb{K}(\mathrm{Ch}^b(\mathbf{E})).
\end{array}$$

Here, the horizontal sequences are homotopy fibration of spectra by the fibration theorem ([7], Thm. 11) and the vertical map h is induced from the inclusion map $\mathcal{E} \to \mathrm{Ch}^b(\mathcal{E})$ (Lem. 4.1). The vertical map g is a homotopy equivalence by the Gillet-Waldhausen theorem ([7]). Hence, it is enough to show the map f is a homotopy equivalence. We have $\mathbb{K}(\mathcal{E}^w) \overset{\sim}{\to} \mathbb{K}(\mathrm{Ch}^b(\mathcal{E}^w), \mathrm{qis})$ from the Gillet-Waldhausen theorem again. Recall that $\mathrm{Ac}^b_w(\mathcal{E}) = \mathrm{Ch}^b(\mathcal{E}^w)_{\mathrm{qis}}$ is the qis-closure of $\mathrm{Ch}^b(\mathcal{E}^w)$. We have $\mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{E}^w) \overset{\sim}{\to} \mathcal{D}^b(\mathcal{E})/\mathcal{T}(\mathrm{Ch}^b(\mathcal{E}^w)_{\mathrm{qis}}, \mathrm{qis})$. By comparing the localization sequences, $\mathbb{K}(\mathrm{Ch}^b(\mathcal{E}^w), \mathrm{qis}) \overset{\sim}{\to} \mathbb{K}(\mathrm{Ac}^b_w(\mathcal{E}), \mathrm{qis})$ and the assertion follows from it.

(ii) Let us consider the endofunctor $F = \bigoplus_{n \in \mathbb{N}} [2n]$ on $\mathrm{Ch}^+(\mathcal{E})$. From the identities $F[2] \oplus \mathrm{id} \overset{\sim}{\to} F$ and $\mathbb{K}(F[2]) = \mathbb{K}(F)$, we notice that $\mathbb{K}(\mathrm{id}) = 0$ by the additivity theorem.

(iii) The truncation functor is defined by $\tau^{\geq k} x := \cdots \to 0 \to \mathrm{Im}(\partial^{k-1}) \to x^k \to x^{k+1} \to \cdots$, for any $x \in \mathrm{Ch}(\mathcal{E})$, where $\partial^{k-1} : x^{k-1} \to x^k$ is the differential map. The kernel of the quotient map $x \to \tau^{\geq k} x$ is denoted by $\tau^{<k} x$. For any $x \in \mathrm{Ch}^b(\mathcal{E}) \cap \mathrm{Ch}^+(\mathcal{E})^{qw}$, we have quasi-isomorphisms $x \overset{\sim}{\leftarrow} y \overset{\sim}{\to} z$ with $y \in \mathrm{Ch}^+(\mathcal{E})$ and $z \in \mathrm{Ch}^b(\mathcal{E}^w)$. Since $\mathcal{E}$ is idempotent complete, the canonical map $\tau^{<k} y \to y$ becomes a quasi-isomorphism for some k ([1], Lem. 2.6). Hence, we have $\mathrm{Ch}^b(\mathcal{E})^{qw} = \mathrm{Ch}^b(\mathcal{E}) \cap \mathrm{Ch}^+(\mathcal{E})^{qw}$. On the other hand, for any $x \in \mathrm{Ch}^b(\mathcal{E}) \cap \mathrm{Ch}^-(\mathcal{E})^{qw}$, we have quasi-isomorphisms $x \overset{\sim}{\to} y \overset{\sim}{\leftarrow} z$ with $y \in \mathrm{Ch}^-(\mathcal{E})$ and $z \in \mathrm{Ch}^b(\mathcal{E}^w)$. The canonical map $y \to \tau^{\geq k} y$ becomes a quasi-isomorphism and thus $\mathrm{Ch}^b(\mathcal{E})^{qw} = \mathrm{Ch}^b(\mathcal{E}) \cap \mathrm{Ch}^-(\mathcal{E})^{qw}$. Similarly, we have $\mathrm{Ch}^+(\mathcal{E})^{qw} = \mathrm{Ch}^+(\mathcal{E}) \cap \mathrm{Ch}(\mathcal{E})^{qw}$, and $\mathrm{Ch}^-(\mathcal{E})^{qw} = \mathrm{Ch}^-(\mathcal{E}) \cap \mathrm{Ch}(\mathcal{E})^{qw}$. We have the square of fully faithful inclusions which induces on category

equivalences on the quotient ([7], Proof of Lem. 7):

$$\begin{array}{ccc} \mathcal{D}^b(\mathcal{E}) & \longrightarrow & \mathcal{D}^+(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{D}^-(\mathcal{E}) & \longrightarrow & \mathcal{D}(\mathcal{E}) \end{array} .$$

The diagram extends to

$$\begin{array}{ccc} \mathcal{D}^b(\mathbf{E}) & \longrightarrow & \mathcal{D}^+(\mathbf{E}) \\ \downarrow & & \downarrow \\ \mathcal{D}^-(\mathbf{E}) & \longrightarrow & \mathcal{D}(\mathbf{E}) \end{array} . \tag{6}$$

Here, all functors are fully faithful (Lem. 4.3 below) and the induced functors on quotients are equivalences. Thus the assertion follows from the localization theorem and (i) and (ii). □

Lemma 4.3 ([5], Lem. 10.3)**.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ and $\mathcal{N}$ full triangulated subcategories of $\mathcal{T}$. If each morphism $x \to y$ in $\mathcal{T}$ with $x \in \mathcal{N}$ and $y \in \mathcal{S}$ factors through some object in $\mathcal{S} \cap \mathcal{N}$, then the canonical functor $\mathcal{S}/\mathcal{S} \cap \mathcal{N} \to \mathcal{T}/\mathcal{N}$ is fully faithful.*

Let us denote n-th times iteration of $\mathrm{Ch}^{\#}$ for $\mathbf{E}$ by $\mathrm{Ch}_n^{\#}(\mathbf{E})$ and $\mathcal{D}_n^{\#}(\mathbf{E}) := \mathcal{T}(\mathrm{Ch}_n^{\#}(\mathbf{E}))$ the *n-th higher derived category* of $\mathbf{E}$. As the following corollary, we can consider the negative K-groups as obstruction group of idempotent completeness of the higher derived categories:

Corollary 4.4. *We assume that $\mathcal{E}$ is complicial or w is just the class of isomorphisms in $\mathcal{E}$. For any positive integer n, we have*
(i) $\mathbb{K}_{-n}(\mathbf{E}) \simeq \mathbb{K}_0(\mathcal{D}_n(\mathbf{E}))$.
(ii) $\mathbb{K}_{-n}(\mathbf{E})$ *is trivial if and only if* $\mathcal{D}_n(\mathbf{E})$ *is idempotent complete.*

Proof. Suppose that $\mathbf{E} = (\mathcal{E}, w)$ is complicial. By Theorem 4.2 (iii), we have $\mathbb{K}_{-n}(\mathbf{E}) \simeq \mathbb{K}_0(\mathrm{Ch}_n(\mathbf{E})) \simeq \mathbb{K}_0(\mathcal{D}_n(\mathbf{E}))$. Then Proposition 4.5 below leads the desired assertion. If $\mathbf{E} = \mathcal{E}$ is an exact category; it may not be complical, but $w =$ the class of isomorphisms). From the Gillet-Waldhausen and Lemma 7 and Corollary 6 in [7] as already refered in Introduction, we have $\mathbb{K}_{-n}(\mathcal{E}) \overset{\simeq}{\to} \mathbb{K}_{-n}(\mathrm{Ch}^b(\mathcal{E}), \mathrm{qis}) \overset{\simeq}{\to} \mathbb{K}_{-n+1}(\mathrm{Ch}(\mathcal{E}), \mathrm{qis})$. Hence, one reduce to the case of complicial. □

Proposition 4.5. (i) *For an essentially small triangulated category $\mathcal{T}$, if $\mathbb{K}_0(\mathcal{T}) = K_0(\mathcal{T}^\sim)$ is trivial, then $\mathcal{T}$ is idempotent complete.*

(ii) *The derived category $\mathcal{D}(\mathbf{E})$ is idempotent complete if and only if the Grothendieck group $\mathbb{K}_0(\mathcal{D}(\mathbf{E})) = K_0(\mathcal{D}(\mathbf{E})^\sim)$ is trivial.*

Proof. (i) Since the map $K_0(\mathcal{T}) \to K_0(\mathcal{T}^\sim)$ is injective by [9] Corollary 2.3, now $K_0(\mathcal{T})$ is also trivial. Applying the Thomason classification theorem of (strictly) dense triangulated subcategories in essentially small triangulated categories [9] Theorem 2.1 for $\mathcal{T}^\sim$, the inclusion functor $\mathcal{T} \to \mathcal{T}^\sim$ must be an equivalence.

(ii) From the diagram (6) in the proof of Theorem 4.2, we have a surjection $0 = K_0(\mathcal{D}^+(\mathbf{E})) \oplus K_0(\mathcal{D}^-(\mathbf{E})) \to K_0(\mathcal{D}(\mathbf{E}))$. Therefore $K_0(\mathcal{D}(\mathbf{E})) = 0$. If $\mathcal{D}(\mathbf{E})$ is idempotent complete, that is, $\mathcal{D}(\mathbf{E}) \overset{\sim}{\to} \mathcal{D}(\mathbf{E})^\sim$, then we have $\mathbb{K}_0(\mathcal{D}(\mathbf{E})) = K_0(\mathcal{D}(\mathbf{E})^\sim) = K_0(\mathcal{D}(\mathbf{E})) = 0$. The converse is followed from (i). □

References

[1] P. Balmer and M. Schlichting, *Idempotent completion of triangulated categories*, J. Algebra **236** (2001), 819–834.

[2] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

[3] T. Gunnarsson, *Abstract homotopy theory and related topics*, thesis (1978).

[4] WC. Hsiang, *Geometric applications of algebraic K-theory*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), PWN, 1984, pp. 99–118.

[5] B. Keller, *Derived categories and their uses*, Handbook of algebra, Vol. 1, North-Holland, Amsterdam, 1996, pp. 671–701.

[6] M. Schlichting, *Delooping the K-theory of exact categories*, Topology **43** (2004), 1089–1103.

[7] ———, *Negative K-theory of derived categories*, Math. Z. **253** (2006), 97–134.

[8] ______, *Higher algebraic K-theory (after Quillen, Thomason, and others)*, preprint available at
http://www.math.lsu.edu/~mschlich/research/prelim.html
(2009).

[9] R. W. Thomason, *The classification of triangulated subcategories*, Compositio Math. **105** (1997), 1–27.

[10] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[11] C. A. Weibel, *K-theory and analytic isomorphisms*, Invent. Math. **61** (1980), 177–197.

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