NAVIER–STOKES EQUATIONS ON THE β-PLANE

M.A.H. AL-JABOORI AND D. WIROSOETISNO

Abstract. We show that, given a sufficiently regular forcing, the solution of the two-dimensional Navier–Stokes equations on the periodic β-plane (i.e. with the Coriolis force varying as $f_0 + \beta y$) will become nearly zonal: with the vorticity $\omega(x, y, t) = \bar{\omega}(y, t) + \tilde{\omega}(x, y, t)$, one has $|\tilde{\omega}|^2_{H^s} \le \beta^{-1} M_s(\cdots)$ as $t \to \infty$. We use this show that, for sufficiently large β, the global attractor of this system reduces to a point.

1. Introduction

The two-dimensional Navier–Stokes equations (2d NSE) have been the subject of many studies and its basic mathematical properties (existence, uniqueness, regularity, etc.) are now well understood; see, e.g., [5, 11] for reviews. As a tool to understand various geophysical flows, it is often desirable to include the effect of planetary rotation, but a constant rotation rate (the so-called f-plane approximation) has no effect on the dynamics when periodic boundary conditions are used. To feel the effect of rotation, we need to go to the so-called β-plane approximation, in which the rotation is given by $f_0 + \beta y$.

Simple physical arguments and numerical studies [7, 15] suggest that a rotation rate that varies as βy tends to force the solution to become more zonal (a zonal flow is one that does not depend on x). In this article, we prove that this is indeed the case, by obtaining a bound $|\tilde{\omega}(t)|^2_{H^s} \le \beta^{-1} M_s(f; \cdots)$, valid for large time t, on the non-zonal part $\tilde{\omega}$ of the flow in terms of the forcing f.

With the further assumption that the forcing is independent of time, it has been shown that the Navier–Stokes equations possess a global attractor $\mathcal{A}$ of finite Hausdorff dimension. The long-known and nearly optimal bound on this dimension [4] also applies to our rotating case, but it does not take into account the effect of the rotation. Using our bounds on $\tilde{\omega}$, we show that the dimension of $\mathcal{A}$ is zero for ε sufficiently small, reducing the long-time dynamics to a single steady (and stable) flow determined completely by the forcing. This is to be contrasted with the situation for larger (but still small) ε, where the solution, although nearly zonal, evolves in time even though $\partial_t f = 0$.

Among the works similar in spirit to the present article, we mention [6] where weak convergence to zonal flow is proved for the (more difficult) β-plane shallow-water equations. A related result for the inviscid Euler equation can be found in [10]. The technique of using rapid oscillations to obtain better bounds have been used in different contexts in, e.g., [2, 9].

2000 *Mathematics Subject Classification.* Primary: 35B40, 35B41, 76D05.

Key words and phrases. Navier–Stokes equations, beta plane, global attractor.

This research was partially supported by ...

In the rest of this section, we describe the problem and set up the notation. In Section 2, we review basic results on 2d NSE which will be needed later. The heart of this article is Section 3, where L^2 and H^s bounds are obtained for the non-zonal component of the flow. An application of these bounds to the dimension of the global attractor follows in Section 4. The proof of an L^∞ Agmon inequality is presented in the Appendix.

In dimensional form, the two-dimensional Navier–Stokes equations read

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \beta y \boldsymbol{v}^\perp + \nabla p = \mu\Delta\boldsymbol{v} + f_{\boldsymbol{v}} \tag{1.1}$$

where the constant rotation f_0 has been dropped since it has no effect (i.e. in 2d NSE, there is no difference between equatorial and mid-latitude β-planes). Here $\boldsymbol{v} = (u, v)$ is the velocity with $\boldsymbol{v}^\perp := (-v, u)$ and p is the pressure obtained by enforcing the incompressibility constraint $\nabla\cdot\boldsymbol{v} = 0$. In what follows, we will work with the dimensionless form

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{Y}{\varepsilon}\boldsymbol{v}^\perp + \nabla p = \mu\Delta\boldsymbol{v} + f_{\boldsymbol{v}},\\ &\nabla\cdot\boldsymbol{v} = 0. \end{aligned} \tag{1.2}$$

We work with $\boldsymbol{x} = (x, y) \in \mathcal{M} := [0, L_1]\times[-L_2/2, L_2/2]$, with periodic boundary conditions in both directions. Note that we have replaced βy in (1.1) by $Y(y)/\varepsilon$, where $Y(-L_2/2) = L_2/2$ and $Y(y) = y$ for $y \in (-L_2/2, L_2/2]$. Furthermore, we assume the following symmetry on the velocity

$$u(x, -y, t) = u(x, y, t) \quad\text{and}\quad v(x, -y, t) = -v(x, y, t). \tag{1.3}$$

It is readily verified that if the initial data $\boldsymbol{v}(0)$ and the forcing $f_{\boldsymbol{v}}(t)$ also satisfy this symmetry, which we henceforth assume, it persists for all $t \ge 0$. Note also that periodicity and (1.3) imply that

$$v(x, -L_2/2, t) = v(x, L_2/2, t) = 0. \tag{1.4}$$

With no loss of generality, we require that the integral over $\mathcal{M}$ of $\boldsymbol{v}$ vanishes.

In two dimensions, it is convenient to work with the vorticity $\omega := \nabla^\perp\cdot\boldsymbol{v} = \partial_x v - \partial_y u$, whose evolution equation is

$$\partial_t\omega + \boldsymbol{v}\cdot\nabla\omega + \frac{Y'}{\varepsilon}\,v = \mu\Delta\omega + f. \tag{1.5}$$

Here $f := \nabla^\perp\cdot f_{\boldsymbol{v}}$ and we can recover the velocity using $\boldsymbol{v} = \nabla^\perp\Delta^{-1}\omega$. By our assumption on $\boldsymbol{v}$, the integral of ω over $\mathcal{M}$ is zero; similarly, Δ^{-1} is defined uniquely by the zero-integral condition. The symmetry (1.3) implies that $\omega(x, -y, t) = -\omega(x, y, t)$ and

$$\omega(x, -L_2/2, t) = \omega(x, L_2/2, t) = 0. \tag{1.6}$$

Now $Y'(y) = 1 - L_2\,\delta(y - L_2/2)$, where δ is the Dirac distribution. Using the fact that $v(x, \pm L_2/2, t) = 0$, we replace vY' by v in (1.5) and write

$$\partial_t\omega + \boldsymbol{v}\cdot\nabla\omega + \frac{1}{\varepsilon}\,v = \mu\Delta\omega + f. \tag{1.7}$$

This is the form that we will be mostly working with.

It is also convenient to write (1.7) in the usual functional form

$$\partial_t\omega + B(\omega,\omega) + \frac{1}{\varepsilon}L\omega + \mu A\omega = f, \tag{1.8}$$

where $B(\omega,\omega^\sharp) := (\nabla^\perp\Delta^{-1}\omega)\cdot\nabla\omega^\sharp$, $L\omega := \partial_x\Delta^{-1}\omega$ and $A := -\Delta$. The following properties, valid whenever the expressions make sense, are readily verified by integration by parts and the boundary conditions (1.4)–(1.6)

$$\begin{aligned}&(B(\omega,\omega^\sharp),\omega^\sharp)_{L^2} = 0,\\ &(L\omega,\omega)_{L^2} = 0,\\ &(A\omega,\omega)_{L^2} = |\nabla\omega|_{L^2}^2.\end{aligned} \tag{1.9}$$

2. Preliminary Estimates

The estimates derived in this section are standard from the theory of 2d NSE (see, e.g., [12, 5, 8]), with very minor modifications to handle the Coriolis term. We gather them here for later use.

We start by noting that the vanishing of spatial integrals of $\boldsymbol{v}$ and ω implies the equivalence of the norms $|\omega|_{H^s}$ and $|\nabla^s\omega|_{L^2} := |(-\Delta)^{s/2}\omega|_{L^2}$, which will thus be used interchangeably below. We denote by c_0 the constant in Poincaré inequality

$$c_0\,|\nabla^s\omega|_{L^2} \le |\nabla^{s+1}\omega|_{L^2}. \tag{2.1}$$

Besides the usual Sobolev and interpolation inequalities for two and one dimensions (for functions depending on y only), we note the one-dimensional Agmon inequality

$$|\bar w|_{L^\infty} \le c\,|\bar w|_{L^2}^{1/2}|\nabla\bar w|_{L^2}^{1/2}. \tag{2.2}$$

A version we use for the two-dimensional case is in Appendix A.

The L^2 estimate for the velocity is obtained by multiplying (1.2) by $\boldsymbol{v}$ and using Cauchy–Schwarz,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{v}|_{L^2}^2 + \mu\,|\nabla\boldsymbol{v}|_{L^2}^2 \le \frac{c}{\mu}\,|f_{\boldsymbol{v}}|_{L^2}^2\,. \tag{2.3}$$

Assuming that $f_{\boldsymbol{v}} \in L_t^\infty L_{\boldsymbol{x}}^2$, we thus have $\boldsymbol{v} \in L_t^\infty L_{\boldsymbol{x}}^2 \cap L_t^2 H_{\boldsymbol{x}}^1$. Here and henceforth, $L_t^p L_{\boldsymbol{x}}^q := L^p((0,\infty); L^q(\mathcal{M}))$, and $H_{\boldsymbol{x}}^s$ and $|\nabla^s\omega|$ below are defined in the usual way. We denote

$$[\![w]\!] := \sup_{t>0}\,|w(t)|_{L^2}. \tag{2.4}$$

Here and elsewhere in this article, c denotes a generic constant depending only on $\mathcal{M}$ whose value may not be the same each time it appears, while numbered constants such as c_1 have fixed values.

Now let $\varphi \in L^\infty$ be such that $\varphi'(t) = 1$ for $t \in [0,\frac{1}{2}]$ and $\varphi(t) = 1$ for $t > 2$, so $\varphi(t) \simeq \tanh t$. Multiplying (1.7) by $\varphi\omega$ in L^2, or equivalently, multiplying (1.2a) by $-\varphi\Delta\boldsymbol{v}$ in L^2, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi|\omega|_{L^2}^2\big) + \mu\,\varphi\,|\nabla\omega|_{L^2}^2 \le \varphi'\,|\omega|^2 + \frac{c}{\mu}\,\varphi\,|f_{\boldsymbol{v}}|_{L^2}^2\,. \tag{2.5}$$

Assuming henceforth that $[\![f_{\boldsymbol{v}}]\!] < \infty$, we have $\omega \in L_t^\infty L_{\boldsymbol{x}}^2 \cap L_t^2 H_{\boldsymbol{x}}^1$ and, for $t \ge T_0(|\boldsymbol{v}(0)|_{L^2}, [\![f_{\boldsymbol{v}}]\!]; \mu)$,

$$|\omega(t)|_{L^2} \le \frac{c}{\mu}\,[\![f_{\boldsymbol{v}}]\!]. \tag{2.6}$$

Note that T_0 does not depend on $\omega(0)$ and that the requirement $f_{\boldsymbol{v}} \in L_t^\infty L_{\boldsymbol{x}}^2$ can be weakened in t, but we shall not do so here.

A bound in H^m is obtained as follows. Fix a multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| := \alpha_1 + \alpha_2 = m$, and multiply (1.7) by $\mathsf{D}^{2\alpha}\omega := \partial_x^{2\alpha_1}\partial_y^{2\alpha_2}\omega$ in L^2,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{D}^\alpha\omega|^2 + (\boldsymbol{v}\cdot\nabla\omega, \mathsf{D}^{2\alpha}\omega) + \frac{1}{\varepsilon}(v, \mathsf{D}^{2\alpha}\omega) + \mu\,|\nabla\mathsf{D}^\alpha\omega|^2 = (f, \mathsf{D}^{2\alpha}\omega) \tag{2.7}$$

where here and henceforth $|\cdot|$ and $(\cdot,\cdot)$ denote L^2 norm and inner product. The linear term involving $1/\varepsilon$ vanishes, and one then proceeds as usual: Using the fact that $(\boldsymbol{v}\cdot\nabla\mathsf{D}^\alpha\omega, \mathsf{D}^\alpha\omega) = 0$, the nonlinear term is bounded as

$$\begin{aligned} \big|(\boldsymbol{v}\cdot\nabla\omega, \mathsf{D}^{2\alpha}\omega)\big| &\le \sum_{1\le|\beta|\le|\alpha|} \big|((\mathsf{D}^\beta\boldsymbol{v})\cdot\nabla\mathsf{D}^{\alpha-\beta}\omega, \mathsf{D}^\alpha\omega)\big| \\ &\le c\textstyle\sum_\beta \big|\mathsf{D}^{\beta-1}\omega\big|_{L^4}\big|\mathsf{D}^{\alpha-\beta+1}\omega\big|_{L^4}\big|\mathsf{D}^\alpha\omega\big|_{L^2} \\ &\le c\textstyle\sum_\beta \big|\mathsf{D}^{\beta-1}\omega\big|_{H^{1/2}}\big|\mathsf{D}^{\alpha-\beta+1}\omega\big|_{H^{1/2}}\big|\mathsf{D}^\alpha\omega\big|_{L^2} \\ &\le c(m)\textstyle\sum_{l=1}^m |\omega|_{H^{l-1/2}}|\omega|_{H^{m-l+3/2}}|\omega|_{H^m} \end{aligned} \tag{2.8}$$

where we have used Sobolev inequalities for the second and third line, and where $l := |\beta|$ in the last line. Using the interpolation inequalities

$$\begin{aligned} |\omega|_{H^{l-1/2}} &\le c\,|\omega|_{L^2}^{(2m-2l+3)/(2m+2)}|\omega|_{H^{m+1}}^{(2l-1)/(2m+2)} \\ |\omega|_{H^{m-l+3/2}} &\le c\,|\omega|_{L^2}^{(2l-1)/(2m+2)}|\omega|_{H^{m+1}}^{(2m-2l+3)/(2m+2)}, \end{aligned} \tag{2.9}$$

followed by Cauchy–Schwarz and summing over α, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|_{H^m}^2 + \frac{3\mu}{2}\,|\omega|_{H^{m+1}}^2 \le \frac{c(m)}{\mu}\,|\omega|_{L^2}^2|\omega|_{H^m}^2 + \frac{c'(m)}{\mu}\,|f|_{H^{m-1}}^2 \tag{2.10}$$

for $m = 1, 2, \cdots$. Proceeding by Gronwall and induction on (2.10), we have the following uniform bounds independent of the initial data

$$\begin{aligned} &|\omega(t)|_{H^m}^2 \le \Big(\frac{c(m)}{\mu}\,[\![\nabla^{m-1}f]\!]^2 + 1\Big)^{m+1} \\ &\mathrm{e}^{-\nu t'}\int_t^{t+t'} \mathrm{e}^{\nu\tau}|\omega(\tau)|_{H^{m+1}}^2\,\mathrm{d}\tau \le \frac{c}{\mu}\Big(\frac{c(m)}{\mu}\,[\![\nabla^{m-1}f]\!]^2 + 1\Big)^{m+1} \end{aligned} \tag{2.11}$$

valid for all $t \ge T_m(|\boldsymbol{v}(0)|_{L^2}, [\![\nabla^{m-1}f]\!]; \mu)$. Here $\nu = c_0^2\mu$ with c_0 the constant in Poincaré inequality (2.1). Note that for T_m to depend only on $|\boldsymbol{v}(0)|_{L^2}$, and for the validity of (2.11) for $\boldsymbol{v}(0) \notin H^1$, we need to multiply by φ as in (2.5), but this was not done explicitly for conciseness.

3. Bounds on the Non-zonal Component

Assuming sufficient regularity for f, which implies that for ω for any $t > 0$, we expand them in Fourier series

$$\begin{aligned} \omega(\boldsymbol{x}, t) &= \textstyle\sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\Omega_{\boldsymbol{k}}t/\varepsilon} \\ f(\boldsymbol{x}, t) &= \textstyle\sum_{\boldsymbol{k}} f_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \end{aligned} \tag{3.1}$$

where $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}_L := \{(2\pi l_1/L_1, 2\pi l_2/L_2) : (l_1, l_2) \in \mathbb{Z}^2\}$ and $\Omega_{\boldsymbol{k}} := -k_1/|\boldsymbol{k}|^2$ is i times the eigenvalue of the linear operator L for wavenumber $\boldsymbol{k}$. Since ω and f have vanishing integrals over $\mathcal{M}$, $\omega_{\boldsymbol{0}} = 0$ and $f_{\boldsymbol{0}} = 0$. Here and in what follows, sums over wavenumbers are understood to be taken over $\mathbb{Z}_L$. In terms of Fourier components, (1.7) reads

$$\frac{\mathrm{d}\omega_{\boldsymbol{l}}}{\mathrm{d}t} + \textstyle\sum_{\boldsymbol{jk}} B_{\boldsymbol{jkl}}\omega_{\boldsymbol{j}}\omega_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}(\Omega_{\boldsymbol{l}}-\Omega_{\boldsymbol{j}}-\Omega_{\boldsymbol{k}})t/\varepsilon} + \mu\,|\boldsymbol{l}|^2\omega_{\boldsymbol{l}} = f_{\boldsymbol{l}}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{l}}t/\varepsilon} \tag{3.2}$$

where the coefficient of the nonlinear term is

$$B_{\boldsymbol{jkl}} = (B(\mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}},\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}),\mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}}) = |\mathcal{M}|\,\frac{\boldsymbol{j}\wedge\boldsymbol{k}}{|\boldsymbol{j}|^2}\,\delta_{\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l}} \tag{3.3}$$

with $\boldsymbol{j}\wedge\boldsymbol{k} := j_1k_2 - j_2k_1$. We note that the linear term $\varepsilon^{-1}L\omega$ has been removed from (3.2) by including $\exp(-\mathrm{i}\Omega_{\boldsymbol{k}}t/\varepsilon)$ in (3.1a).

Let us split ω into a slow part $\bar\omega$, for which $\Omega_{\boldsymbol{k}} = 0$, and the remaining fast part $\tilde\omega := \omega - \bar\omega$, viz.,

$$\begin{aligned}\bar\omega(\boldsymbol{x},t) &= \textstyle\sum_{k_1=0}\omega_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\\ \tilde\omega(\boldsymbol{x},t) &= \textstyle\sum_{k_1\neq 0}\omega_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\Omega_{\boldsymbol{k}}t/\varepsilon}.\end{aligned} \tag{3.4}$$

We note that, also having zero integrals over $\mathcal{M}$, $\tilde\omega$ and $\bar\omega$ are orthogonal in H^m for $m = 0, 1, \cdots$. For convenience, we also define

$$\bar\omega_{\boldsymbol{k}} := \begin{cases}\omega_{\boldsymbol{k}} & \text{if } k_1 = 0\\ 0 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad \tilde\omega_{\boldsymbol{k}} := \begin{cases}\omega_{\boldsymbol{k}} & \text{if } k_1 \neq 0\\ 0 & \text{otherwise.}\end{cases} \tag{3.5}$$

Our objective in this section is to obtain long-time bounds for $\tilde\omega$ that tend to zero as $\varepsilon \to 0$.

3.1. **Bound in L^2.** The development in this subsection largely follows that in [13] for the primitive equations, the main difference being the absence of a spectral gap (that is, the eigenvalues of the antisymmetric operator L accumulate at zero in the present case).

We start by multiplying (1.8) by $\tilde\omega$ in L^2,

$$(\partial_t\omega,\tilde\omega) + (B(\omega,\omega),\tilde\omega) + \frac{1}{\varepsilon}(L\omega,\tilde\omega) + \mu\,(A\omega,\tilde\omega) = (f,\tilde\omega). \tag{3.6}$$

Now, using (1.9a) twice and the fact that $B(\bar\omega,\bar\omega) = 0$,

$$\begin{aligned}(B(\omega,\omega),\tilde\omega) &= (B(\omega,\tilde\omega),\tilde\omega) + (B(\omega,\bar\omega),\tilde\omega)\\ &= (B(\bar\omega,\bar\omega),\tilde\omega) + (B(\tilde\omega,\bar\omega),\tilde\omega)\\ &= -(B(\tilde\omega,\tilde\omega),\bar\omega).\end{aligned} \tag{3.7}$$

Thus (3.6) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde\omega|^2 + \mu\,|\nabla\tilde\omega|^2 = (B(\tilde\omega,\tilde\omega),\bar\omega) + (f,\tilde\omega). \tag{3.8}$$

Dropping the nonlinear term for the moment, the fact that $\tilde\omega$ is rapidly varying while f is slow implies that the effective forcing from the rhs becomes weaker for smaller ε. This essentially is the mechanism for the attenuation of the fast part $\tilde\omega$; the nonlinear term will be handled in the proof below. Recalling the definition (2.4), we state the result of this subsection.

Theorem 1. *Assume that the initial data* $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ *and that the forcing is bounded as* $[\![\nabla^2 f]\!] + [\![\partial_t f]\!] < \infty$. *Then there exist* $\mathcal{T}_0(|\boldsymbol{v}(0)|_{L^2}, [\![\nabla^2 f]\!], [\![\partial_t f]\!]; \mu)$ *and* $M_0([\![\nabla^2 f]\!], [\![\partial_t f]\!]; \mu)$ *such that, for* $t \geq \mathcal{T}_0$,

$$
\begin{aligned}
&|\tilde\omega(t)|^2_{L^2} \leq \varepsilon\, M_0,\\
&\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau} |\nabla\tilde\omega(\tau)|^2 \,\mathrm{d}\tau \leq \varepsilon\, M_0.
\end{aligned}
\tag{3.9}
$$

Proof. Recalling that $\nu = \mu c_0^2$, we obtain from (3.8)

$$
\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{\nu t}|\tilde\omega|^2\big) + \mu \mathrm{e}^{\nu t}|\nabla\tilde\omega|^2 \leq 2\mathrm{e}^{\nu t}(B(\tilde\omega,\tilde\omega),\bar\omega) + 2\mathrm{e}^{\nu t}(f,\tilde\omega).
\tag{3.10}
$$

We integrate the last term from 0 to t by parts,

$$
\begin{aligned}
\int_0^t \mathrm{e}^{\nu\tau}(f,\tilde\omega)\,\mathrm{d}\tau &= |\mathcal{M}| {\textstyle\sum'_{\boldsymbol{k}}} \int_0^t f_{\boldsymbol{k}}(\tau)\overline{\tilde\omega_{\boldsymbol{k}}(\tau)}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}}\tau/\varepsilon+\nu\tau}\,\mathrm{d}\tau\\
&= -\mathrm{i}\varepsilon|\mathcal{M}| \sum\nolimits'_{\boldsymbol{k}} \frac{1}{\Omega_{\boldsymbol{k}}}\Big[f_{\boldsymbol{k}}\overline{\tilde\omega_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}}\tau/\varepsilon+\nu\tau}\Big]_0^t\\
&\qquad + \mathrm{i}\varepsilon|\mathcal{M}| \sum\nolimits'_{\boldsymbol{k}} \frac{1}{\Omega_{\boldsymbol{k}}}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big[f_{\boldsymbol{k}}\overline{\tilde\omega_{\boldsymbol{k}}}\,\mathrm{e}^{\nu\tau}\big]\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}}\tau/\varepsilon}\,\mathrm{d}\tau
\end{aligned}
\tag{3.11}
$$

where the prime on the sums indicates that the resonant terms (i.e. those with $\Omega_{\boldsymbol{k}} = 0$) are excluded. Defining the operator ∂_t^* by, for any w for which it makes sense,

$$
\begin{aligned}
&\partial_t^* w := \mathrm{e}^{-tL/\varepsilon}\partial_t\big(\mathrm{e}^{tL/\varepsilon}w\big)\\
\Rightarrow\qquad &\partial_t^*\tilde\omega := \partial_t\tilde\omega + \frac{1}{\varepsilon}L\tilde\omega = -\tilde B(\omega,\omega) - \mu A\tilde\omega + \tilde f,
\end{aligned}
\tag{3.12}
$$

and defining the operator I_Ω by

$$
\mathsf{I}_\Omega \tilde f(\boldsymbol{x},t) := \sum\nolimits'_{\boldsymbol{k}} \frac{1}{\mathrm{i}\Omega_{\boldsymbol{k}}}\, f_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \mathrm{i}\sum\nolimits'_{\boldsymbol{k}} \frac{|\boldsymbol{k}|^2}{k_1}\, f_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},
\tag{3.13}
$$

which being the restricted inverse of L is also antisymmetric, we can write

$$
\begin{aligned}
\int_0^t \mathrm{e}^{\nu\tau}(f,\tilde\omega)\,\mathrm{d}\tau &= \varepsilon\,(\mathsf{I}_\Omega\tilde f,\tilde\omega)(t)\mathrm{e}^{\nu t} - \varepsilon\,(\mathsf{I}_\Omega\tilde f,\tilde\omega)(0)\\
&\qquad - \varepsilon\int_0^t \big[\nu(\mathsf{I}_\Omega\tilde f,\tilde\omega) + (\partial_\tau\mathsf{I}_\Omega\tilde f,\tilde\omega) + (\mathsf{I}_\Omega\tilde f,\partial_\tau^*\tilde\omega)\big]\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.
\end{aligned}
\tag{3.14}
$$

Using (3.13), the endpoint terms can be bounded as

$$
|(\mathsf{I}_\Omega\tilde f,\tilde\omega)| \leq c\,|\nabla\tilde f|\,|\nabla\tilde\omega|.
\tag{3.15}
$$

We now bound the terms in the integrand. First,

$$
\big|(\partial_\tau\mathsf{I}_\Omega\tilde f,\tilde\omega)\big| = \big|(\partial_\tau\tilde f,\mathsf{I}_\Omega\tilde\omega)\big| \leq c\,|\partial_\tau\tilde f|\,|\Delta\tilde\omega|.
\tag{3.16}
$$

Next, using (3.12b) and noting the fact that $(\mathsf{I}_\Omega\tilde f,\tilde f) = 0$, we bound the last term in (3.14) by

$$
\big|(\mathsf{I}_\Omega\tilde f,\mu\Delta\tilde\omega)\big| \leq \mu c\,|\Delta\tilde f|\,|\Delta\tilde\omega|;
\tag{3.17}
$$

and, using Sobolev and interpolation inequalities,

$$
\begin{aligned}
\big|(\mathsf{I}_\Omega\tilde f, B(\omega,\omega))\big| &\le c\,|\nabla\tilde f|_{L^2}|\nabla B(\tilde\omega,\tilde\omega)|_{L^2}\\
&\le c\,|\nabla\tilde f|_{L^2}|\omega|_{L^4}|\nabla\omega|_{L^4} + c\,|\nabla\tilde f|_{L^2}|\nabla^{-1}\omega|_{L^\infty}|\Delta\omega|_{L^2}\\
&\le c\,|\nabla\tilde f|\,|\nabla\omega|\,|\Delta\omega|.
\end{aligned}\tag{3.18}
$$

Thus the integral in (3.14) is bounded as

$$
\begin{aligned}
&\left|\int_0^t [\nu(\mathsf{I}_\Omega\tilde f,\tilde\omega) + (\partial_\tau\mathsf{I}_\Omega\tilde f,\tilde\omega) + (\mathsf{I}_\Omega\tilde f,\partial_\tau^*\tilde\omega)]\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\right|\\
&\qquad\le c\int_0^t \big[\mu\,|\Delta\tilde f|\,|\Delta\tilde\omega| + |\partial_\tau\tilde f|\,|\Delta\tilde\omega| + |\nabla\tilde f|\,|\nabla\omega|\,|\Delta\omega|\big]\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\\
&\qquad\le c\int_0^t \big\{(1+\mu)\,|\Delta\tilde\omega|^2 + \mu\,|\Delta\tilde f|^2 + |\partial_\tau\tilde f|^2 + |\Delta\omega|\,|\nabla\omega|\,|\nabla\tilde f|\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.
\end{aligned}\tag{3.19}
$$

We now treat the penultimate term in (3.10). First, we write

$$
\begin{aligned}
(B(\tilde\omega,\tilde\omega),\bar\omega) &= \textstyle\sum_{\boldsymbol{jkl}} B_{\boldsymbol{jkl}}\tilde\omega_{\boldsymbol{j}}\tilde\omega_{\boldsymbol{k}}\overline{\omega_{\boldsymbol{l}}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})t/\varepsilon}\\
&= \frac12\textstyle\sum_{\boldsymbol{jkl}}(B_{\boldsymbol{jkl}}+B_{\boldsymbol{kjl}})\,\tilde\omega_{\boldsymbol{j}}\tilde\omega_{\boldsymbol{k}}\overline{\omega_{\boldsymbol{l}}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})t/\varepsilon}
\end{aligned}\tag{3.20}
$$

and then note that $B_{\boldsymbol{jkl}}+B_{\boldsymbol{kjl}}=0$ in the resonant case, i.e. when $\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}}=0$ and $l_1=0$. Furthermore, we have

$$
\begin{aligned}
B_{\boldsymbol{jkl}}+B_{\boldsymbol{kjl}} &= \left(\frac{\boldsymbol{j}\wedge\boldsymbol{k}}{|\boldsymbol{j}|^2}+\frac{\boldsymbol{k}\wedge\boldsymbol{j}}{|\boldsymbol{k}|^2}\right)|\mathcal{M}| \qquad = (\boldsymbol{j}\wedge\boldsymbol{k})\left(\frac{1}{|\boldsymbol{j}|^2}-\frac{1}{|\boldsymbol{k}|^2}\right)|\mathcal{M}|\\
&= j_1 l_2\left(\frac{1}{|\boldsymbol{j}|^2}-\frac{1}{|\boldsymbol{k}|^2}\right)|\mathcal{M}| \qquad = -l_2\,(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})\,|\mathcal{M}|
\end{aligned}\tag{3.21}
$$

whenever $\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}$ and $l_1=0$. Motivated by (3.20), we introduce the bilinear symmetric operator B_Ω by

$$
\begin{aligned}
(B_\Omega(\tilde\omega^\sharp,\tilde\omega^\flat),\bar\omega) &:= \frac{\mathrm{i}}{2}\sum\nolimits'_{\boldsymbol{jkl}}\frac{B_{\boldsymbol{jkl}}+B_{\boldsymbol{kjl}}}{\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}}}\,\tilde\omega^\sharp_{\boldsymbol{j}}\tilde\omega^\flat_{\boldsymbol{k}}\overline{\omega_{\boldsymbol{l}}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})t/\varepsilon}\\
&= \frac{|\mathcal{M}|}{2\mathrm{i}}\sum\nolimits'_{\boldsymbol{jkl}} l_2\,\tilde\omega^\sharp_{\boldsymbol{j}}\tilde\omega^\flat_{\boldsymbol{k}}\overline{\omega_{\boldsymbol{l}}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})t/\varepsilon}
\end{aligned}\tag{3.22}
$$

for any $\tilde\omega^\sharp$, $\tilde\omega^\flat$ and $\bar\omega$, where the prime on the sum again indicates that resonant terms (for which $\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}}=0$) are omitted. We note that, thanks to (3.21), the resonant terms are also absent in $(B(\tilde\omega,\tilde\omega),\bar\omega)$. Integrating by parts, we have

$$
\begin{aligned}
\int_0^t \mathrm{e}^{\nu\tau}(B(\tilde\omega,\tilde\omega),\bar\omega)\,\mathrm{d}\tau &= \varepsilon\,\mathrm{e}^{\nu t}(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega)(t) - \varepsilon\,(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega)(0)\\
&\quad + \varepsilon\int_0^t \mathrm{e}^{\nu\tau}\big[\nu(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega) + 2\,(B_\Omega(\partial_\tau^*\tilde\omega,\tilde\omega),\bar\omega)\\
&\qquad + (B_\Omega(\tilde\omega,\tilde\omega),\partial_\tau\bar\omega)\big]\,\mathrm{d}\tau.
\end{aligned}\tag{3.23}
$$

For the last term in the integrand, we use the fact that $\bar B(\tilde\omega,\bar\omega)=\bar B(\bar\omega,\tilde\omega)=\bar B(\bar\omega,\bar\omega)=0$ to write

$$
\partial_\tau\bar\omega = -\bar B(\tilde\omega,\tilde\omega) - \mu A\bar\omega + \bar f \tag{3.24}
$$

and estimate, using $H^1 \subset L^\infty$ for $\bar f$ and (A.2) for the L^∞ estimates,

$$\begin{aligned}\big|(B_\Omega(\tilde\omega,\tilde\omega),\partial_\tau\bar\omega)\big| &\le c\,|\tilde\omega|_{L^2}|\partial_y\tilde\omega|_{L^2}|\bar f|_{L^\infty} + \mu c\,|\tilde\omega|_{L^4}|\partial_y\tilde\omega|_{L^4}|\Delta\bar\omega|_{L^2}\\ &\qquad + c\,|\tilde\omega|_{L^\infty}|\partial_y\tilde\omega|_{L^2}|\nabla^{-1}\tilde\omega|_{L^\infty}|\nabla\tilde\omega|_{L^2}\\ &\le c\,|\tilde\omega|\,|\nabla\tilde\omega|\,|\bar f'| + \mu c\,|\tilde\omega|^{1/2}|\nabla\tilde\omega|\,|\Delta\omega|^{3/2} + c\,|\nabla\omega|^3|\tilde\omega|\Big(\log\frac{|\Delta\omega|}{c_0|\nabla\omega|}+c'\Big).\end{aligned} \tag{3.25}$$

For the term involving $(B_\Omega(\partial_\tau^*\tilde\omega,\tilde\omega),\bar\omega)$, we bound, using $\partial_\tau^*\tilde\omega + \tilde B(\omega,\omega) + \mu A\tilde\omega = \tilde f$ and the inequality $|\bar\omega|_{L^\infty} \le c\,|\bar\omega|^{1/2}|\bar\omega'|^{1/2}$,

$$\begin{aligned}\big|(B_\Omega(\partial_\tau^*\tilde\omega,\tilde\omega),\bar\omega)\big| &\le c\,|\partial_y\tilde f|\,|\tilde\omega|\,|\bar\omega|_{L^\infty} + c\,|\tilde f|\,|\partial_y\tilde\omega|\,|\bar\omega|_{L^\infty}\\ &\qquad + \mu c\,|\Delta\omega|\,|\tilde\omega|\,|\bar\omega'|_{L^\infty} + c\,|\nabla^{-1}\omega|_{L^\infty}|\nabla\omega|\,|\tilde\omega|\,|\bar\omega'|_{L^\infty}\\ &\le c\,|\nabla\tilde f|\,|\tilde\omega|\,|\bar\omega'| + c\,|\tilde f|\,|\nabla\tilde\omega|\,|\bar\omega'| + \mu c\,|\tilde\omega|\,|\bar\omega'|^{1/2}|\Delta\omega|^{3/2}\\ &\qquad + c\,|\omega|^2|\nabla\omega|^{3/2}|\bar\omega''|^{1/2}\Big(\log\frac{|\nabla\omega|}{c_0|\omega|}+1\Big)^{1/2}\end{aligned} \tag{3.26}$$

where all unadorned norms are L^2. Finally, we bound

$$\begin{aligned}\big|(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega)\big| &\le c\,|\tilde\omega|\,|\partial_y\tilde\omega|\,|\bar\omega|_{L^\infty}\\ &\le c\,|\tilde\omega|\,|\nabla\tilde\omega|\,|\bar\omega|^{1/2}|\bar\omega'|^{1/2}.\end{aligned} \tag{3.27}$$

Using these also to bound the endpoint terms, the integral in (3.23) is bounded as

$$\begin{aligned}&\Big|\int_0^t \mathrm{e}^{\nu\tau}(B(\tilde\omega,\tilde\omega),\bar\omega)\,\mathrm{d}\tau\Big|\\ &\qquad\le \varepsilon c\,\big[|\tilde\omega|\,|\nabla\tilde\omega|\,|\bar\omega|^{1/2}|\bar\omega'|^{1/2}\big](t)\,\mathrm{e}^{\nu t} + \varepsilon c\,\big[|\tilde\omega|\,|\nabla\tilde\omega|\,|\bar\omega|^{1/2}|\bar\omega'|^{1/2}\big](0)\\ &\qquad + \varepsilon\int_0^t\Big\{c\,|\nabla f|\,|\tilde\omega|\,|\nabla\omega| + c\,|\tilde f|\,|\nabla\omega|^2 + \mu c\,|\omega|^{1/2}\,|\nabla\omega|\,|\Delta\omega|^{3/2}\\ &\qquad\qquad + c\,|\omega|\,|\nabla\omega|^{5/2}|\Delta\omega|^{1/2}\Big(\log\frac{|\Delta\omega|}{c_0|\nabla\omega|}+c'\Big)\Big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.\end{aligned} \tag{3.28}$$

Putting together (3.15), (3.19) and (3.28), we have

$$\begin{aligned}|\tilde\omega(t)|^2 &+ \mu\int_0^t |\nabla\tilde\omega|^2\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau \le \mathrm{e}^{-\nu t}|\tilde\omega(0)|^2\\ &+ \varepsilon c_2\,(1+\mathrm{e}^{-\nu t})\sup_{0\le t'\le t}\big\{|\nabla\tilde f|\,|\nabla\tilde\omega| + |\omega|^{3/2}|\nabla\omega|^{3/2}\big\}\\ &+ \varepsilon c_3(\mu)\int_0^t\Big\{|\Delta\tilde f|^2 + |\partial_\tau\tilde f|^2 + |\nabla\tilde f|\,|\nabla\omega|\,|\Delta\omega| + |\Delta\omega|^2\big(1+|\nabla\omega|\big)\\ &\qquad + |\omega|\,|\nabla\omega|^{5/2}|\Delta\omega|^{1/2}\Big(\log\frac{|\Delta\omega|}{c_0|\nabla\omega|}+c'\Big)\Big\}\,\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau.\end{aligned} \tag{3.29}$$

We now shift the origin of time such that $t = 0$ corresponds to T_2 in (2.11). The hypothesis that $[\![\nabla^2 f]\!] + [\![\partial_t f]\!] < \infty$ then implies that both the endpoints and the integral in (3.29) are bounded uniformly for all $t > 0$, independently of the initial data provided that $\boldsymbol{v} \in L^2$ initially. Rewriting the bound in (3.29) as

$$|\tilde\omega(t)|^2 + \mu\int_0^t |\nabla\tilde\omega|^2\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau \le \mathrm{e}^{-\nu t}|\tilde\omega(0)|^2 + \frac{\varepsilon}{2}\,M_0([\![\nabla^2 f]\!],[\![\partial_t f]\!];\mu), \tag{3.30}$$

the proof is complete.

We also note from (3.29) that the hypothesis $f \in L_t^\infty H_{\boldsymbol{x}}^2$ and $\partial_t f \in L_t^\infty L_{\boldsymbol{x}}^2$ can be weakened to $f \in L_t^2 H_{\boldsymbol{x}}^2 \cap L_t^\infty H_{\boldsymbol{x}}^1$ and $\partial_t f \in L_t^2 L_{\boldsymbol{x}}^2$.

3.2. **Bounds in H^s.** With a little extra work, H^s bounds for $\tilde\omega$ that scales as $\sqrt{\varepsilon}$ can also be obtained. We do this explicitly for $|\nabla\tilde\omega|$ and sketch the computation for $s = 2, 3, \cdots$.

For the H^1 bound, we multiply (1.8) by $A\tilde\omega$ in L^2 to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\tilde\omega|^2 + \mu\,|\Delta\tilde\omega|^2 + (B(\omega,\omega), A\tilde\omega) = (\tilde f, A\tilde\omega), \tag{3.31}$$

which implies [cf. (3.10)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{\nu t}|\nabla\tilde\omega|^2\big) + \mu\,\mathrm{e}^{\nu t}|\Delta\tilde\omega|^2 \le 2\mathrm{e}^{\nu t}\,(B(\omega,\omega),\Delta\tilde\omega) - 2\mathrm{e}^{\nu t}(\tilde f,\Delta\tilde\omega). \tag{3.32}$$

As in the L^2 case, we integrate from 0 to t,

$$\begin{aligned}\mathrm{e}^{\nu t}|\nabla\tilde\omega(t)|^2 - |\nabla\tilde\omega(0)|^2 + \mu\int_0^t |\Delta\tilde\omega|^2\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\\ \le 2\int_0^t \big\{(B(\omega,\omega),\Delta\tilde\omega) - (\tilde f,\Delta\tilde\omega)\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.\end{aligned} \tag{3.33}$$

The forcing term gives

$$\begin{aligned}\int_0^t \mathrm{e}^{\nu\tau}(f,\Delta\tilde\omega)\,\mathrm{d}\tau &= \varepsilon\,(\mathsf{I}_\Omega\tilde f,\Delta\tilde\omega)(t)\mathrm{e}^{\nu t} - \varepsilon\,(\mathsf{I}_\Omega\tilde f,\Delta\tilde\omega)(0)\\ &\quad + \varepsilon\int_0^t \big[\nu(\mathsf{I}_\Omega\tilde f,\Delta\tilde\omega) + (\partial_\tau\mathsf{I}_\Omega\tilde f,\Delta\tilde\omega) + (\mathsf{I}_\Omega\tilde f,\Delta\partial_\tau^*\tilde\omega)\big]\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau,\end{aligned} \tag{3.34}$$

which can be bounded as in the L^2 case, giving

$$\begin{aligned}-2\int_0^t (\tilde f,\Delta\tilde\omega)\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau &\le \varepsilon c\,\big[|\Delta\tilde f|\,|\Delta\tilde\omega|\big](t)\,\mathrm{e}^{\nu t} + \varepsilon c\,\big[|\Delta\tilde f|\,|\Delta\tilde\omega|\big](0)\\ &\quad + \varepsilon c\int_0^t \big\{(1+\mu)\,|\nabla^3\omega|^2 + \mu\,|\nabla^3\tilde f|^2 + |\nabla\partial_\tau\tilde f|^2 + |\nabla^3\omega|\,|\nabla\omega|\,|\Delta\tilde f|\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.\end{aligned} \tag{3.35}$$

For the nonlinear term, we use the fact that $B(\bar\omega,\bar\omega) = 0$ to write

$$(B(\omega,\omega), A\tilde\omega) = (B(\bar\omega,\tilde\omega), A\tilde\omega) + (B(\tilde\omega,\bar\omega), A\tilde\omega) + (B(\tilde\omega,\tilde\omega), A\tilde\omega), \tag{3.36}$$

and, using $(B(\omega^\sharp,\tilde\omega), A\tilde\omega) = (B(\nabla\omega^\sharp,\tilde\omega),\nabla\tilde\omega)$, previously used in (2.8), we bound

$$\begin{aligned}\big|(B(\tilde\omega,\tilde\omega), A\tilde\omega)\big| &= \big|(B(\nabla\tilde\omega,\tilde\omega),\nabla\tilde\omega)\big| \le c\,|\tilde\omega|_{L^2}|\nabla\tilde\omega|_{L^4}^2 \le c\,|\tilde\omega|\,|\nabla\tilde\omega|\,|\Delta\tilde\omega|\\ &\le \frac{\mu}{4}\,|\Delta\tilde\omega|^2 + \frac{c}{\mu}\,|\tilde\omega|^2|\nabla\tilde\omega|^2\\ \big|(B(\bar\omega,\tilde\omega), A\tilde\omega)\big| &= \big|(B(\bar\omega',\tilde\omega),\partial_y\tilde\omega)\big| \le c\,|\bar\omega|_{L^\infty}|\nabla\tilde\omega|_{L^2}^2\\ \big|(B(\tilde\omega,\bar\omega), A\tilde\omega)\big| &\le |\Delta\tilde\omega|_{L^2}|\nabla^{-1}\tilde\omega|_{L^\infty}|\bar\omega'|_{L^2} \le |\Delta\tilde\omega|_{L^2}|\nabla\tilde\omega|_{L^2}|\bar\omega'|_{L^2}\\ &\le \frac{\mu}{4}\,|\Delta\tilde\omega|^2 + \frac{c}{\mu}\,|\nabla\tilde\omega|^2|\bar\omega'|^2.\end{aligned} \tag{3.37}$$

Using Poincaré inequality on the last term in (3.37c), we obtain

$$
\begin{aligned}
(3.38)\quad & 2\int_0^t (B(\omega,\omega),\Delta\tilde\omega)\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau \\
&\qquad \le c_2(\mu)\int_0^t \Big\{\big[|\bar\omega|_{L^\infty} + |\nabla\omega|^2\big]\,|\nabla\tilde\omega|^2 + \frac{\mu}{2}\,|\Delta\tilde\omega|^2\Big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.
\end{aligned}
$$

After moving the $|\Delta\tilde\omega|^2$ to the left-hand side, a factor of ε can be obtained by pulling the square bracket outside the integral and using (3.9b). Collecting, we have

$$
\begin{aligned}
(3.39)\quad & |\nabla\tilde\omega(t)|^2 + \frac{\mu}{2}\int_0^t \mathrm{e}^{\nu(\tau-t)}|\Delta\tilde\omega|^2\,\mathrm{d}\tau \le \mathrm{e}^{-\nu t}|\nabla\tilde\omega(0)|^2 + c\varepsilon\sup_{t'>0}|\Delta\tilde f(t')|\,|\Delta\omega(t')| \\
&\quad + c\varepsilon\int_0^t \big\{(1+\mu)\big(|\nabla^3\omega|^2 + |\nabla^3\tilde f|^2\big) + |\nabla\partial_\tau\tilde f|^2 + |\nabla\omega|^2|\Delta\tilde f|^2\big\}\,\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau \\
&\quad + \varepsilon\,c_3(\mu)\,M_0\sup_{t'>0}\big\{|\bar\omega(t')|_{L^\infty} + |\nabla\omega(t')|^2\big\}.
\end{aligned}
$$

Arguing as in the L^2 case, $f\in L^\infty_t H^3_{\boldsymbol{x}}$ and $\partial_t\tilde f\in L^\infty_t H^1_{\boldsymbol{x}}$ gives us an $\mathsf{O}(\sqrt{\varepsilon})$ bound for $\tilde\omega(t)$ in $L^\infty_t H^1_{\boldsymbol{x}}$ uniform for large t.

Bounds in H^s can now be obtained inductively. Assuming that Theorem 2 below holds for $s-1$ (we just showed that it does for $s=2$), we multiply (1.8) by $A^s\tilde\omega$ and integrate the resulting equation in time as above. We bound the nonlinear term $(B(\omega,\omega),A^s\tilde\omega) = (B(\tilde\omega,\tilde\omega),A^s\tilde\omega) + (B(\bar\omega,\tilde\omega),A^s\tilde\omega) + (B(\tilde\omega,\bar\omega),A^s\tilde\omega)$ as follows. The first term is bounded exactly as in (2.8)–(2.9),

$$
(3.40)\quad \big|(B(\tilde\omega,\tilde\omega),A^s\tilde\omega)\big| \le \frac{\mu}{4}\,|\nabla^{s+1}\tilde\omega|^2 + \frac{c(s)}{\mu}\,|\tilde\omega|^2|\nabla^s\tilde\omega|^2.
$$

We bound the next term by [cf. (2.8)], with $|\alpha| = s$ and $1\le|\beta| = r\le s$,

$$
\begin{aligned}
(3.41)\quad \big|(B(\bar\omega,\tilde\omega),A^s\tilde\omega)\big| &\le c\textstyle\sum_{\alpha\beta}|\mathsf{D}^\beta\bar{\boldsymbol{v}}|_{L^\infty}|\mathsf{D}^{\alpha-\beta}\nabla\tilde\omega|_{L^2}|\mathsf{D}^\alpha\tilde\omega|_{L^2} \\
&\le c(s)\textstyle\sum_{r=1}^s|\nabla^r\bar\omega|\,|\nabla^{s-r+1}\tilde\omega|\,|\nabla^s\tilde\omega| \\
&\le c(s)\,|\nabla^s\bar\omega|\,|\nabla^s\tilde\omega|^2.
\end{aligned}
$$

Finally, we bound the last term as, where now $0\le|\beta| = r\le s = |\alpha|$,

$$
\begin{aligned}
(3.42)\quad & \big|(B(\tilde\omega,\bar\omega),A^s\tilde\omega)\big| \le c\textstyle\sum_{\alpha\beta}|\mathsf{D}^\beta\tilde{\boldsymbol{v}}|_{L^4}|\mathsf{D}^{\alpha-\beta}\nabla\bar\omega|_{L^2}|\mathsf{D}^\alpha\tilde\omega|_{L^4} \\
&\quad \le c(s)\textstyle\sum_{r=0}^s|\tilde\omega|_{H^{r-1/2}}\,|\bar\omega|_{H^{s-r+1}}\,|\tilde\omega|_{H^{s+1/2}} \\
&\quad \le \frac{\mu}{4}\,|\nabla^{s+1}\tilde\omega|^2 + c(s,\mu)\textstyle\sum_{r=0}^s|\nabla^{r-1}\tilde\omega|^{2/3}|\nabla^r\tilde\omega|^{2/3}|\nabla^s\tilde\omega|^{2/3}|\nabla^{s-r+1}\bar\omega|^{4/3} \\
&\quad \le \frac{\mu}{4}\,|\nabla^{s+1}\tilde\omega|^2 + c(s,\mu)\,|\nabla^s\tilde\omega|^2\,|\nabla^{s+1}\bar\omega|^{4/3}.
\end{aligned}
$$

Moving the $|\nabla^{s+1}\tilde\omega|^2$ in (3.40) and (3.42) to the left-hand side of the main inequality, the right-hand side depends at most on $|\nabla^s\tilde\omega|^2$, which is of $\mathsf{O}(\varepsilon)$ in L^2_t from step $s-1$, and on $|\nabla^{s+1}\bar\omega|^2$. As before, the worst term (i.e. that requires the highest derivative on f) in fact comes from bounding $(f,A^s\tilde\omega)$.

We summarise our results as:

Theorem 2. *Let the initial data* $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ *and the forcing be bounded as*

$$K_s(f) := [\![\nabla^{s+2} f]\!] + [\![\nabla^s \partial_t f]\!] < \infty. \tag{3.43}$$

Then there exist $\mathcal{T}_s(|\boldsymbol{v}(0)|_{L^2}, K_s; \mu)$ *and* $M_s(K_s; \mu)$ *such that*

$$\begin{aligned} &|\nabla^s \tilde{\omega}(t)|_{L^2}^2 \le \varepsilon\, M_s, \\ &\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau} |\nabla^s \tilde{\omega}(\tau)|^2 \,\mathrm{d}\tau \le \varepsilon\, M_s \end{aligned} \tag{3.44}$$

for all $t \ge \mathcal{T}_s$.

3.3. **Higher-order Bounds.** As in [13], one can obtain bounds that scale as $\varepsilon^{n/2}$ for $\tilde{\omega}$ when the force f is independent of time; see [1].

4. Stability and the Global Attractor

When the forcing f is independent of time, the existence of the global attractor $\mathcal{A}$ follows, just as for the non-rotating 2d Navier–Stokes equations, from the uniform long-time bounds in Section 2, where the planetary rotation does not appear at all. In the non-rotating case, the Hausdorff dimension of $\mathcal{A}$ is bounded by

$$\dim_{\mathrm{H}} \mathcal{A} \le c\, G^{2/3} (1 + \log G)^{1/3} \tag{4.1}$$

where in our notation the Grashof number is

$$G := |\nabla^{-1} f|_{L^2} / \mu^2. \tag{4.2}$$

The rotation not posing any extra essential difficulty, the usual analysis, e.g. [5, §9.2], carries over essentially line-by-line to our case, giving the bound (4.1) also for the rotating case (1.7).

As discussed in the introduction, and following our results that the flow becomes more zonal ("ordered") as $\varepsilon \to 0$, we expect the dimension of the attractor to decrease as $\varepsilon \to 0$. In this section, we use a simple computation similar to that used for Theorem 1 to show that $\dim_{\mathrm{H}} \mathcal{A} = 0$ for ε sufficiently small.

Theorem 3. *Let the forcing* f *be time independent,* $\partial_t f = 0$*, and assume the hypotheses of Theorem 1, i.e.* $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ *and*

$$|\nabla^2 f|_{L^2} < \infty. \tag{4.3}$$

Then there exists an $\varepsilon_*(|\nabla^2 f|; \mu)$ *such that, for all* $\varepsilon < \varepsilon_*$,

$$dim_H\, \mathcal{A} = 0. \tag{4.4}$$

Since $\mathcal{A}$ is connected, (4.4) implies that $\mathcal{A}$ consists of a single point, that is, a steady flow ω_* to which all bounded solutions converge. Following Theorems 1 and 2, this steady flow is nearly, but not exactly, zonal (except in the non-generic case when $\tilde{f} = 0$). Heuristically, an approximation to ω_* is the steady flow

$$\omega_*^{(1)} = -\mu^{-1} \Delta^{-1} \bar{f} + \varepsilon L^{-1} \tilde{f}, \tag{4.5}$$

which satisfies

$$\frac{1}{\varepsilon} L \omega_*^{(1)} + B(\omega_*^{(1)}, \omega_*^{(1)}) + \mu A \omega_*^{(1)} = f \tag{4.6}$$

up to $\mathsf{O}(\varepsilon)$. More careful work would be needed to determine ω_* exactly.

In turbulence parlance, the smallness of ε demanded by Theorem 3 implies that the Rhines scale [14] is so large that it overwhelms the entire spectral range, rendering the dynamics trivial.

A general result related to ours is described in [3, ch. 18], where the trajectory attractor $\mathcal{A}_\epsilon$ of a dynamical system depending on t/ϵ (formally, in our case $\mathcal{A}_\epsilon$ would simply be the attractor $\mathcal{A}$ for $\varepsilon > 0$) converges weakly to the attractor $\mathcal{A}_0$ of the corresponding averaged system. Formally averaging our equations following this construction (which does not apply directly to our case, in which the oscillations have an infinite number of frequencies which accumulate at zero), we obtain purely zonal NSE, whose dynamics is trivial and whose attractor thus has dimension zero. This is of course consistent with our results: strong convergence at finite ε of $\mathcal{A}$ to a point (which becomes zonal as $\varepsilon \to 0$).

Proof. Fix a solution $\omega(t)$ of (1.8) that lives on $\mathcal{A}$, so the bounds (3.44) hold for all t. We consider a nearby solution $\omega(t) + \phi(t)$. The linearised evolution equation for ϕ is then

$$\begin{aligned}\partial_t\phi &= -(\nabla^\perp\Delta^{-1}\omega)\cdot\nabla\phi - (\nabla^\perp\Delta^{-1}\phi)\cdot\nabla\omega(t) - \frac{1}{\varepsilon}\partial_x\Delta^{-1}\phi + \mu\Delta\phi \\ &= -B(\omega,\phi) - B(\phi,\omega) - \frac{1}{\varepsilon}L\phi - \mu A\phi =: \mathcal{L}(t)\phi.\end{aligned} \tag{4.7}$$

Multiplying this by ϕ in L^2 and noting that $(B(\omega,\phi),\phi) = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\phi|^2 + \mu\,|\nabla\phi|^2 &= (B(\phi,\phi),\omega) \\ &= (B(\phi,\phi),\bar\omega) + (B(\phi,\phi),\tilde\omega).\end{aligned} \tag{4.8}$$

For the first term, we split $\phi = \bar\phi + \tilde\phi$ in analogy with $\omega = \bar\omega + \tilde\omega$ to get

$$(B(\phi,\phi),\bar\omega) = (B(\tilde\phi,\tilde\phi),\bar\omega) \tag{4.9}$$

using the (now familiar) facts that $B(\bar\phi,\bar\phi) = 0$ and all tilde-bar-bar terms vanish. Using Poincaré inequality in (4.8) gives us [cf. (3.10)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{\nu t}|\phi|^2\big) + \mu\mathrm{e}^{\nu t}|\nabla\phi|^2 \le 2\,\mathrm{e}^{\nu t}(B(\tilde\phi,\tilde\phi),\bar\omega) + 2\,\mathrm{e}^{\nu t}(B(\phi,\phi),\tilde\omega), \tag{4.10}$$

which integrates to

$$\begin{aligned}&|\phi(t)|^2\,\mathrm{e}^{\nu t} + \mu\int_0^t|\nabla\phi|^2\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau \\ &\qquad\le |\phi(0)|^2 + 2\int_0^t\big\{(B(\tilde\phi,\tilde\phi),\bar\omega) + (B(\phi,\phi),\tilde\omega)\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.\end{aligned} \tag{4.11}$$

We bound the last term of the integrand using

$$\begin{aligned}(B(\phi,\phi),\tilde\omega) &\le c\,|\nabla^{-1}\phi|_{L^\infty}|\nabla\phi|_{L^2}|\tilde\omega|_{L^2} \\ &\le c_4\,|\nabla\phi|^2|\tilde\omega|_{L^2}.\end{aligned} \tag{4.12}$$

The other term needs to be integrated by parts,

$$\begin{aligned}\int_0^t(B(\tilde\phi,\tilde\phi),\bar\omega)\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau &= \varepsilon\,(B_\Omega(\tilde\phi,\tilde\phi),\bar\omega)(t)\,\mathrm{e}^{\nu t} - \varepsilon\,(B_\Omega(\tilde\phi,\tilde\phi),\bar\omega)(0) \\ &\quad - \varepsilon\int_0^t\big\{\nu\,(B_\Omega(\tilde\phi,\tilde\phi),\bar\omega) + (B_\Omega(\tilde\phi,\tilde\phi),\partial_\tau\bar\omega) + 2\,(B_\Omega(\partial_\tau^*\tilde\phi,\tilde\phi),\bar\omega)\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\end{aligned} \tag{4.13}$$

where $\partial_t^* \phi = -B(\omega,\phi) - B(\phi,\omega) - \mu A\phi$. We bound the endpoint terms using

$$2\left|(B_\Omega(\tilde\phi,\tilde\phi),\bar\omega)\right| \le c_5 |\tilde\phi|^2 |\bar\omega'|_{L^\infty}. \tag{4.14}$$

It remains to bound the integrand in (4.13):

$$\begin{aligned}\left|(B_\Omega(\tilde\phi,\tilde\phi),\bar\omega)\right| &\le c\,|\partial_y\tilde\phi|_{L^2}|\tilde\phi|_{L^4}|\bar\omega|_{L^4}\\ &\le c\,|\nabla\tilde\phi|^2|\bar\omega|_{L^4}\end{aligned} \tag{4.15}$$

$$\begin{aligned}\left|(B_\Omega(\tilde\phi,\tilde\phi),\partial_t\bar\omega)\right| &\le c\,|\partial_y\tilde\phi|_{L^2}|\tilde\phi|_{L^{10}}|\partial_t\bar\omega|_{L^{5/2}}\\ &\le c\,|\nabla\tilde\phi|^2|\partial_t\bar\omega|_{L^{5/2}}\end{aligned} \tag{4.16}$$

Recalling (4.7) for the last term in (4.13), we bound

$$\begin{aligned}\left|(B_\Omega(\tilde B(\phi,\omega),\tilde\phi),\bar\omega)\right| &\le c\,|\nabla^{-1}\phi|_{L^\infty}|\nabla\omega|_{L^2}|\tilde\phi|_{L^{10}}|\bar\omega'|_{L^{5/2}}\\ &\le c\,|\nabla\phi|^2|\bar\omega'|_{L^{5/2}}|\nabla\omega|_{L^2}\\ \left|(B_\Omega(\tilde B(\omega,\phi),\tilde\phi),\bar\omega)\right| &\le c\,|\nabla^{-1}\omega|_{L^\infty}|\nabla\phi|_{L^2}|\tilde\phi|_{L^{10}}|\bar\omega'|_{L^{5/2}}\\ &\le c\,|\nabla\phi|^2|\bar\omega'|_{L^{5/2}}|\nabla\omega|_{L^2}\\ \left|(B_\Omega(\Delta\tilde\phi,\tilde\phi),\bar\omega)\right| &= \left|(B_\Omega(\nabla\tilde\phi,\nabla\tilde\phi),\bar\omega)\right| + \left|(B_\Omega(\partial_y\tilde\phi,\tilde\phi),\bar\omega')\right|\\ &\le c\,|\nabla\tilde\phi|^2|\bar\omega'|_{L^\infty} + c\,|\nabla\tilde\phi|_{L^2}|\tilde\phi|_{L^{10}}|\bar\omega''|_{L^{5/2}}\\ &\le c\,|\nabla\tilde\phi|^2|\bar\omega''|_{L^{5/2}}.\end{aligned} \tag{4.17}$$

Collecting, (4.11) now implies

$$\begin{aligned}|\phi(t)|^2\bigl(1-\varepsilon c_5|\bar\omega'(t)|_{L^\infty}\bigr) + \int_0^t \bigl\{\mu - \varepsilon\,N(\tau) - c_4|\tilde\omega(\tau)|_{L^2}\bigr\}\,|\nabla\tilde\phi|^2\,\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau\\ \le \mathrm{e}^{-\nu t}|\phi(0)|^2\bigl(1+\varepsilon c_5|\bar\omega'(0)|_{L^\infty}\bigr)\end{aligned} \tag{4.18}$$

where

$$N(t) := c_6\bigl\{\mu\,|\bar\omega''|_{L^{5/2}} + |\bar\omega'|_{L^{5/2}}|\nabla\omega|_{L^2} + |\partial_t\bar\omega|_{L^{5/2}} + |\bar\omega|_{L^4}\bigr\}(t). \tag{4.19}$$

By (2.11), $f \in H^2$ implies that $\omega \in H^3$ with uniform bound in t since we are already on the attractor, and by Theorem 1 we can find an ε_* so small that, for $\varepsilon < \varepsilon_*$,

$$\sup_{t>0}\bigl\{\varepsilon N(t) + c_4|\tilde\omega(t)|_{L^2}\bigr\} < \mu. \tag{4.20}$$

Requiring furthermore that ε_* also implies, for all $\varepsilon < \varepsilon_*$,

$$\varepsilon c_5 \sup_{t>0} |\bar\omega'(t)|_{L^\infty} < 1. \tag{4.21}$$

These and (4.18) then imply that

$$|\phi(t)|^2 \le C(\cdots)\,\mathrm{e}^{-\nu t}|\phi(0)|^2, \tag{4.22}$$

in other words, all phase space volumes contract and thus the global attractor has dimension zero.

It is clear from the above proof that our solution $\omega(t)$ is linearly stable. Since (4.7) only differs by $B(\phi,\phi)$ from the nonlinear system, the fact that $(B(\phi,\phi),\phi) = 0$ implies that stability also holds under the same hypotheses for the full nonlinear system.

Appendix A. An L^∞ Inequality

Lemma 1. *Let u and $v \in H^2(\mathcal{M})$ have zero integrals and are L^2 orthogonal,*

$$(u, v)_{L^2} = 0, \tag{A.1}$$

and let $w = u + v$. Then the following Agmon inequality holds,

$$|u|_{L^\infty} \le c\,|\nabla w| \Big(\log \frac{|\Delta w|}{c_0|\nabla w|} + 1\Big)^{1/2}. \tag{A.2}$$

Before the proof, we note that the interpolation inequality

$$|\nabla w|^2 \le c_9\,|w|\,|\Delta w| \tag{A.3}$$

can be written as

$$\begin{aligned} & 2\log|\nabla w| \le \log|w| + \log|\Delta w| + \log c_9 \\ \Leftrightarrow\quad & \log|\nabla w| - \log|w| \le \log|\Delta w| - \log|\nabla w| + \log c_9 \\ \Leftrightarrow\quad & \log\frac{|\nabla w|}{c_0|w|} \le \log\frac{|\Delta w|}{c_0|\nabla w|} + \log c_9, \end{aligned} \tag{A.4}$$

which can be used to simplify, e.g., $|w|_{L^\infty}|\nabla w|_{L^\infty}$ when bounded using (A.2).

Proof. For most of this proof, up to (A.7) below, we follow [5, Lemma 7.1] exactly. For conciseness, we put $L_1 = L_2 = 1$ but keep the Poincaré constant c_0. With $\kappa > 0$, we expand u in Fourier series

$$u(\boldsymbol{x}) = \textstyle\sum_{|\boldsymbol{k}|<\kappa} u_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + \sum_{|\boldsymbol{k}|\ge\kappa} u_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} =: u^<(\boldsymbol{x}) + u^>(\boldsymbol{x}), \tag{A.5}$$

and analogously for v and w. Then

$$\begin{aligned} |u|_{L^\infty} &= \sup_{\boldsymbol{x}} \big|\textstyle\sum_{\boldsymbol{k}} u_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\big| \le \sum_{|\boldsymbol{k}|<\kappa} |u_{\boldsymbol{k}}| + \sum_{|\boldsymbol{k}|\ge\kappa} |u_{\boldsymbol{k}}| \\ &=: \textstyle\sum^< |\boldsymbol{k}|^{-1}|\boldsymbol{k}|\,|u_{\boldsymbol{k}}| + \sum^> |\boldsymbol{k}|^{-2}|\boldsymbol{k}|^2|u_{\boldsymbol{k}}| \\ &\le \Big(\textstyle\sum^< |\boldsymbol{k}|^{-2}\Big)^{1/2}\Big(\sum^< |\boldsymbol{k}|^2|u_{\boldsymbol{k}}|^2\Big)^{1/2} \\ &\qquad + \Big(\textstyle\sum^> |\boldsymbol{k}|^{-4}\Big)^{1/2}\Big(\sum^> |\boldsymbol{k}|^4|u_{\boldsymbol{k}}|^2\Big)^{1/2} \end{aligned} \tag{A.6}$$

Now on the right-hand side, $\sum^< |\boldsymbol{k}|^{-2} \le c\,\log\kappa$ and $\sum^> |\boldsymbol{k}|^{-4} \le c/\kappa^2$, so fixing

$$\kappa = |\Delta w|/(c_0|\nabla w|), \tag{A.7}$$

the lemma follows from

$$\begin{aligned} |u|_{L^\infty} &\le c\,|\nabla u^<|\Big(\log\frac{|\Delta w|}{c_0|\nabla w|}\Big)^{1/2} + c\,|\Delta u^>|\,\frac{|\nabla w|}{|\Delta w|} \\ &\le c\,|\nabla w|\Big(\log\frac{|\Delta w|}{c_0|\nabla w|}\Big)^{1/2} + c\,|\nabla w|. \end{aligned} \tag{A.8}$$

References

[1] M. A. H. Al-Jaboori, *T.B.D.*, Ph.D. thesis, University of Durham, 2011

[2] A. Babin, A. Mahalov, and B. Nicolaenko, *Regularity and integrability of 3d Euler and Navier–Stokes equations*, Asymp. Anal., 15 (1997), pp. 103–150.

[3] V. V. Chepyzhov and M. I. Vishik, *Attractors for equations of mathematical physics*, American Math. Soc., 2002.

[4] P. Constantin, C. Foias, and R. Temam, *On the dimension of the attractors in two-dimensional turbulence*, Physica, D 30 (1988), pp. 284–296.

[5] C. R. Doering and J. D. Gibbon, *Applied analysis of the Navier–Stokes equations*, Cambridge Univ. Press, 1995.
[6] I. Gallagher and L. Saint-Raymond, *Mathematical study of the betaplane model: equatorial waves and convergence results*, Mém. Soc. Math. France, 107 (2006), pp. vi+116 pp.
[7] P. B. Rhines, *Waves and turbulence on a beta-plane*, J. Fluid Mech., 69 (1975), pp. 417–443.
[8] J. C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge Univ. Press, 2001.
[9] S. Schochet, *Fast singular limits of hyperbolic PDEs*, J. Diff. Eq., 114 (1994), pp. 476–512.
[10] T. G. Shepherd, *Non-ergodicity of inviscid two-dimensional flow on a beta-plane and on the surface of a rotating sphere*, J. Fluid Mech., 184 (1987), pp. 289–302.
[11] R. Temam, *Navier–Stokes equations and nonlinear functional analysis*, SIAM, 2 ed., 1995.
[12] ———, *Infinite-dimensional dynamical systems in mechanics and physics, 2ed*, Springer-Verlag, 1997.
[13] R. Temam and D. Wirosoetisno, *Stability of the slow manifold in the primitive equations*, SIAM J. Math. Anal., 42 (2010), pp. 427–458. arXiv:0808.2878.
[14] G. K. Vallis, *Atmospheric and oceanic fluid dynamics*, Cambridge Univ. Press, 2006.
[15] G. K. Vallis and M. E. Maltrud, *Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence*, J. Fluid Mech., 228 (1991), pp. 321–342.

E-mail address: mustafa.hussain@durham.ac.uk

E-mail address: djoko.wirosoetisno@durham.ac.uk
URL: http://www.maths.dur.ac.uk/~dma0dw

Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom