TRIVIALITY OF THE DRESSING ISOTROPY FOR A SMYTH-TYPE POTENTIAL AND NONCLOSING OF THE RESULTING CMC SURFACES

JOSEF F. DORFMEISTER AND WAYNE ROSSMAN

1. Introduction

In [2] we have investigated spacelike surfaces in Minkowski 3-space which are generated by the potential

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} dz \, .$$

This was motivated by the quantum cohomology of $\mathbb{C}P^1$. It turns out that this potential yields, via the loop group method [4], spacelike CMC surfaces in Minkowski 3-space, for which the metric is invariant under a 1-parameter family of isometries of the domain. In $\mathbb{R}^3$, Delaunay surfaces and Smyth surfaces are known to have this property. Moreover, these surfaces can be defined on a (punctured) disk, i.e. the immersion closes around the fixed point of the rotations under which the metric is invariant (and, in the Smyth case, is even defined at the fixed point). We were therefore interested in finding out whether among the immersions generated by the potential ξ above (which is different from the Delaunay potentials and the Smyth potentials) there are immersions closing around $z = 0$. In this note we prove, as already partially announced in [2], that in every integrable surface class for which the potential ξ above makes sense[1], there do not exist any closing immersions (Theorem 4.1). The conclusion comes quite easily from the fact that the isotropy group of the dressing action is trivial (Theorem 3.1). The proof of the latter statement covers most of this note.

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2. The DPW method

Let G denote any of the real Lie groups $\mathrm{SU}(2)$, $\mathrm{SU}(1,1)$ or $\mathrm{SL}_2\,\mathbb{R}$. Clearly, G is a real form of $Sl_2\mathbb{C}$. Let $\Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ denote the twisted loop group of smooth maps from the unit circle $\mathbb{S}^1$ to $\mathrm{SL}_2\,\mathbb{C}$, where "twisted" means that the two diagonal elements of the image of the map are even functions of $\lambda \in \mathbb{S}^1$ and the two off-diagonal elements are odd functions of λ. Let ΛG_σ denote the subgroup of maps from $\mathbb{S}^1$ to G. Let $\Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, resp. $\Lambda_+^{\mathbb{R}}\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, denote the subgroup of unnormalized, resp. normalized, positive loop groups – in other words, $B \in \Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, resp. $B \in \Lambda_+^{\mathbb{R}}\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, if B

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[1]Following Kobayashi [6], we are only interested in the almost compact cases, and hence we are interested in the involutions C1, C2, C3 and C4 in that paper. However, we omit the case C4, since there we have a situation completely different from the other three. The groups under consideration for the cases C1, C2, C3 are $\mathrm{SU}(2)$, $\mathrm{SU}(1,1)$, and $\mathrm{SL}_*(2,\mathbb{R})$ (isomorphic to $\mathrm{SL}_2\,\mathbb{R}$ under conjugation by the diagonal matrix with diagonal entries $\sqrt{i}^{-1}$ and $\sqrt{i}$), respectively. The Gauss maps, respectively, go into the symmetric spaces S^2 equal to $\mathrm{SU}(2)$ modulo diagonal matrices, H^2 equal to $\mathrm{SU}(1,1)$ modulo diagonal matrices, and $S^{1,1}$ equal to $\mathrm{SL}_*(2,R)$ modulo diagonal matrices. Therefore, harmonic maps into these spaces all have normalized potentials which are arbitrary meromorphic and off-diagonal.

can be extended smoothly to a map defined on the unit disk (with boundary $\mathbb{S}^1$) and the diagonal matrix $B|_{\lambda=0}$ does not necessarily, resp. does necessarily, have positive reals on the diagonal.

At this point, the DPW method requires an Iwasawa splitting relative to ΛG_σ. It turns out that in the case $G = \mathrm{SU}(2)$ every element $g(\lambda)$ in $\Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ can be written in the form $g = FB$ with $F \in \Lambda SU(2)_\sigma$ and $B \in \Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma$. In the other two cases, however, this is not true. More precisely, in the case of $G = \mathrm{SL}_2\,\mathbb{R}$ there is a cell that is open and dense in $\Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ and on which the Iwasawa splittings exist, but which cannot be all of $\Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$. In the case of $G = \mathrm{SU}(1,1)$ there are two open cells, $\mathcal{B}_1 = \Lambda G_\sigma \cdot \Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ and $\mathcal{B}_2 = \Lambda G_\sigma \cdot \omega \cdot \Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ with $\omega = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$, and the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ is open and dense in $\Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, so that Iwasawa splittings exist for any element of $\mathcal{B}_1 \cup \mathcal{B}_2$ [5], [1]. With this notation the Iwasawa splitting of an element $L \in \mathcal{B}_1 \subset \Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, respectively $L \in \mathcal{B}_2 \subset \Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$, is

$$L = F \cdot B \;, \quad \text{for some } \; F \in \Lambda\,\mathrm{SU}(1,1)_\sigma \; \text{ and } \; B \in \Lambda_+^{\mathbb{R}}\,\mathrm{SL}_2\,\mathbb{C}_\sigma \;,$$

respectively,

$$L = F \cdot \omega \cdot B \;, \quad \text{for some } \; F \in \Lambda\,\mathrm{SU}(1,1)_\sigma \; \text{ and } \; B \in \Lambda_+^{\mathbb{R}}\,\mathrm{SL}_2\,\mathbb{C}_\sigma \;.$$

Now, the DPW method can start with an equation of the form

$$dL = L \cdot \xi \;, \quad \xi = \lambda^{-1} \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix} dz$$

defined on some domain of $\mathbb{C}$ over which g and h are holomorphic and g is nonzero. One then Iwasawa splits a solution L into $L = FB$, or $F\omega B$ and inserts F or $\omega^{-1}F\omega$ into the Sym-Bobenko formula. In the cases $G = \mathrm{SU}(2)$ and $G = \mathrm{SU}(1,1)$ this formula is

$$f = \frac{-i}{2H}\left[F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^{-1} - 2\lambda(\partial_\lambda F)F^{-1} \right]$$

to obtain (for every $\lambda \in S^1$) a conformal spacelike CMC $H \neq 0$ immersion f into Euclidean 3-space $\mathbb{R}^3$ and Minkowski 3-space $\mathbb{L}^3$ respectively.

In the case $G = \mathrm{SL}_2\,\mathbb{R}$, the Sym formula

$$f = -i \left[\lambda(\partial_\lambda F)F^{-1} \right]$$

yields a conformal timelike constant negative Gauß curvature $K < 0$ immersion f into $\mathbb{L}^3$. See [6].

To obtain the Smyth surfaces in $\mathbb{R}^3$, we can take the domain to be all of $\mathbb{C}$ and the potential to be

$$\xi_k = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^k & 0 \end{pmatrix} dz \;, \quad k \in \mathbb{Z} \;, \quad k \geq 0 \;,$$

where $c \in \mathbb{C} \setminus \{0\}$, and we can take the initial condition for the solution L to be $L|_{z=0} = I$.

The case we will consider in this note is

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^{-1} & 0 \end{pmatrix} dz$$

where $c \in \mathbb{C} \setminus \{0\}$, and now we cannot specify any initial condition at $z = 0$.

3. The result on dressing isotropy

As pointed out in the introduction, we consider all integrable surface classes for which the potential ξ makes sense. In the classification of [6] these are the CMC surfaces in $\mathbb{R}^3$, spacelike CMC surfaces in Minkowsi 3-space $\mathbb{L}^3$ and the timelike surfaces of constant negative Gauß curvature in $\mathbb{L}^3$. The group G in each case is $\mathrm{SU}(2)$, $\mathrm{SU}(1,1)$ and $\mathrm{SL}_2\,\mathbb{R}$, respectively. However, for the claim and the proof of Theorem 3.1 this is of no importance. It is only important to note that our potential is contained in the complexification $p^{\mathbb{C}}$, where $g = k + p$ (g denotes the Lie algebra in any of the three cases, and k is the diagonal part and p is the off-diagonal part) is the Cartan decomposition corresponding to the target space of the Gauß map. In all three cases $p^{\mathbb{C}}$ consists of all 2×2 off-diagonal matrices with complex entries.

Theorem 3.1. *Let L be any solution of $dL = L\xi$ for the potential*

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ cz^{-1} & 0 \end{pmatrix} dz,$$

where $c \in \mathbb{C} \setminus \{0\}$. Then the isotropy group of L relative to the dressing action is $\{\pm I\}$.

We now consider how to prove this theorem. Solving $dL = L\xi$, where ξ is as in the theorem and $c \in \mathbb{C} \setminus \{0\}$, we have solutions of the form

$$L = \begin{pmatrix} X' & \lambda^{-1}X \\ \lambda Y' & Y \end{pmatrix} ,$$

and X and Y satisfy

$$zX'' - \lambda^{-2}cX = 0 \ , \quad zY'' - \lambda^{-2}cY = 0 \ . \tag{1}$$

The Frobenius method leads us to one particular solution

$$\tilde{L} = \hat{L} \cdot P \ , \tag{2}$$

where

$$\hat{L} = \begin{pmatrix} 1 & 0 \\ \lambda^{-1}c\log z & 1 \end{pmatrix} = e^{\log z \cdot D} \ , \quad D = \begin{pmatrix} 0 & 0 \\ \lambda^{-1}c & 0 \end{pmatrix} ,$$

and, for appropriate constants η_{ij},

$$P = \begin{pmatrix} 1 & 0 \\ -\lambda\eta_{2,1} - \lambda^{-1}c & 1 \end{pmatrix} \cdot \begin{pmatrix} \sum_{j=0}^{\infty}(j+1)\,\eta_{1,j}z^j & \lambda^{-1}z\sum_{j=0}^{\infty}\eta_{1,j}z^j \\ \lambda\left\{\sum_{j=1}^{\infty} j\eta_{2,j}z^{j-1} + \lambda^{-2}c\sum_{j=0}^{\infty}\eta_{1,j}z^j\right\} & \sum_{j=0}^{\infty}\eta_{2,j}z^j \end{pmatrix} .$$

Note that in this solution we can assume $\eta_{1,0} = \eta_{2,0} = 1$, i.e. that $\lim_{z\to 0} P = I$.

Let us consider the isotropy group of $\tilde{L}$:

Definition 3.2. *An element $h \in \Lambda\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ is in the* isotropy group *of $\tilde{L}$ if there exists a possibly z-dependent function $W_+ \in \Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma$ so that*

$$h\tilde{L} = \tilde{L}W_+ \ .$$

Lemma 3.3. *If h is in the isotropy group of $\tilde{L}$, then*

$$h \in \Lambda_+\,\mathrm{SL}_2\,\mathbb{C}_\sigma \ , \tag{3}$$

and $\lim_{\lambda\to 0} h = \pm I$, which implies either h or $-h$ lies in $\Lambda_+^{\mathbb{R}}\,\mathrm{SL}_2\,\mathbb{C}_\sigma$.

Proof. If $h = (h_{ij})_{i,j=1}^2$ is in the isotropy group, then $\tilde{L}^{-1} h \tilde{L} = P^{-1} e^{-\log z \cdot D} h e^{\log z \cdot D} P \in \Lambda_+ \mathrm{SL}_2 \mathbb{C}_\sigma$. We have

$$\tilde{L}^{-1} h \tilde{L} = \begin{pmatrix} h_{11} + \frac{ch_{12}}{\lambda} \log z & h_{12} \\ h_{21} - \frac{c(h_{11}-h_{22})}{\lambda} \log z - \frac{c^2 h_{12}^2}{\lambda^2} (\log z)^2 & h_{22} - \frac{ch_{12}}{\lambda} \log z \end{pmatrix} + \mathcal{O} \, ,$$

where $\mathcal{O}$ denotes terms converging to 0 as $z \to 0$. This matrix lies in $\Lambda_+ \mathrm{SL}_2 \mathbb{C}_\sigma$. Considering the $(1,2)$-entry we observe that $h_{12} \in \mathcal{A}_+$, the algebra of positive Wiener functions. Hence we have the expansion $h_{12} = h_{12,1}\lambda^1 + h_{12,3}\lambda^3 + h_{12,5}\lambda^5 + \cdots$ for certain constants $h_{12,k}$.

Substituting this into the diagonal terms of the matrix above we see that h_{11} and h_{22} do not contain any negative powers of λ in their Fourier expansions. Moreover, the terms independent of λ are $h_{11,0} + ch_{12,1}\log(z) + \mathcal{O}$ and $h_{22,0} - ch_{12,1}\log(z) + \mathcal{O}$. Since the matrix under consideration has determinant equal to 1 and is in $\Lambda_+ \mathrm{SL}_2 \mathbb{C}_\sigma$, we obtain $(h_{11,0} + ch_{12,1}\log(z) + \mathcal{O})(h_{22,0} - ch_{12,1}\log(z) + \mathcal{O}) = 1$. Hence $h_{12,1} = 0$.

Substituting this into the $(2,1)$-entry of the matrix above we infer that the third term is of order 4 in λ, while the term $(c(h_{11,0} - h_{22,0})/\lambda)\log(z)$ cannot be cancelled by $h_{21,-1}$, since the latter does not depend on z. Hence $h_{11,0} = h_{22,0}$ and h_{21} does not contain any negative powers of λ. In particular, we have shown that h is in $\Lambda_+ \mathrm{SL}_2 \mathbb{C}_\sigma$. Moreover, using the last equality and the determinant being 1, we obtain that the λ-independent summand of h is $\pm I$. This proves the claim. $\square$

The next result closely follows arguments from [3]:

Lemma 3.4. *Suppose h is in the isotropy group of $\tilde{L}$, and so there exists a W_+ as in Definition 3.2. If the upper-right entry of W_+ is identically zero, then $h = I$ or $h = -I$.*

Proof. We note that the constant c in ξ can be removed by some constant gauge and a coordinate transformation. We therefore assume that $c = 1$. We have $h\tilde{L} = \tilde{L}W_+$, and we write the components of W_+ as (now the notation "c" plays a different role)

$$W_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Then

$$W_+ \cdot \lambda^{-1} \begin{pmatrix} 0 & 1 \\ \frac{1}{z} & 0 \end{pmatrix} = W_+ \cdot \tilde{L}^{-1} \partial_z \tilde{L} = W_+ \cdot (h\tilde{L})^{-1} \partial_z (h\tilde{L}) =$$

$$W_+ \cdot (\tilde{L}W_+)^{-1} \partial_z (\tilde{L}W_+) = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ \frac{1}{z} & 0 \end{pmatrix} W_+ + \partial_z W_+ .$$

Thus

$$\lambda \partial_z a = -\lambda \partial_z d = \frac{b}{z} - c \tag{4}$$

and

$$\lambda \partial_z b = -\lambda z \partial_z c = a - d \tag{5}$$

hold. Now if $b = 0$, then Equations (4) and (5) give that $a = d = \pm 1$ and $c = 0$. Therefore, $W_+ = \pm I$ and so $h = \pm I$. $\square$

We now give a proof of Theorem 3.1.

Proof. It suffices to prove the result for one particular solution, as different solutions give conjugate isotropy groups, so let us take the solution $\tilde{L}$ as given above. We give a proof by contradiction.

Suppose h is in the isotropy group, but not in $\{\pm I\}$. The previous two lemmas imply we may assume that $h \in \Lambda_+ \operatorname{SL}_2 \mathbb{C}_\sigma$ and that b is not identically zero for the matrix

$$W_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponding to h. Equations (4) and (5) imply that

$$\frac{\lambda^2}{2}\partial_z^3 b + \frac{1}{z^2} b - \frac{2}{z}\partial_z b = 0 \tag{6}$$

holds. Now we consider the expansion $b = \sum_{n=0}^{\infty} b_n(z)\lambda^n$ and choose $N \in \mathbb{Z}_+ \cup \{0\}$ so that $b_n = 0$ for all $n < N$ and $b_N \neq 0$. Then Equation (6) implies

$$2z\partial_z b_N = b_N$$

and thus $b_N = c_1\sqrt{z}$ for some constant c_1. However, $W_+ = \tilde{L}^{-1} h \tilde{L}$, and $\tilde{L}^{-1} h \tilde{L}$ is comprized of only products and sums of holomorphic functions and $\log z$ and $(\log z)^2$, so in particular $b_N = c_1\sqrt{z} = f_1(z) + f_2(z)\log z + f_3(z)(\log z)^2$ for functions f_j that are holomorphic at $z = 0$. That this is a contradiction can be seen as follows: Letting τ be the deck transformation associated to a counterclockwise loop about the origin in $\mathbb{C} \setminus \{0\}$, and applying τ twice, $\sqrt{z}$ is invariant, and so

$$f_1 + f_2 \cdot \log z + f_3 \cdot (\log z)^2 = f_1 + f_2 \cdot (\log z + 4\pi i) + f_3 \cdot (\log z + 4\pi i)^2 \ .$$

Thus

$$f_2 + 4\pi i f_3 + 2f_3 \cdot \log z = 0 \ .$$

However, $\log z$ is not holomorphic at $z = 0$, so f_3 must be identically zero, and then f_2 must be zero as well. Thus $\sqrt{z} = f_1$, but $\sqrt{z}$ is also not holomorphic at $z = 0$, providing the contradiction. □

4. Nonclosing of the resulting surfaces

In this section we prove a result about the corresponding surfaces. We recall that the potential ξ considered throughout produces surfaces of three different types via the DPW method.

Theorem 4.1. *For the potential*

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^{-1} & 0 \end{pmatrix} dz,$$

where $c \in \mathbb{C} \setminus 0$, no resulting immersion of any of the three integrable surface types can be well-defined on any annular domain Σ in $\mathbb{C} \setminus \{0\}$ with nontrivial winding order about $z = 0$, for any value of the associated spectral parameter in $\mathbb{S}^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$.

To prove this theorem, we again assume without loss of generality that $c = 1$ (as in the proof of Lemma 3.4), and we will suppose there exists a solution $L = C\tilde{L}$ defined on $\tilde{\Sigma}$ (the universal cover of Σ), for some $C \in \Lambda \operatorname{SL}_2 \mathbb{C}_\sigma$, of $dL = L\xi$ with Iwasawa splitting (with respect to G)

$$L = F \cdot B$$

so that the frame F produces, at some $\lambda = \lambda_0 \in \mathbb{S}^1$, a well-defined CMC or CGC immersion on an annular region Σ with nontrivial winding number about $z = 0$, and then find a contradiction.

Remark 4.2. *If the ambient space is $\mathbb{R}^3$, then the Iwasawa splitting for $\operatorname{SU}(2)$ is global* [4]*, and so the domain of definition of the surface can be extended from Σ to all of $\mathbb{C} \setminus \{0\}$. In the other two cases (surfaces in $\mathbb{L}^3$), however, the Iwasawa splitting is not global and one can expect to encounter singularities on the surface, equivalently one can expect to leave the region where Iwasawa splittings exist, as one extends Σ to larger domains within $\mathbb{C} \setminus \{0\}$ (see* [1]*). For this reason the restriction to annular regions Σ smaller than $\mathbb{C} \setminus \{0\}$ is necessary in Theorem 4.1.*

Remark 4.3. *In the case of $G = \mathrm{SU}(1,1)$, the splitting can take two possible forms: either $L = FB$ or $L = F\omega B$, where ω is as in Section 2, see* [1], [2]. *However, if we find that L satisfies the second form, we can replace L with $\omega^{-1}L$ (this changes the resulting surface only by a rigid motion) and $\omega^{-1}F\omega$ with F to switch over to the first form. So without loss of generality we may assume the first form.*

Now, C can be Iwasawa decomposed into parts $C_u \in \Lambda G_\sigma$ and $C_+ \in \Lambda_+^{\mathbb{R}} \mathrm{SL}_2 \mathbb{C}_\sigma$, i.e.

$$C = C_u \cdot C_+ \quad \text{or} \quad C = C_u\omega \cdot C_+ \ .$$

The C_u or $C_u\omega$ part only moves the resulting surface in $\mathbb{R}^3$ or $\mathbb{L}^3$ by a rigid motion, so we can take C to be in $\Lambda_+^{\mathbb{R}} \mathrm{SL}_2 \mathbb{C}_\sigma$ without loss of generality (the possibility of $C_u\omega$ occurs only in the $G = \mathrm{SU}(1,1)$ case.).

Let τ be the deck transformation associated to a counterclockwise loop about the origin in Σ. Let $\mathcal{M}_L$, $\mathcal{M}_{\tilde{L}}$ ($\tilde{L}$ as in Equation (2)) and $\mathcal{M}_F$ be the monodromies of L, $\tilde{L}$ and F, respectively, with respect to the deck transformation τ. That is to say, under the deck transformation τ, we have the following transformations:

$$L \to \tau^* L = \mathcal{M}_L \cdot L \ , \quad \tilde{L} \to \tau^* \tilde{L} = \mathcal{M}_{\tilde{L}} \cdot \tilde{L} \ , \quad F \to \tau^* F = \mathcal{M}_F \cdot F \ .$$

Because $\tau^* P = P$, we have

$$\mathcal{M}_{\tilde{L}} = \hat{L}\left(\tau(1), \lambda\right) \cdot \left(\hat{L}(1,\lambda)\right)^{-1} = \hat{L}\left(\tau(1), \lambda\right) = \begin{pmatrix} 1 & 0 \\ 2\pi i \lambda^{-1} & 1 \end{pmatrix} \ . \tag{7}$$

The monodromy $\mathcal{M}_L$ is conjugate to $\mathcal{M}_{\tilde{L}}$ under conjugation by C, i.e.

$$\mathcal{M}_L = C \cdot \mathcal{M}_{\tilde{\mathcal{L}}} \cdot C^{-1} \ .$$

Remark 4.4. *Note that $\mathcal{M}_F$ is a monodromy because the resulting immersion is assumed to be well-defined on $\mathbb{C} \setminus \{0\}$, that is, $\mathcal{M}_F$ does not depend on z. In other words, we have the following statement: Given a CMC surface well-defined on Σ, the Maurer-Cartan form $F^{-1}dF$ of its extended frame F is invariant under τ.*

Lemma 4.5. *$\mathcal{M}_F = \pm\mathcal{M}_L$. In particular, $\mathcal{M}_L$ lies in ΛG_σ. Moreover, $\tau^* B = \pm B$.*

Proof. $\tau^* L = \mathcal{M}_L \cdot L = \tau^* F \cdot \tau^* B = \mathcal{M}_F \cdot L B^{-1} \cdot \tau^* B$ implies $\mathcal{M}_F^{-1} \mathcal{M}_L \cdot L = LW$, where $W = B^{-1} \cdot \tau^* B$ is a positive loop. By Theorem 3.1, the isotropy group is $\{\pm I\}$, implying the lemma. □

Note that

$$\tau^* f = \mathcal{M}_L f \mathcal{M}_L^{-1} + H^{-1} \partial_t \mathcal{M}_L \cdot \mathcal{M}_L^{-1}$$

for $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1)$, and for $\lambda = e^{it}$. The term $\mathcal{M}_L f \mathcal{M}_L^{-1}$ represents a rotation of f, and the term $\partial_t \mathcal{M}_L \cdot \mathcal{M}_L^{-1}$ represents a translation. Thus if $\tau^* f = f$ for $\lambda = \lambda_0 = e^{it_0}$, we have

$$\mathcal{M}_L|_{\lambda=\lambda_0} = \pm I \ , \quad \partial_t \mathcal{M}_L|_{\lambda=\lambda_0} = 0 \ . \tag{8}$$

When $G = \mathrm{SL}_2 \mathbb{R}$, we have

$$\tau^* f = \mathcal{M}_L f \mathcal{M}_L^{-1} - \partial_t \mathcal{M}_L \cdot \mathcal{M}_L^{-1} \ .$$

We finally prove Theorem 4.1.

Proof. By way of contradiction, assume there exists a solution L that gives such a surface for the case of $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1)$. Then $\mathcal{M}_{\tilde{L}}$ as in (7) implies $\mathcal{M}_L = C \cdot \mathcal{M}_{\tilde{L}} \cdot C^{-1}$ is never $\pm I$ at any λ_0. But if $f|_{\lambda_0}$ closes to be well-defined on an annulus, then Equation (8) implies that $\mathcal{M}_L$ must be $\pm I$ at λ_0. This contradiction proves the theorem for these two cases.

In the third case $G = \mathrm{SL}_2 \mathbb{R}$, $\mathcal{M}_{\tilde{L}}$ as in (7) is conjugated by an element $C \in \Lambda_+^{\mathbb{R}} \mathrm{SL}_2 \mathbb{C}_\sigma$ to $\mathcal{M}_F \in \Lambda G_\sigma$, which is a contradiction. □

5. A remark on the more general potentials ξ_k

Finally, in this section we remark on the behavior that results with the more general potential

$$\xi_k = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^k & 0 \end{pmatrix} dz \ , \quad k \in \mathbb{Z} \ , \quad c \in \mathbb{C} \setminus \{0\} \ .$$

We will see that any value of k other than -1 can produce CMC surfaces in $\mathbb{R}^3$ that close on annular domains in $\mathbb{C} \setminus \{0\}$ with nontrivial winding order about $z = 0$. In the case of the symmetric spaces G/K with $G = \mathrm{SU}(1, 1)$ or $G = \mathrm{SL}_2 \mathbb{R}$, and K the subgroup of diagonal matrices in G, the question addressed in Theorem 5.1 seems to be more complicated and shall not be discussed here.

Theorem 5.1. *Up to gauge transformations and admissible coordinate changes of the resulting CMC surfaces in $\mathbb{R}^3$, the potential ξ_k can be taken so that $k \geq -2$. Furthermore, amongst the cases $k \in Z \cap (-2, \infty)$, any case other than $k = -1$ will produce CMC surfaces that close on annular domains in $\mathbb{C} \setminus \{0\}$ with nontrivial winding order about $z = 0$.*

Proof. Choose a solution L to $L^{-1}dL = \xi$ and a gauge $p_+ \in \Lambda_+ \mathrm{SL}_2 \mathbb{C}$, where p_+ is allowed to depend on z. Changing L to $\tilde{L} = Lp_+$ gives a solution to $\tilde{L}^{-1}d\tilde{L} = \tilde{\xi}$, where

$$\tilde{\xi} = p_+^{-1}\xi p_+ + p_+^{-1}dp_+ \ . \tag{9}$$

Thus the holomorphic potentials ξ and $\tilde{\xi}$ make the same collection of surfaces via the DPW method.

Applying the transformation $z \to \frac{1}{z}$, ξ changes to

$$\lambda^{-1} \begin{pmatrix} 0 & -z^{-2} \\ -cz^{-k-2} & 0 \end{pmatrix} dz \ , \tag{10}$$

and gauging this resulting potential with

$$p_+ = \begin{pmatrix} iz^{-1} & 0 \\ -i\lambda & -iz \end{pmatrix} \in \mathrm{SL}_2 \mathbb{C} \ ,$$

we get the potential

$$p_+^{-1}\xi p_+ + p_+^{-1}dp_+ = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^{-k-4} & 0 \end{pmatrix} dz \ . \tag{11}$$

This implies that the cases k and $-k-4$ produce the same surfaces. Also, by gauging and coordinate changes, we can see that we may assume $c > 0$. So without loss of generality, we can restrict to

$$c > 0$$

and

$$k \geq -2 \ .$$

For $k \geq 0$, Smyth surfaces can be produced, which of course are well-defined immersions on annular regions with nontrivial winding order about $z = 0$. (And in fact, they can extend to $z = 0$.)

The situation for ξ_{-1} is already established in Theorem 4.1.

For ξ_{-2}, gauging with

$$p_{+,1} = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda}{2z} & 1 \end{pmatrix} \in \mathrm{SL}_2 \mathbb{C} \ ,$$

and then

$$p_{+,2} = \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \frac{1}{\sqrt{z}} \end{pmatrix} \in \mathrm{SL}_2 \mathbb{C} \ ,$$

and then by an appropriate constant diagonal $p_{+,3}$ gives the potential

$$\sqrt{c}\begin{pmatrix} 0 & \lambda^{-1} \\ \lambda^{-1}+\frac{\lambda}{4c} & 0 \end{pmatrix}\frac{dz}{z}\ .$$

Further gauging by

$$p_{+,4}=\begin{pmatrix} \Omega^{3/4} & \frac{\lambda}{2\sqrt{c}}\Omega^{-1/4} \\ \frac{\lambda}{2\sqrt{c}}\Omega^{1/4} & \Omega^{1/4} \end{pmatrix}^{-1}\ ,\quad \Omega = 1+\frac{\lambda^2}{4c}\ ,$$

and then by

$$p_{+,5}=\exp\left(\lambda^{-1}\sqrt{c}(1-\sqrt{\Omega})\log z\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

shows that we may assume the potential is

$$\sqrt{c}\lambda^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{dz}{z}\ ,$$

and we can choose the CMC surface to be a cylinder in $\mathbb{R}^3$, which can of course close to become annular.

This completes the proof. □

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