CLASSIFICATION OF NON-SYMPLECTIC AUTOMORPHISMS ON $K3$ SURFACES WHICH ACT TRIVIALLY ON THE NÉRON-SEVERI LATTICE.

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ABSTRACT. We treat non-symplectic automorphisms on $K3$ surfaces which act trivially on the Néron-Severi lattice. In this paper, we classify non-symplectic automorphisms of prime-power order, especially 2-power order on $K3$ surfaces, i.e., we describe their fixed locus.

1. INTRODUCTION

Let X be a $K3$ surface. In the following, we denote by S_X, T_X and ω_X the Néron-Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form on X, respectively.

An automorphism of X is *symplectic* if it acts trivially on $\mathbb{C}\omega_X$. This paper is devoted to study of *non*-symplectic automorphisms of prime-power order for which act trivially on S_X. The study of non-symplectic automorphisms of $K3$ surfaces was pioneered by V.V. Nikulin.

We suppose that g is a non-symplectic automorphism of order I on X such that $g^*\omega_X = \zeta_I\omega_X$ where ζ_I is a primitive I-th root of unity. Then g^* has no non-zero fixed vectors in $T_X \otimes \mathbb{Q}$ and hence $\phi(I)$ divides $\operatorname{rank} T_X$, where ϕ is the Euler function. In particular $\phi(I) \leq \operatorname{rank} T_X$ and hence $I \leq 66$ [9, Theorem 3.1 and Corollary 3.2].

The following proposition was announced by Vorontsov [18] and then it was proved by Kondo [6].

Proposition 1.1. Let φ be a non-symplectic automorphism on X which acts trivially on S_X. Then the order of φ is prime-power; $p^k = 2^\alpha$ $(1 \leq \alpha \leq 4)$, 3^β $(1 \leq \beta \leq 3)$, 5^γ $(1 \leq \gamma \leq 2)$, 7, 11, 13, 17 or 19. Moreover S_X is a p-elementary lattice, that is, S_X^*/S_X is a p-elementary group where $S_X^* = \operatorname{Hom}(S_X, \mathbb{Z})$.

Non-symplectic automorphisms of prime order have been studied by several authors e.g. Nikulin [11], Oguiso, Zhang [12], [13], Artebani,

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Sarti [1] and Taki [16]. Recently, we have the classification of non-symplectic automorphisms of prime order on $K3$ surfaces [2].

Theorem 1.2. We assume that S_X is p-elementary. Let r be the Picard number of X and let a be the minimal number of generators of S_X^*/S_X.

Then there exists a non-symplectic automorphism φ of order p on X if and only if $22 - r - (p-1)a \in 2(p-1)\mathbb{Z}_{\geq 0}$.

Moreover if X has a non-symplectic automorphism φ of order p which acts trivially on S_X. then the fixed locus $X^\varphi := \{x \in X | \varphi(x) = x\}$ has the form

$$X^\varphi = \begin{cases} \phi & \text{if } S_X = U(2) \oplus E_8(2), \\ C^{(1)} \amalg C^{(1)} & \text{if } S_X = U \oplus E_8(2), \\ \{P_1, \dots, P_M\} \amalg C^{(g)} \amalg E_1 \amalg \cdots \amalg E_N & \text{otherwise,} \end{cases}$$

and

$$g = \frac{22 - r - (p-1)a}{2(p-1)},$$

$$M = \begin{cases} 0 & \text{if } p = 2, \\ \dfrac{(p-2)r + 22}{p-1} & \text{if } p = 17, 19, \\ \dfrac{(p-2)r - 2}{p-1} & \text{otherwise,} \end{cases}$$

$$N = \begin{cases} \dfrac{r - s}{2} & \text{if } p = 2, \\ 0 & \text{if } p = 17, 19, \\ \dfrac{2 + r - (p-1)a}{2(p-1)} & \text{otherwise,} \end{cases}$$

where P_j is an isolated point, $C^{(g)}$ is a non-singular curve with genus g and E_k is a non-singular rational curve.

On the other hand, studies of prime power order have progressed, too. Schütt [15] classified $K3$ surfaces with non-symplectic automorphisms whose the order is 2-power and equals $\operatorname{rank} T_X$. Machida and Oguiso [8] or Oguiso and Zhang [12] have proved that the $K3$ surface with non-symplectic automorphisms of order 25 or 27, respectively, is unique. Recently, Taki [17] classified non-symplectic automorphisms of 3-power order. The following theorem is known.

Theorem 1.3. (1) X has a non-symplectic automorphism φ of order 9 acting trivially on S_X if and only if $S_X = U \oplus A_2$, $U \oplus E_8$,

$U \oplus E_6 \oplus A_2$ or $U \oplus E_8 \oplus E_6$. Moreover the fixed locus X^φ has the form

$$X^\varphi = \begin{cases} \{P_1, P_2, \ldots, P_6\} & \text{if } S_X = U \oplus A_2, \\ \{P_1, P_2, \ldots, P_{10}\} \amalg E_1 & \text{if } S_X = U \oplus E_8 \text{ or } U \oplus E_6 \oplus A_2, \\ \{P_1, P_2, \ldots, P_{14}\} \amalg E_1 \amalg E_2 & \text{if } S_X = U \oplus E_8 \oplus E_6. \end{cases}$$

(2) X has a non-symplectic automorphism φ of order 27 acting trivially on S_X if and only if $S_X = U \oplus A_2$. Moreover the fixed locus X^φ has the form $X^\varphi = \{P_1, P_2, \ldots, P_6\}$.

Here we denote by P_i an isolated point and by E_j a non-singular rational curve.

By Proposition 1.1, if the order of a non-symplectic automorphism is non-prime-power then S_X is unimodular. The cases are studied by Kondo [6].

Theorem 1.4. Let φ be a non-symplectic automorphism on X and ϕ the Euler function.

(1) If $S_X = U$, then $\operatorname{ord}\varphi | 66$, 44 or 12.
(2) If $S_X = U \oplus E_8$, then $\operatorname{ord}\varphi | 42$, 36 or 28.
(3) If $S_X = U \oplus E_8^{\oplus 2}$, then $\operatorname{ord}\varphi | 12$.
(4) If $\phi(\varphi) = \operatorname{rank} T_X$, then $\operatorname{ord}\varphi = 66$, 44, 42, 36, 28 or 12. Moreover for $m = 66$, 44, 42, 36, 28 or 12, there exists a unique (up to isomorophisms) $K3$ surface with $\operatorname{ord}\varphi = m$.

Hence, in order to classify non-symplectic automorphisms on X which act trivially on S_X, we need the complete classification of non-symplectic automorphisms of 2-power order, i.e., generalization of Schütt's result. The main purpose of this paper is to prove the following theorem.

Main Theorem. We assume that S_X is 2-elementary.

(1) X has a non-symplectic automorphism φ of order 4 acting trivially on S_X if and only if S_X has $\delta = 0$ and $S_X \neq U \oplus E_8(2)$, $U(2) \oplus E_8(2)$, $U \oplus D_4^{\oplus 3}$ and $U \oplus D_8^{\oplus 2}$. Moreover the fixed locus X^φ has the form

$$X^\varphi = \begin{cases} \{P_1, P_2, \ldots, P_4\} & \text{if } \operatorname{rank} S_X = 2, \\ \{P_1, P_2, \ldots, P_6\} \amalg E_1 & \text{if } \operatorname{rank} S_X = 6, \\ \{P_1, P_2, \ldots, P_8\} \amalg E_1 \amalg E_2 & \text{if } \operatorname{rank} S_X = 10, \\ \{P_1, P_2, \ldots, P_{10}\} \amalg E_1 \amalg E_2 \amalg E_3 & \text{if } \operatorname{rank} S_X = 14, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg E_1 \amalg E_2 \amalg E_3 \amalg E_4 & \text{if } \operatorname{rank} S_X = 18. \end{cases}$$

(2) X has a non-symplectic automorphism φ of order 8 acting trivially on S_X if and only if $S_X = U \oplus D_4$, $U(2) \oplus D_4$ or $U \oplus D_4 \oplus E_8$. Moreover the fixed locus X^φ has the form

$$X^\varphi = \begin{cases} \{P_1, P_2, \ldots, P_6\} \amalg E_1 & \text{if rank } S_X = 6, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg E_1 \amalg E_2 & \text{if rank } S_X = 14. \end{cases}$$

(3) X has a non-symplectic automorphism φ of order 16 acting trivially on S_X if and only if $S_X = U \oplus D_4$ or $U \oplus D_4 \oplus E_8$. Moreover the fixed locus X^φ has the form

$$X^\varphi = \begin{cases} \{P_1, P_2, \ldots, P_6\} \amalg E_1 & \text{if } S_X = U \oplus D_4, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg E_1 \amalg E_2 & \text{if } S_X = U \oplus D_4 \oplus E_8. \end{cases}$$

Here, P_i is an isolated point and E_j is a non-singular rational curve.

We summarize the contents of this paper. In Section 2, we review the classification of even indefinite 2-elementary lattices. And we check that the non-existence of lattice isometries of order 4. As a result, we get the Néron-Severi lattice of $K3$ surfaces with non-symplectic automorphisms of order 4, 8 or 16 which act trivially on S_X. Section 3 is a preliminary section. We recall some basic results about non-symplectic automorphisms on $K3$ surfaces. Section 4 is the main part of this paper. Here, we classify non-symplectic automorphisms of order 4. By using the Lefschetz formula and the classification of non-symplectic involution, we study fixed locus of non-symplectic automorphisms of order 4. In Section 5 and Section 6, we treat non-symplectic automorphisms of order 8 and 16, respectively. In Section 7 we collect examples of $K3$ surface with a non-symplectic automorphism of 2-power order.

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2. The Néron-Severi and p-elementary lattices

A lattice L is a free abelian group of finite rank r equipped with a non-degenerate symmetric bilinear form, which will be denoted by $\langle\ ,\ \rangle$. The bilinear form $\langle\ ,\ \rangle$ determines a canonical embedding $L \subset L^* = \mathrm{Hom}(L, \mathbb{Z})$. We denote by A_L the factor group L^*/L which is a finite abelian group. $L(m)$ is the lattice whose bilinear form is the one on L multiplied by m.

We denote by U the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature $(1, 1)$, and by A_m, D_n or E_l

an even negative definite lattice associated with the Dynkin diagram of type A_m, D_n or E_l ($m \geq 1$, $n \geq 4$ and $l = 6, 7, 8$).

Let p be a prime number. A lattice L is called *p-elementary* if $A_L \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$, where a is the minimal number of generator of A_L. For a p-elementary lattice we always have the inequality $a \leq r$, since $\mid L^*/L \mid = p^a$, $\mid L^*/pL^* \mid = p^r$ and $pL^* \subset L \subset L^*$.

Example 2.1. For all p, lattices E_8, $E_8(p)$, U and $U(p)$ are p-elementary. A_1, D_4, D_8 and E_7 are 2-elementary.

Definition 2.2. For a 2-elementary lattice L, we put

$$\delta_L = \begin{cases} 0 & \text{if } x^2 \in \mathbb{Z}, \forall x \in L^*, \\ 1 & \text{otherwise.} \end{cases}$$

Even indefinite 2-elementary lattices were classified by [10, Theorm 3.6.2].

Theorem 2.3. An even indefinite 2-elementary lattice L is determined by the invariants (δ_L, t_+, t_-, a) where the pair (t_+, t_-) is the signature of L.

By the Theorem, we can get the Néron-Severi lattice of $K3$ surfaces with a non-symplectic automorphism of order 2^k acting trivially on S_X. See Table [11, Table 1].

If $k \geq 2$ then $\phi(2^k)$ is even. Since $\phi(2^k)$ divides $\operatorname{rank} T_X$, $\operatorname{rank} T_X$ is even. Hence if X has a non-symplectic automorphisms of 2-power order then $\operatorname{rank} S_X$ is even. Moreover we have the following.

Proposition 2.4. Let L be a 2-elementary lattice. If $\delta_L = 1$ then L has no non-trivial isometries f of order 4 which act trivially on A_L and do not have eigenvalues 1 or -1.

Proof. Let $f : L \to L$ be an isometry of order 4 which acts trivially on A_L and does not have eigenvalues 1 or -1. Since the induced isometry $A_L \to A_L$ ($\bar{x} \mapsto \overline{f^*(x)}$) is identity, for all $x \in L^*$, there exists an $l \in L$ such that $f^*(x) = x + l$.

By the assumption, we have $f^* + f^{*3} = 0$. This implies $0 = \langle f^*(x) + f^{*3}(x), x\rangle = \langle f^*(x), x\rangle + \langle f^{*3}(x), x\rangle = 2\langle f^*(x), x\rangle = 2(\langle x, x\rangle + \langle l, x\rangle)$. Thus we have $\langle x, x\rangle = -\langle l, x\rangle \in \mathbb{Z}$. Hence $\delta_L = 0$. □

The following tables are lists of 2-elementary lattices with and $\delta = 0$. Hence if X has a non-symplectic automorphisms of order 4, 8 or 16 which act trivially on S_X then S_X is one of the lattices in the following table. (See also Lemma 3.1 (1).)

rank S_X	a	S_X	T_X
2	0	U	$U^{\oplus 2} \oplus E_8^{\oplus 2}$
2	2	$U(2)$	$U \oplus U(2) \oplus E_8^{\oplus 2}$
6	2	$U \oplus D_4$	$U^{\oplus 2} \oplus E_8 \oplus D_4$
6	4	$U(2) \oplus D_4$	$U(2)^{\oplus 2} \oplus E_8 \oplus D_4$
10	0	$U \oplus E_8$	$U^{\oplus 2} \oplus E_8$
10	2	$U \oplus D_8$	$U^{\oplus 2} \oplus D_8$
10	4	$U \oplus D_4^{\oplus 2}$	$U^{\oplus 2} \oplus D_4^{\oplus 2}$
10	6	$U(2) \oplus D_4^{\oplus 2}$	$U \oplus U(2) \oplus D_4^{\oplus 2}$
10	8	$U \oplus E_8(2)$	$U^{\oplus 2} \oplus E_8(2)$
10	10	$U(2) \oplus E_8(2)$	$U \oplus U(2) \oplus E_8(2)$
14	2	$U \oplus E_8 \oplus D_4$	$U^{\oplus 2} \oplus D_4$
14	4	$U \oplus D_8 \oplus D_4$	$U \oplus U(2) \oplus D_4$
14	6	$U \oplus D_4^{\oplus 3}$	$U(2)^{\oplus 2} \oplus D_4$
18	0	$U \oplus E_8^{\oplus 2}$	$U^{\oplus 2}$
18	2	$U \oplus E_8 \oplus D_8$	$U \oplus U(2)$
18	4	$U \oplus D_8^{\oplus 2}$	$U(2)^{\oplus 2}$

Table 1: 2-elementary lattices

Remark 2.5. Let $\{e, f\}$ be a basis of U (resp. $U(2)$) with $\langle e, e\rangle = \langle f, f\rangle = 0$ and $\langle e, f\rangle = 1$ (resp. $\langle e, f\rangle = 2$) . If necessary replacing e by $\varphi(e)$, where φ is a composition of reflections induced from non-singular rational curves on X, we may assume that e is represented by the class of an elliptic curve F and the linear system $|F|$ defines an elliptic fibration $\pi : X \to \mathbb{P}^1$. Note that π has a section $f - e$ in case U. In case $U(2)$, there are no (-2)-vectors r with $\langle r, e\rangle = 1$, and hence π has no sections.

It follows from Remark 2.5 and Table 1 that X has an elliptic fibration $\pi : X \to \mathbb{P}^1$. In the following, we fix such an elliptic fibration.

The following lemma follows from [14, §3 Corollary 3] and the classification of singular fibers of elliptic fibrations [5].

Lemma 2.6. Assume that $S_X = U(m) \oplus K_1 \oplus \cdots \oplus K_r$, where $m = 1$ or 2, and K_i is a lattice isomorphic to A_m, D_n or E_l. Then π has a reducible singular fiber with corresponding Dynkin diagram K_i.

3. Preliminaries

Lemma 3.1. Let φ be a non-symplectic automorphism of 2-power order on X. Then we have :

(1) $\varphi^* \mid T_X \otimes \mathbb{C}$ can be diagonalized as:

$$\begin{pmatrix} \zeta I_q & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \zeta^3 I_q & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \zeta^n I_q & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \zeta^{2k-1} I_q \end{pmatrix},$$

where I_q is the identity matrix of size q, ζ is a primitive 2^k-th root of unity, n is a odd number.

(2) Let P be an isolated fixed point of φ on X. Then φ^* can be written as

$$\begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix} \qquad (i + j \equiv 1 \mod 2^k)$$

under some appropriate local coordinates around P.

(3) Let C be an irreducible curve in X^φ and Q a point on C. Then φ^* can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

under some appropriate local coordinates around Q. In particular, fixed curves are non-singular.

Proof. (1) This follows form [9, Theorem 3.1].

(2), (3) Since φ^* acts on $H^0(X, \Omega_X^2)$ as a multiplication by ζ, it acts on the tangent space of a fixed point as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix}$$

where $i + j \equiv 1 \pmod{2^k}$. □

Thus the fixed locus of φ consists of disjoint union of non-singular curves and isolated points. Hence we can express the irreducible decomposition of X^φ as

$$X^\varphi = \{P_1, \dots, P_M\} \amalg C_1 \amalg \cdots \amalg C_N,$$

where P_j is an isolated point and C_k is a non-singular curve.

In the following, we assume that $k \geq 2$. Hence we treat non-symplectic automorphisms of order 4, 8 and 16.

Lemma 3.2. Let r be the Picard number of X. Then $\chi(X^\varphi) = r + 2$.

Proof. We apply the topological Lefschetz formula:

$$\chi(X^{\varphi}) = \sum_{i=0}^{4}(-1)^i \mathrm{tr}(\varphi^*|H^i(X,\mathbb{R})).$$

Since φ^* acts trivially on S_X, $\mathrm{tr}(\varphi^*|S_X) = r$. By Lemma 3.1 (1), $\mathrm{tr}(\varphi^*|T_X) = q(\zeta+\zeta^3+\cdots+\zeta^n+\cdots+\zeta^{2k-1}) = -q(1+\zeta^2+\cdots+\zeta^{2k-2}) = 0$. Hence we can calculate the right -hand side of the Lefschetz formula as follows: $\sum_{i=0}^{4}(-1)^i \mathrm{tr}(\varphi^*|H^i(X,\mathbb{R})) = 1 - 0 + \mathrm{tr}(\varphi^*|S_X) + \mathrm{tr}(\varphi^*|T_X) - 0 + 1 = r + 2$. □

4. Order 4

We shall study the fixed locus of non-symplectic automorphisms of order 4. In this section, let φ be a non-symplectic automorphism of order 4.

Proposition 4.1. Let r be the Picard number of X. Then the number of isolated points M is $(r+6)/2$.

Proof. First we calculate the holomorphic Lefschetz number $L(\varphi)$ in two ways as in [3, page 542] and [4, page 567]. That is

$$L(\varphi) = \sum_{i=0}^{2} \mathrm{tr}(\varphi^*|H^i(X,\mathcal{O}_X)),$$

$$L(\varphi) = \sum_{j=1}^{M} a(P_j) + \sum_{l=1}^{N} b(C_l).$$

Here

$$a(P_j) := \frac{1}{\det(1-\varphi^*|T_{P_j})} = \frac{1}{\det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\zeta^2 & 0\\ 0 & \zeta^3\end{pmatrix}\right)},$$

$$b(C_l) := \frac{1-g(C_l)}{1-\zeta} - \frac{\zeta C_l^2}{(1-\zeta)^2},$$

where T_{P_j} is the tangent space of X at P_j, $g(C_l)$ is the genus of C_l.

Using the Serre duality $H^2(X,\mathcal{O}_X) \simeq H^0(X,\mathcal{O}_X(K_X))^{\vee}$, we calculate from the first formula that $L(\varphi) = 1+\zeta^3$. From the second formula, we obtain

$$L(\varphi) = \frac{M}{(1-\zeta^2)(1-\zeta^3)} + \sum_{l=1}^{N} \frac{(1+\zeta)(1-g(C_l))}{(1-\zeta)^2}.$$

Combing these two formulae, we have $M = 4 + \sum_{l=1}^{N}(2 - 2g(C_l)$. By $\chi(X^{\varphi}) = M + \sum_{l=1}^{N}(2 - 2g(C_l))$ and Lemma 3.2, we have $M = (r + 6)/2$. □

Proposition 4.2. If $S_X = U \oplus E_8(2)$, $U(2) \oplus E_8(2)$, $U \oplus D_4^{\oplus 3}$ or $U \oplus D_8^{\oplus 2}$ then X has no non-symplectic automorphisms of order 4 which act trivially on S_X.

Proof. We will check the statement for each S_X individually.

We assume $S_X = U \oplus E_8(2)$ or $U(2) \oplus E_8(2)$. If X has a non-symplectic automorphism φ of order 4 which acts trivially on S_X then X^{φ} contains non-singular rational curves by Lemma 3.2 and Proposition 4.1. Although these curves are fixed by φ^2, it is a contradiction by Theorem 1.2. This settles Proposition 4.2 in cases $S_X = U \oplus E_8(2)$ and $U(2) \oplus E_8(2)$.

We assume $S_X = U \oplus D_4^{\oplus 3}$ and X has a non-symplectic automorphism φ of order 4 which acts trivially on S_X. Then $X^{\varphi^2} = C^{(1)} \amalg E_1 \amalg \cdots \amalg E_4$ by Theorem 1.2.

Since φ acts trivially on S_X, φ preserves reducible singular fibers of an elliptic fibration π. Hence the automorphism φ acts trivially on the base of π and the section (c.f. Remark 2.5) is fixed by φ. By Lemma 2.6, π has three singular fibers of type I_0^*. The component with multiplicity 2 is pointwisely fixed by φ. Hence X^{φ} contains at least four non-singular rational curves.

On the other hand $\chi(C^{(g)} \amalg E_1 \amalg \cdots \amalg E_N) = 16 - 10 = 6$ by Lemma 3.2 and Proposition 4.1. Thus X^{φ} contains non-singular curve $C^{(g)}$ with $g \geq 2$. But this is a contradiction because X^{φ^2} does not contain $C^{(2)}$. This settles Proposition 4.2 in cases $S_X = U \oplus D_4^{\oplus 3}$.

By [15, Theorem 1], X with $S_X = U \oplus D_8^{\oplus 2}$ has no non-symplectic automorphisms of order 4 . □

In other cases of Table 1, there exist $K3$ surfaces with a non-symplectic automorphism of order 4. See Section 7.

Proposition 4.3. Assume S_X is 2-elementary and $\delta = 0$. If $S_X \neq U \oplus E_8(2)$, $U(2) \oplus E_8(2)$, $U \oplus D_4^{\oplus 3}$ or $U \oplus D_8^{\oplus 2}$ then X^{φ} has the form

$$X^{\varphi} = \begin{cases} \{P_1, P_2, \ldots, P_4\} & \text{if } \operatorname{rank} S_X = 2, \\ \{P_1, P_2, \ldots, P_6\} \amalg E_1 & \text{if } \operatorname{rank} S_X = 6, \\ \{P_1, P_2, \ldots, P_8\} \amalg E_1 \amalg E_2 & \text{if } \operatorname{rank} S_X = 10, \\ \{P_1, P_2, \ldots, P_{10}\} \amalg E_1 \amalg E_2 \amalg E_3 & \text{if } \operatorname{rank} S_X = 14, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg E_1 \amalg E_2 \amalg E_3 \amalg E_4 & \text{if } \operatorname{rank} S_X = 18. \end{cases}$$

Proof. We will check the form of X^{φ} for each S_X individually.

Assume $S_X = U$. By Theorem 1.2, $X^{\varphi^2} = C^{(10)} \amalg E_1$. If X^{φ} contains a non-singular rational curve E_2 or a non-singular curve $C^{(1)}$ then E_2 or $C^{(1)}$ are also contained X^{φ^2}. This is a contradiction. Thus X^{φ} contains at most one non-singular rational curve and no non-singular curves with genus 1. We remark that $\chi(C^{(g)} \amalg E_1 \amalg \cdots \amalg E_N) = 4-4 = 0$ by Lemma 3.2 and Proposition 4.1. If X^{φ} contains E_1 then X^{φ} contains a non-singular curve $C^{(2)}$. But this is a contradiction because X^{φ^2} does not contain $C^{(2)}$. Hence $X^{\varphi} = \{P_1, P_2, \ldots, P_4\}$. This settles Proposition 4.3 in the case $S_X = U$.

Assume $S_X = U \oplus E_8 \oplus D_4$. Then $X^{\varphi^2} = C^{(3)} \amalg E_1 \amalg \cdots \amalg E_6$ by Theorem 1.2. We remark that $\chi(C^{(g)} \amalg E_1 \amalg \cdots \amalg E_N) = 16 - 10 = 6$ by Lemma 3.2 and Proposition 4.1. If X^{φ} contains $C^{(3)}$ then $X^{\varphi} = \{P_1, P_2, \ldots, P_{10}\} \amalg C^{(3)} \amalg E_1 \amalg \cdots \amalg E_5$. Since E_6 is not fixed by φ, isolated fixed points P_i lie on E_6. But this is a contradiction because a non-singular rational curve has exactly two fixed points. Hence $X^{\varphi} = \{P_1, P_2, \ldots, P_{10}\} \amalg E_1 \amalg E_2 \amalg E_3$. This settles Proposition 4.3 in the case $S_X = U \oplus E_8 \oplus D_4$.

In the other case we can check the claim by similar arguments. □

5. Order 8

In this section, let φ be a non-symplectic automorphism of order 8. And we shall describe $X^{\varphi} = \{P_1, \ldots, P_M\} \amalg C^{(g)} \amalg E_1 \amalg \cdots \amalg E_N$.

Proposition 5.1. Let r be the Picard number of X. Then the number of isolated points M is $(3r + 6)/4$.

Proof. By the holomorphic Lefschetz formulae, we have

$$(\sharp) \qquad \begin{cases} 0 &= 2m_{3,6} - m_{4,5} - \sum_{l=1}^{N}(2 - 2g(C_l)), \\ 2 &= m_{2,7} - m_{3,6} + m_{4,5} - \sum_{l=1}^{N}(2 - 2g(C_l)). \end{cases}$$

We remark that $\varphi^2(P^{u,v})$ is a fixed point of a non-symplectic automorphism of order 4. It is easy to see that $\varphi^2(P^{2,7})$ and $\varphi^2(P^{3,6})$ are isolated fixed points of φ^2. By proposition 4.1 and Lemma 5.2, we have

$$(1) \qquad m_{2,7} + m_{3,6} = \frac{r+6}{2}.$$

By $(\sharp)$, (1) and Lemma 3.2, we have $M = (3r + 6)/4$. □

Lemma 5.2. Let P be an isolated fixed point of φ^2. Then $\varphi(P) = P$.

Proof. Let $m \neq 0$ be the number of such P. Then m satisfies $m_{2,7} + m_{3,6} + m = (r + 6)/2$. By the equation and $(\sharp)$, we have $m_{2,7} =$

$(r+14)/4-3m/2$, $m_{3,6}=(r-2)/4+m/2$, $m_{4,5}=(r-6)/4+3m/2$ and $\sum_{l=1}^{N}(2-2g(C_l))=(r+2)/4-m/2$.

Since $m_{2,7}+m_{3,6}$ is even by ($\sharp$), m is even, $m_{2,7}$ and $m_{3,6}$ are odd. Hence we have $m\le(r+6)/2-1-1=(r+2)/2$. By the parity of $m_{2,7}$, $m_{3,6}$ and $m_{4,5}$, if $r=2$, 10 and 18 (resp. 6 and 14) then $m=2\times$ odd number (resp. $2\times$ even number).

Assume $r=10$. Then $m=2$ or 6. If $m=6$ then $m_{2,7}=6-9<0$. This is a contradiction. If $m=2$ then $m_{4,5}=4$ and $\sum_{l=1}^{N}(2-2g(C_l))=2$. Since $\varphi^2(P^{4,5})$ is a point on a irreducible fixed curve by φ^2, these two equations imply that φ^2 has 3 fixed non-singular rational curves. This is a contradiction by Proposition 4.3. This settles Lemma 5.2 in the case $r=10$.

In other cases we can check the claim by similar the argument. $\square$

Remark 5.3. $m_{2,7}=(r+14)/4$, $m_{3,6}=(r-2)/4$, $m_{4,5}=(r-6)/4$.

Corollary 5.4. If X has a non-symplectic automorphism of order 8 then $\operatorname{rank} S_X=6$ or 14.

Proof. If $\operatorname{rank} S_X=2$, 10 or 18 then M is odd by Proposition 5.1. But $\chi(X^\varphi)=M+\sum_{l=1}^{N}(2-2g(C_l))$ is even by Lemma 3.2. $\square$

If $S_X=U\oplus D_4$ or $U(2)\oplus D_4$ then there exist $K3$ surfaces with non-symplectic automorphisms of order 8 by Example 7.3 and 7.4. And Schütt [15, Theorem 1] proved that the $K3$ surface with a non-symplectic automorphism of order 8 and $\operatorname{rank} S_X=14$ is unique.

Proposition 5.5. X has a non-symplectic automorphism φ of order 8 acting trivially on S_X if and only if $S_X=U\oplus D_4$, $U(2)\oplus D_4$ or $U\oplus D_4\oplus E_8$. Moreover the fixed locus X^φ has the form

$$X^\varphi=\begin{cases}\{P_1,P_2,\dots,P_6\}\amalg E_1 & \text{if } \operatorname{rank} S_X=6,\\ \{P_1,P_2,\dots,P_{12}\}\amalg E_1\amalg E_2 & \text{if } \operatorname{rank} S_X=14.\end{cases}$$

Proof. Note $\chi(C^{(g)}\amalg E_1\amalg\cdots\amalg E_N)=(2+r)/4$ by Lemma 3.2 and Proposition 5.1. We remark that X^{φ^2} does not contain non-singular curve with genus ≥ 1 by Proposition 4.3. Thus $N=(2+r)/8$. $\square$

6. Order 16

In this section, let φ be a non-symplectic automorphism of order 16. And we shall describe $X^\varphi=\{P_1,\dots,P_M\}\amalg C^{(g)}\amalg E_1\amalg\cdots\amalg E_N$. We remark that if X has a non-symplectic automorphism of order 16 then $\operatorname{rank} S_X=6$ or 14.

Proposition 6.1. Let r be the Picard number of X. Then the number of isolated points M is $(3r+6)/4$.

Proof. It is similar to the proof of Proposition 5.1. □

Remark 6.2. $m_{2,15}=(r+10)/4$, $m_{3,14}=(r+2)/8$, $m_{4,13}=(r-6)/8$, $m_{5,12}=(r-6)/8$, $m_{6,11}=(r-6)/8$, $m_{7,10}=1$, $m_{8,9}=0$.

Schütt [15, Theorem 1] proved that the $K3$ surface with a non-symplectic automorphism of order 16 and $\operatorname{rank} S_X=6$ is unique. That is $S_X=U\oplus D_4$.

By Proposition 5.5, if X has a non-symplectic automorphism of order 16 and $\operatorname{rank} S_X=14$ then $S_X=U\oplus D_4\oplus E_8$. Indeed there exists a $K3$ surface with non-symplectic automorphisms of order 16 and $S_X=U\oplus D_4\oplus E_8$. See Example 7.9.

Proposition 6.3. X has a non-symplectic automorphism φ of order 16 acting trivially on S_X if and only if $S_X=U\oplus D_4$ or $U\oplus D_4\oplus E_8$. Moreover the fixed locus X^φ has the form

$$X^\varphi=\begin{cases}\{P_1,P_2,\dots,P_6\}\amalg E_1 & \text{if } S_X\simeq U\oplus D_4,\\ \{P_1,P_2,\dots,P_{12}\}\amalg E_1\amalg E_2 & \text{if } S_X\simeq U\oplus D_4\oplus E_8.\end{cases}$$

Proof. It is similar to the proof of Proposition 5.5. □

7. Examples

In this section, we give examples of $K3$ surfaces with a non-symplectic automorphism of 2-power order. We remark that $K3$ surfaces have an elliptic fibration from Remark 2.5 and Table 1.

Example 7.1. [[6, (3.1)]](**Case:** $S_X=U$)
$X: y^2=x^3+x+t^{11}$, $\varphi(x,y,t)=(\zeta_{44}^{22}x,\zeta_{44}^{11}y,\zeta_{44}^{2}t)$.

Example 7.2. (**Case:** $S_X=U(2)$) Let λ_i be distinct 4 complex numbers. Let $([x_0:x_1],[y_0:y_1])$ be the bi-homogeneous coordinates on $\mathbb{P}^1\times\mathbb{P}^1$. Consider a smooth divisor C in $\mathbb{P}^1\times\mathbb{P}^1$ of bidegree $(4,4)$ given by

$$(y_0^4+y_1^4)\cdot\prod_{i=1}^{4}(x_0-\lambda_i x_1)=0.$$

Let ι be an involution of $\mathbb{P}^1\times\mathbb{P}^1$ given by

$$([x_0:x_1],[y_0:y_1])\to([x_0:x_1],[y_1:y_0])$$

which preserves C.

Let X the double cover of $\mathbb{P}^1\times\mathbb{P}^1$ branched along C. Then X is a $K3$ surface with $S_X=U(2)$. And the involution ι induces an automorphism φ which satisfies $\varphi^*\omega_X=\zeta_4\omega_X$.

Example 7.3. [[15]](**Case:** $S_X = U \oplus D_4$)
$X : y^2 = x^3 + t^2x + t^{11}$, $\varphi(x, y, t) = (\zeta_{16}^2 x, \zeta_{16}^3 y, \zeta_{16}^2 t)$.

Example 7.4. [[8, Propositon 4 (15)]](**Case:** $S_X = U(2) \oplus D_4$)
Let X be the minimal resolution of the surface $\widetilde{X} := \{z^2 = x_0(x_0^4 x_2 + x_1^5 - x_2^5)\}$ having 5 ordinary double points $[0 : 1 : \zeta_5^i : 0]$ $(i = 0, 1, 2, 3, 4)$ and $\varphi([x_0 : x_1 : x_2 : z]) = [x_0 : \zeta_{20} x_1 : \zeta_4 x_2 : \zeta_8^5 z]$.

Example 7.5. [[6, (3.2)]](**Case:** $S_X = U \oplus E_8$)
$X : y^2 = x^3 - t^5 \prod_{i=1}^{6}(t - \zeta_6^i)$, $\varphi(x, y, t) = (\zeta_{36}^2 x, \zeta_{36}^3 y, \zeta_{36}^{30} t)$.

Example 7.6. (**Case:** $S_X = U \oplus D_8$)
$X : y^2 = x^3 + t \prod_{i=1}^{6}(t - \zeta_6^i)x^2 + t \prod_{i=1}^{6}(t - \zeta_6^i)$, $\varphi(x, y, t) = (-x, \zeta_4 y, -t)$.

Example 7.7. (**Case:** $S_X = U \oplus D_4^{\oplus 2}$)
$X : y^2 = x^3 - t^3 \prod_{i=1}^{6}(t - \zeta_6^i)$, $\varphi(x, y, t) = (-x, \zeta_4 y, -t)$.

Example 7.8. [[7, §2.1]](**Case:** $S_X = U(2) \oplus D_4^{\oplus 2}$)
Let $\{[\lambda_i : 1]\}$ be a set of distinct 8 points on the projective line. Let $([x_0 : x_1], [y_0 : y_1])$ be the bi-homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Consider a smooth divisor C in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4, 2)$ given by

$$y_0^2 \cdot \prod_{i=1}^{4}(x_0 - \lambda_i x_1) + y_1^2 \cdot \prod_{i=5}^{8}(x_0 - \lambda_i x_1) = 0.$$

Let L_0 (resp. L_1) be the divisor defined by $y_0 = 0$ (resp. $y_1 = 0$). Let ι be an involution of $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$([x_0 : x_1], [y_0 : y_1]) \to ([x_0 : x_1], [y_0 : -y_1])$$

which preserves C, L_0 and L_1.

Note that the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $C + L_0 + L_1$ has 8 rational double points of type A_1 and its minimal resolution X is a $K3$ surface. The involution ι lifts to an automorphism φ which satisfies $\varphi^* \omega_X = \zeta_4 \omega_X$.

Example 7.9. (**Case:** $S_X = U \oplus E_8 \oplus D_4$)
$X : y^2 = x^3 + t^2x + t^7$, $\varphi(x, y, t) = (\zeta_{16}^{10} x, \zeta_{16}^7 y, \zeta_{16}^2 t)$.

Example 7.10. (**Case:** $S_X = U \oplus D_8 \oplus D_4$)
$X : y^2 = x^3 + t \prod_{i=1}^{4}(t - \zeta_4^i)x^2 + t^3 \prod_{i=1}^{4}(t - \zeta_4^i)$, $\varphi(x, y, t) = (-x, \zeta_4 y, -t)$.

Example 7.11. [[6, (3.4)]](**Case:** $S_X = U \oplus E_8^{\oplus 2}$)
$X : y^2 = x^3 - t^5(t - 1)(t + 1)$, $\varphi(x, y, t) = (\zeta_{12}^2 x, \zeta_{12}^3 y, -t)$.

Example 7.12. [[15]](**Case:** $S_X = U \oplus E_8 \oplus D_8$)
$X : y^2 = x^3 + tx^2 + t^7$, $\varphi(x, y, t) = (-x, \zeta_4 y, -t)$.

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