On one Laurent series ring over an extension of $\mathbb{Q}$

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September 24, 2010

Abstract

In this paper, using the general Mal'cev-Neumann construction of Laurent series rings, we construct a ring with a base ring which is an extension of the field $\mathbb{Q}$ of rational numbers. Further, we establish some useful properties of such a ring and as direct consequences, we obtain the negative answers to five problems arising from the work [3].

Key words: Laurent series ring.
Mathematics Subject Classification 2010: 16K20

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In this short note, we are interesting in the construction of some special division ring D which is not algebraic over its center F. However, D contains some maximal subfield K, algebraic over F. Such a division ring can be taken as a counterexample for some questions, arising from the work [3].

1 The construction of a ring $K((G, \Phi))$

In this section, following the general Mal'cev-Neumann construction of Laurent series rings, we construct a ring with a base ring which is an extension of the field $\mathbb{Q}$ of rational numbers. Thus, let us denote by $G = \mathbb{Z}^\infty$ the set of all infinite sequences of integers of the form $(n_1, n_2, n_3, \ldots)$ with only finitely many non-zeros n_i. Clearly G is an abelian group with the addition defined by the obvious way. For any positive integer i, denote by $x_i = (0, \ldots, 0, 1, 0, \ldots)$ the element of G with 1 in the i-th position and 0 elsewhere. Then G is a free abelian group generated by all x_i and every element $x \in G$ is written uniquely in the form

$$x = \sum_{i \in I} n_i x_i, \tag{1}$$

with $n_i \in \mathbb{Z}$ and some finite set I.

Now, we define an order in G as the following:

For elements $x = (n_1, n_2, n_3, \ldots)$ and $y = (m_1, m_2, m_3, \ldots)$ in G, define $x < y$ if either $n_1 < m_1$ or there exists $k \in \mathbb{N}$ such that $n_1 = m_1, \ldots, n_k = m_k$ and $n_{k+1} < m_{k+1}$. Clearly, with this order G is a totally ordered set.

Suppose that $p_1 < p_2 < \ldots < p_n < \ldots$ is a sequence of prime numbers and $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots)$ is the subfield of the field $\mathbb{R}$ of real numbers generated by $\mathbb{Q}$ and $\sqrt{p_1}, \sqrt{p_2}, \ldots$, where $\mathbb{Q}$ is the field of rational numbers. For any $i \in \mathbb{N}$, suppose that $f_i : K \longrightarrow K$ is $\mathbb{Q}$-isomorphism satisfying the following condition:

$$f_i(\sqrt{p_j}) = \begin{cases} \sqrt{p_j}, & \text{if } j \neq i; \\ -\sqrt{p_i}, & \text{if } j = i. \end{cases}$$

It is easy to verify that $f_i f_j = f_j f_i, \forall i, j \in \mathbb{N}$. Moreover, we have the following lemma:

Lemma 1.1 *Suppose that $x \in K$. Then, $f_i(x) = x, \forall i \in \mathbb{N}$ if and only if $x \in \mathbb{Q}$.*

Proof. The converse is obvious. Now, suppose that $x \in K$ such that $f_i(x) = x, \forall i \in \mathbb{N}$. If $x \notin \mathbb{Q}$, then there exists $i \in \mathbb{N}$ such that x can be written in the form

$$x = a + b\sqrt{p_i},$$

where $a, b \in K, b \neq 0$ and $\sqrt{p_i}$ does not appear in the formal expressions of a and b. Therefore $0 = x - f_i(x) = 2b\sqrt{p_i}$ that is a contradiction. Hence $x \in \mathbb{Q}$. ■

For an element $x = (n_1, n_2, ...) = \sum_{i\in I} n_i x_i \in G$, define $\Phi_x := \prod_{i\in I} f_i^{n_i}$. Clearly $\Phi_x \in Gal(K/\mathbb{Q})$ and the map

$$\Phi : G \longrightarrow Gal(K/\mathbb{Q}),$$

defined by $\Phi(x) = \Phi_x$ is a group homomorphism. It is easy to prove the following proposition:

Proposition 1.1 i) $\Phi(x_i) = f_i, \forall i \in \mathbb{N}$.

ii) *If* $x = (n_1, n_2, \ldots) \in G$, *then* $\Phi_x(\sqrt{p_i}) = (-1)^{n_i}\sqrt{p_i}$.

For the convenience, from now on we write the operation in G multiplicatively. For G and K as above, consider formal sums of the form

$$\alpha = \sum_{x\in G} a_x x, a_x \in K.$$

For such an α, define the support of α by $supp(\alpha) = \{x \in G : a_x \neq 1\}$. Put

$$D = K((G, \Phi)) = \Big\{\alpha = \sum_{x\in G} a_x x, a_x \in K \mid supp(\alpha) \text{ is well-ordered } \Big\}.$$

For $\alpha = \sum_{x\in G} a_x x$ and $\beta = \sum_{x\in G} b_x x$ from D, define

$$\begin{aligned} \alpha + \beta &= \sum_{x\in G} (a_x + b_x)x; \\ \alpha.\beta &= \sum_{z\in G} \Big(\sum_{xy=z} a_x \Phi_x(b_y) \Big) z. \end{aligned}$$

In [[2], p.243], it is proved that these operations are well-defined. Moreover, the following theorem holds:

Theorem 1.1 ([[2], Th.(14.21), p.244]) $D = K((G, \Phi))$ *with the operations, defined as above is a division ring.*

Remarks. i) For any $x \in G, a \in K$, we have $xa = \Phi_x(a)x$.

ii) For any $i \neq j$, we have $x_i\sqrt{p_i} = -\sqrt{p_i}x_i$ and $x_j\sqrt{p_i} = \sqrt{p_i}x_j$.

iii) Generally, $\forall i \neq j$ and $\forall n \in \mathbb{N}$, we have $x_i^n\sqrt{p_i} = (-1)^n\sqrt{p_i}x_i^n$ and $x_j^n\sqrt{p_i} = \sqrt{p_i}x_j^n$.

Put $H = \{x^2 : x \in G\}$ and

$$\mathbb{Q}((H)) = \Big\{\alpha = \sum_{x\in H} a_x x, a_x \in \mathbb{Q}, supp(\alpha) \text{ is well-ordered } \Big\}.$$

It is easy to check that H is a subgroup of G and for every $x \in H$, $\Phi_x = Id_K$.

Theorem 1.2 *$\mathbb{Q}((H))$ is the center of D.*

Proof. Denote by F the center of D. Suppose that $\alpha = \sum_{x\in H} a_x x \in \mathbb{Q}((H))$. Then, for every $\beta = \sum_{y\in G} b_y y \in D$, we have $\Phi_x(b_y) = b_y$ and $\Phi_y(a_x) = a_x$. Hence

$$\begin{aligned} \alpha.\beta &= \sum_{z\in G}\Big(\sum_{xy=z} a_x\Phi_x(b_y)\Big)z = \sum_{z\in G}\Big(\sum_{xy=z} a_x b_y\Big)z, \\ \beta.\alpha &= \sum_{z\in G}\Big(\sum_{xy=z} b_y\Phi_y(a_x)\Big)z = \sum_{z\in G}\Big(\sum_{xy=z} a_x b_y\Big)z. \end{aligned}$$

Thus, $\alpha\beta = \beta\alpha, \forall \beta \in D$. Therefore $\alpha \in F$.

Conversely, suppose that $\alpha = \sum_{x\in G} a_x x \in F$. Denote by S the set of all elements x appeared in the expression of α. Then, it suffices to prove that $x \in H$ and $a_x \in \mathbb{Q}, \forall x \in S$. In fact, since $\alpha \in F, \forall i \geq 1$, we have

$$\begin{cases} \sqrt{p_i}\alpha &= \alpha\sqrt{p_i}, \\ \alpha x_i &= x_i\alpha; \end{cases}$$

i.e.

$$\begin{cases} \sum_{x\in S} \sqrt{p_i}a_x x &= \sum_{x\in S} \Phi_x(\sqrt{p_i})a_x x, \\ \sum_{x\in S} a_x(xx_i) &= \sum_{x\in S} \Phi_{x_i}(a_x)(x_i x). \end{cases}$$

Therefore, $\forall x = (n_1, n_2, \ldots) \in S$, we have

$$\begin{cases} \sqrt{p_i}a_x &= \Phi_x(\sqrt{p_i})a_x = (-1)^{n_i}\sqrt{p_i}a_x \text{ (by Proposition 1.1)}, \\ a_x &= \Phi_{x_i}(a_x) = f_i(a_x). \end{cases}$$

From the first equality it follows that n_i is even for any $i \geq 1$. Therefore $x \in H$. From the second equality it follows that $a_x = f_i(a_x)$ for any $i \geq 1$. So by Lemma 1.1, we have $a_x \in \mathbb{Q}$. Therefore $\alpha \in \mathbb{Q}((H))$. ■

2 Some properties of $K((G, \Phi))$

In the precedent section we have constructed the division ring $D = K((G, \Phi))$ with the center $F = \mathbb{Q}((H))$. In this section we investigate the properties of $D = K((G, \Phi))$. Further, using these properties we give the negative answers for five problems arising from the work [3].

Theorem 2.1 *The division ring D is not algebraic over its center F.*

Proof. Suppose that $\alpha = x_1^{-1} + x_2^{-1} + \ldots$ is an infinite sum. Since $x_1^{-1} < x_2^{-1} < \ldots$, $supp(\alpha)$ is well-ordered. Hence $\alpha \in D$. Consider the equality

$$a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n = 0, \quad a_i \in F. \tag{2}$$

Note that $X = x_1^{-1}x_2^{-1}...x_n^{-1}$ does not appear in the expressions of $\alpha, \alpha^2, ..., \alpha^{n-1}$ and the coefficient of X in the expression of α^n is $n!$. Therefore, the coefficient of X in the expression on left side of the equality (2) is $a_n.n!$. It follows that $a_n = 0$. By induction, it is easy to see that $a_0 = a_1 = \ldots = a_n = 0$. Hence, for any $n \in \mathbb{N}$, the set $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ is independent over F. Consequently, α is not algebraic over F. ■

Denote by $K_\infty = F(\sqrt{p_1}, \sqrt{p_2}, \ldots)$ the subfield of D generated by $\sqrt{p_1}, \sqrt{p_2}, \ldots$ over F and for any $n \geq 1$ denote by $L_n := F(\sqrt{p_1}, \ldots, \sqrt{p_n}, x_1, \ldots, x_n)$. Then, $L_n \subseteq L_{n+1}$ and $L_\infty := \bigcup_{n=1}^{\infty} L_n$ is the division subring generated by all $\sqrt{p_i}$ and all x_i over F.

The following theorem gives the negative answers for problems 30, 31 and 32 in [3].

Theorem 2.2 *K_∞ is a maximal subfield of D, algebraic, separable over F and it is not a simple extension of F.*

Proof. In view of [[2], Prop. (15.7),p.254], we have to only prove that $C_D(K_\infty) = K_\infty$. Thus, suppose that $\alpha \in C_D(K_\infty) \setminus K_\infty$. Then, there exists some i such that x_i appears in the expression of α as a formal sum. Since $x_i^2 \in F$, α can be expressed in the form $\alpha = \beta x_i + \gamma$, where $\beta \neq 0$ and x_i does not appear in the formal expressions of β and γ. Therefore, $\sqrt{p_i}\alpha - \alpha\sqrt{p_i} = 2\beta\sqrt{p_i}x_i \neq 0$. It follows that α does not commute with $\sqrt{p_i} \in K_\infty$ that is a contradiction. Hence, K_∞ is a maximal subfield of D.

Now, for any $n \geq 1$, put $K_n = F(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$. Clearly, K_n is a field and $[K_{n+1} : K_n] = 2$. Therefore, $[K_n : F] = 2^n$ and $[K_\infty : F] = \infty$. Moreover, $K_\infty = F(\sqrt{p_1}, \sqrt{p_2}, \ldots) = \bigcup_{n=1}^{\infty} K_n$. Hence, for any $c \in K_\infty$, there exists some $n \in \mathbb{N}$ such that $c \in K_n$. Consequently

$$[F(c) : F] \leq [K_n : F] = 2^n.$$

It follows that $K_\infty \neq F(c)$ and K_∞ is an algebraic extension of F. Since $\mathbb{Q} \subseteq F, K_\infty$ is separable over F. ■

Lemma 2.1 i) $[L_n : F] = 2^{2n}$.

ii) *For any $\alpha \in L_n$, we have $\alpha x_{n+1} = x_{n+1}\alpha$.*

iii) $x_{n+1} \notin L_n$.

Proof. i) Put $S_n = \{\sqrt{p_1}, ..., \sqrt{p_n}, x_1, ..., x_n\}$. Since for any $i \neq j$, $x_i^2, (\sqrt{p_i})^2 \in F$, $x_i x_j = x_j x_i$, $\sqrt{p_i}\sqrt{p_j} = \sqrt{p_j}\sqrt{p_i}$, $x_i\sqrt{p_j} = \sqrt{p_j}x_i$, $x_i\sqrt{p_i} = -\sqrt{p_i}x_i$, every element from $F[S_n]$ can be expressed in the form

$$\alpha = \sum_{0\leq \varepsilon_i,\mu_i\leq 1} a_{(\varepsilon_1,...,\varepsilon_n,\mu_1,...,\mu_n)}(\sqrt{p_1})^{\varepsilon_1}\dots(\sqrt{p_n})^{\varepsilon_n}x_1^{\mu_1}\dots x_n^{\mu_n}, \quad a_{(\varepsilon_1,...,\varepsilon_n,\mu_1,...,\mu_n)} \in F. \qquad (3)$$

Moreover, the set $\mathcal{B}_n$ consits of products $(\sqrt{p_1})^{\varepsilon_1}\dots(\sqrt{p_n})^{\varepsilon_n}x_1^{\mu_1}\dots x_n^{\mu_n}, 0 \leq \varepsilon_i, \mu_i \leq 1$ is finite of 2^{2n} elements. Hence, $F[S_n]$ is a finite dimensional vector space over F. So, by [[1], Lemma 2.3], we have $F[S_n] = F(S_n) = L_n$. Therefore, every element from L_n can be expressed in the form (3). Moreover, it is easy to prove by induction that the set $\mathcal{B}_n$ is linearly independent. Therefore, $\mathcal{B}_n$ is a basis of L_n over F and $[L_n : F] = 2^{2n}$.

ii) Since x_{n+1} commutes with every element of the form (3), x_{n+1} commutes with every element $\alpha \in L_n$.

iii) If $x_{n+1} \in L_n$, then in view of ii) we have $x_{n+1} \in Z(L_n) = F$, that is impossible. Therefore, $x_{n+1} \notin L_n$. ∎

Theorem 2.3 *For any $n \geq 1$, we have $Z(L_n) = Z(L_\infty) = F$.*

Proof. In the first, we show that $Z(L_1) = F$. Thus, suppose that $\alpha \in Z(L_1)$. Since $x_1^2, (\sqrt{p_1})^2 = p_1 \in F$ and $x_1\sqrt{p_1} = -\sqrt{p_1}x_1$, every element $\alpha \in L_1 = F(\sqrt{p_1}, x_1)$ can be expressed in the following form:

$$\alpha = a + b\sqrt{p_1} + cx_1 + d\sqrt{p_1}x_1, \quad a, b, c, d \in F.$$

Since α commutes with x_1 and $\sqrt{p_1}$, we have

$$ax_1 + b\sqrt{p_1}x_1 + cx_1^2 + d\sqrt{p_1}x_1^2 = ax_1 - b\sqrt{p_1}x_1 + cx_1^2 - d\sqrt{p_1}x_1^2,$$

and

$$a\sqrt{p_1} - c\sqrt{p_1}x_1 = a\sqrt{p_1} + c\sqrt{p_1}x_1.$$

From the first equality it follows that $b = d = 0$, while from the second equality we obtain $c = 0$. Hence, $\alpha = a \in F$ and consequently, $Z(L_1) = F$.

Suppose that $n \geq 1$ and $\alpha \in Z(L_n)$. By (3), α can be expressed in the form

$$\alpha = a_1 + a_2\sqrt{p_n} + a_3x_n + a_4\sqrt{p_n}x_n, \text{ with } a_1, a_2, a_3, a_4 \in L_{n-1}.$$

From the equality $\alpha x_n = x_n\alpha$, it follows that

$$a_1x_n + a_2\sqrt{p_n}x_n + a_3x_n^2 + a_4\sqrt{p_n}x_n^2 = a_1x_n - a_2\sqrt{p_n}x_n + a_3x_n^2 - a_4\sqrt{p_n}x_n^2.$$

Therefore, $a_2+a_4x_n=0$ and consequently we have $a_2=a_4=0$. Now, from the equality $\alpha\sqrt{p_n}=\sqrt{p_n}\alpha$, we have $a_1\sqrt{p_n}-a_3\sqrt{p_n}x_n=a_1\sqrt{p_n}+a_3\sqrt{p_n}x_n$ and it follows that $a_3=0$. Therefore, $\alpha=a_1\in L_{n-1}$ and this means that $\alpha\in Z(L_{n-1})$. Thus, we have proved that $Z(L_n)\subseteq Z(L_{n-1})$. By induction we can conclude that $Z(L_n)\subseteq Z(L_1), \forall n\geq 1$. Since $F\subseteq Z(L_n)\subseteq Z(L_1)=F$, it follows that $Z(L_n)=F, \forall n\geq 1$.

Now, suppose that $\alpha\in Z(L_\infty)$. Then, there exists some n such that $\alpha\in L_n$ and clearly $\alpha\in Z(L_n)=F$. Hence $Z(L_\infty)=F$. ■

The following theorem gives the negative answers to the problems 28 and 29 in [3]:

Theorem 2.4 *The division ring L_∞ contains no maximal subfields that are simple extensions over its center.*

Proof. As we have proved above, $Z(L_\infty)=F$. Now, suppose that there exists some element $c\in L_\infty$ such that $F(c)$ is a maximal subfield of L_∞. Since $L_\infty=\bigcup\limits_{n=1}^{\infty}L_n$, there exists some n such that $c\in L_n$. Therefore, $F(c)\subseteq L_n$. By (3), $x_{n+1}\notin L_n$ and since c commutes with x_{n+1}, $F(c,x_{n+1})$ is a subfield of L_∞ that strictly contains $F(c)$. This contradiction completes the proof of the theorem. ■

Remark. Since $K_\infty\subseteq L_\infty, K_\infty$ is a maximal subfield of L_∞. Moreover, it is easy to see that L_∞ is a locally finite over F. In particular, L_∞ is algebraic over F and consequently K_∞ is algebraic over F (we have proved this fact in Theorem 2.2 above). Recall that the center of L_∞ is F, so L_∞ is a locally centrally finite division ring.

References

[1] Bui Xuan Hai and Nguyen Van Thin, On locally nilpotent subgroups of $GL_1(D)$, *Communitations in Algebra* 37 (2009), no. 2, 712–718.

[2] T.Y. Lam, *A First course in non-commutative rings*, GTM 131 (1991), Springer-Verlag.

[3] Mahdavi-Hezavehi M., Commutators in division rings revisited, *Bull. of the Iranian Math. Soc.*, 26 (2), 7-88 (2000).