Toward Berenstein-Zelevinsky data in affine type A I: Construction of affine analogs

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Abstract

We give (conjectural) analogs of Berenstein-Zelevinsky data for affine type A. Moreover, by using these affine analogs of Berenstein-Zelevinsky data, we realize the crystal basis of the negative part of the quantized universal enveloping algebra of the (Langlands dual) Lie algebra of affine type A.

1 Introduction.

This paper provides the first step in our attempt to construct and describe analogs of Mirković-Vilonen (MV for short) polytopes for affine Lie algebras. In this paper, we concentrate on the case of affine type A, and construct (conjectural) affine analogs of Berenstein-Zelevinsky (BZ for short) data. Furthermore, using these affine analogs of BZ data, we give a realization of the crystal basis of the negative part of the quantized universal enveloping algebra associated to (the Langlands dual Lie algebra of) the affine Lie algebra of affine type A. Here we should mention that in the course of the much more sophisticated discussion toward the (conjectural) geometric Satake correspondence for a Kac-Moody group of affine type A, Nakajima [N] constructed affine analogs of MV cycles by using his quiver varieties; see also [BF1], [BF2].

Let G be a semisimple algebraic group over $\mathbb{C}$ with (semisimple) Lie algebra $\mathfrak{g}$. Anderson [A] introduced MV polytopes for $\mathfrak{g}$ as moment polytopes of MV cycles in the affine Grassmannian $\mathcal{G}r$ associated to G, and, on the basis of the geometric Satake correspondence, used them to count weight multiplicities and tensor product multiplicities for finite-dimensional irreducible representations of the Langlands dual group $G^{\vee}$ of G.

Soon afterward, Kamnitzer [Kam1], [Kam2] gave a combinatorial characterization of MV polytopes in terms of BZ data; a BZ datum is a collection of integers (indexed by the set of chamber weights) satisfying the edge inequalities and tropical Plücker relations. To be more precise, let W_I be the Weyl group of $\mathfrak{g}$, and ϖ_i^I, $i \in I$, the fundamental weights, where I is the index set of simple roots; the set Γ_I of chamber weights is by definition $\Gamma_I := \bigcup_{i\in I} W_I \varpi_i^I$. Then, for a BZ datum $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ with $M_\gamma \in \mathbb{Z}$, the corresponding MV polytope $P(\mathbf{M})$ is given by:

$$P(\mathbf{M}) = \bigl\{ h \in (\mathfrak{h}_I)_{\mathbb{R}} \mid \langle h,\, \gamma \rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma_I \bigr\},$$

where $(\mathfrak{h}_I)_{\mathbb{R}}$ is a real form of the Cartan subalgebra $\mathfrak{h}_I$ of $\mathfrak{g}$, and $\langle \cdot\,,\, \cdot \rangle$ is the canonical pairing between $\mathfrak{h}_I$ and $\mathfrak{h}_I^*$. We denote by $\mathcal{BZ}_I$ the set of all BZ data $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ such that $M_{w_0^I \varpi_i^I} = 0$ for all $i \in I$, where $w_0^I \in W_I$ is the longest element.

Now, let $\widehat{\mathfrak{g}}$ denote the affine Lie algebra of type $A_\ell^{(1)}$ over $\mathbb{C}$ with Cartan subalgebra $\widehat{\mathfrak{h}}$, and $\widehat{A} = (\widehat{a}_{ij})_{i,\,j\in\widehat{I}}$ its Cartan matrix with index set $\widehat{I} = \bigl\{0,\, 1,\, \ldots,\, \ell\bigr\}$, where $\ell \in \mathbb{Z}_{\geq 2}$ is a fixed integer. Before constructing the set of (conjectural) analogs of BZ data for the affine Lie algebra $\widehat{\mathfrak{g}}$, we need to construct the set $\mathcal{BZ}_{\mathbb{Z}}$ of BZ data of type A_∞.

Let $\mathfrak{sl}_\infty(\mathbb{C})$ denote the infinite rank Lie algebra over $\mathbb{C}$ of type A_∞ with Cartan subalgebra $\mathfrak{h}$, and $A_{\mathbb{Z}} = (a_{ij})_{i,\,j\in\mathbb{Z}}$ its Cartan matrix with index set $\mathbb{Z}$. Let $W_{\mathbb{Z}} = \langle s_i \mid i \in \mathbb{Z} \rangle \subset GL(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{sl}_\infty(\mathbb{C})$, and $\Lambda_i \in \mathfrak{h}^*$, $i \in \mathbb{Z}$, the fundamental weights; the set $\Gamma_{\mathbb{Z}}$ of chamber weights for $\mathfrak{sl}_\infty(\mathbb{C})$ is defined to be the set

$$\Gamma_{\mathbb{Z}} := \bigcup_{i\in\mathbb{Z}} (-W_{\mathbb{Z}}\Lambda_i) = \bigl\{ -w\Lambda_i \mid w \in W_{\mathbb{Z}},\, i \in \mathbb{Z} \bigr\},$$

not to be the set $\bigcup_{i\in\mathbb{Z}} W_{\mathbb{Z}}\Lambda_i$. Then, for each finite interval I in $\mathbb{Z}$, we can (and do) identify the set Γ_I of chamber weights for the finite-dimensional simple Lie algebra $\mathfrak{g}_I$ over $\mathbb{C}$ of type $A_{|I|}$ with the subset $\bigl\{ -w\Lambda_i \mid w \in W_I,\, i \in I \bigr\}$, where $|I|$ denotes the cardinality of I, and $W_I = \langle s_i \mid i \in I \rangle \subset W_{\mathbb{Z}}$ is the Weyl group of $\mathfrak{g}_I$ (see §3.1 for details). Here we note that the family $\bigl\{ \mathcal{BZ}_I \mid I \text{ is a finite interval in } \mathbb{Z} \bigr\}$ forms a projective system (cf. Lemma 2.4.1).

Using the projective system $\bigl\{ \mathcal{BZ}_I \mid I \text{ is a finite interval in } \mathbb{Z} \bigr\}$ above, we define the set $\mathcal{BZ}_{\mathbb{Z}}$ of BZ data of type A_∞ to be a kind of projective limit, with a certain stability constraint, of the system $\bigl\{ \mathcal{BZ}_I \mid I \text{ is a finite interval in } \mathbb{Z} \bigr\}$; see Definition 3.2.1 for a precise statement. Because of this stability constraint, we can endow the set $\mathcal{BZ}_{\mathbb{Z}}$ a crystal structure for the Lie algebra $\mathfrak{sl}_\infty(\mathbb{C})$ of type A_∞.

Finally, recall the fact that the Dynkin diagram of type $A_\ell^{(1)}$ can be obtained from that of type A_∞ by the operation of "folding" under the Dynkin diagram automorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$ in type A_∞ given by: $\sigma(i) = i + \ell - 1$ for $i \in \mathbb{Z}$, where $\ell \in \mathbb{Z}_{\geq 2}$. In view of this fact, we consider the fixed point subset $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ of $\mathcal{BZ}_{\mathbb{Z}}$ under a natural action of the Dynkin diagram automorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$. Then, we can endow a crystal structure (canonically induced by that on $\mathcal{BZ}_{\mathbb{Z}}$) for the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{g}}^{\vee})$ associated to the (Langlands) dual Lie algebra $\widehat{\mathfrak{g}}^{\vee}$ of $\widehat{\mathfrak{g}}$.

However, the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ for $U_q(\widehat{\mathfrak{g}}^{\vee})$ may be too big for our purpose. Therefore, we restrict our attention to the connected component $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ of the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ containing the BZ datum $\mathbf{O}$ of type A_{∞} whose γ-component is equal to 0 for each $\gamma \in \Gamma_{\mathbb{Z}}$. Our main result (Theorem 4.4.1) states that the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ is isomorphic, as a crystal for $U_q(\widehat{\mathfrak{g}}^{\vee})$, to the crystal basis $\widehat{\mathcal{B}}(\infty)$ of the negative part $U_q^{-}(\widehat{\mathfrak{g}}^{\vee})$ of $U_q(\widehat{\mathfrak{g}}^{\vee})$. Moreover, for each dominant integral weight $\widehat{\lambda} \in \widehat{\mathfrak{h}}$ for $\widehat{\mathfrak{g}}^{\vee}$, the crystal basis $\widehat{\mathcal{B}}(\widehat{\lambda})$ of the irreducible highest weight $U_q(\widehat{\mathfrak{g}}^{\vee})$-module of highest weight $\widehat{\lambda}$ can be realized as a certain explicit subset of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ (see Theorem 4.4.5). In fact, we first prove Theorem 4.4.5 by using Stembridge's result on a characterization of highest weight crystals for simply-laced Kac-Moody algebras; then, Theorem 4.4.1 is obtained as a corollary.

Unfortunately, we have not yet found an explicit characterization of the connected component $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O}) \subset \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ in terms of the "edge inequalities" and "tropical Plücker relations" in type $A_{\ell}^{(1)}$ in a way analogous to the finite-dimensional case; we hope to mention such a description of the connected component $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O}) \subset \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ in our forthcoming paper [NSS]. However, from our results in this paper, it seems reasonable to think of an element $\mathbf{M} = (M_{\gamma})_{\gamma \in \Gamma_{\mathbb{Z}}}$ of the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ as a (conjectural) analog of a BZ datum in affine type A.

This paper is organized as follows. In Section 2, following Kamnitzer, we review some standard facts about BZ data for the simple Lie algebra $\mathfrak{g}_I$ of type $A_{|I|}$, where $I \subset \mathbb{Z}$ is the index set of simple roots with cardinality m, and then show that the system of sets $\mathcal{BZ}_I$ of BZ data for $\mathfrak{g}_I$, where I runs over all the finite intervals in $\mathbb{Z}$, forms a projective system. In Section 3, we introduce the notion of BZ data of type A_{∞}, and define Kashiwara operators on the set $\mathcal{BZ}_{\mathbb{Z}}$ of BZ data of type A_{∞}. Also, we show a technical lemma about some properties of Kashiwara operators on $\mathcal{BZ}_{\mathbb{Z}}$. In Section 4, we first study the action of the Dynkin diagram automorphism σ in type A_{∞} on the set $\mathcal{BZ}_{\mathbb{Z}}$. Next, we define the set of BZ data of type $A_{\ell}^{(1)}$ to be the fixed point subset $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ of $\mathcal{BZ}_{\mathbb{Z}}$ under σ, and endow a canonical crystal structure on it. Finally, in Subsections 4.4 and 4.5, we state and prove our main results (Theorems 4.4.1 and 4.4.5), which give a realization of the crystal basis $\widehat{\mathcal{B}}(\infty)$ for the (Langlands dual) Lie algebra $\widehat{\mathfrak{g}}^{\vee}$ of type $A_{\ell}^{(1)}$. In the Appendix, we restate Stembridge's result on a characterization of simply-laced crystals in a form that will be used in the proofs of the theorems above.

2 Berenstein-Zelevinsky data of type A_m.

In this section, following [Kam1] and [Kam2], we briefly review some basic facts about Berenstein-Zelevinsky (BZ for short) data for the complex finite-dimensional simple Lie algebra of type A_m.

2.1 Basic notation in type A_m. Let I be a fixed (finite) interval in $\mathbb{Z}$ whose cardinality is equal to $m \in \mathbb{Z}_{\geq 1}$; that is, $I \subset \mathbb{Z}$ is a finite subset of the form:

$$I = \{n+1,\, n+2,\, \dots,\, n+m\} \quad \text{for some } n \in \mathbb{Z}. \tag{2.1.1}$$

Let $A_I = (a_{ij})_{i,j \in I}$ denote the Cartan matrix of type A_m with index set I; the entries a_{ij} are given by:

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1.2}$$

for $i,\, j \in I$. Let $\mathfrak{g}_I$ be the complex finite-dimensional simple Lie algebra with Cartan matrix A_I, Cartan subalgebra $\mathfrak{h}_I$, simple coroots $h_i \in \mathfrak{h}_I$, $i \in I$, and simple roots $\alpha_i \in \mathfrak{h}_I^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_I,\, \mathbb{C})$, $i \in I$; note that $\mathfrak{h}_I = \bigoplus_{i \in I} \mathbb{C}h_i$, and that $\langle h_i,\, \alpha_j \rangle = a_{ij}$ for $i,\, j \in I$, where $\langle \cdot\,,\, \cdot \rangle$ is the canonical pairing between $\mathfrak{h}_I$ and $\mathfrak{h}_I^*$. Denote by $\varpi_i^I \in \mathfrak{h}_I^*$, $i \in I$, the fundamental weights for $\mathfrak{g}_I$, and by $W_I := \langle s_i \mid i \in I \rangle$ $(\subset GL(\mathfrak{h}_I^*))$ the Weyl group of $\mathfrak{g}_I$, where s_i is the simple reflection for $i \in I$, with e and w_0^I the identity element and the longest element of the Weyl group W_I, respectively. Also, we denote by $\leq$ the (strong) Bruhat order on W_I. The (Dynkin) diagram automorphism for $\mathfrak{g}_I$ is a bijection $\omega_I : I \to I$ defined by: $\omega_I(n+i) = n+m-i+1$ for $1 \leq i \leq m$ (see (2.1.1) and (2.1.2)). It is easy to see that for $i \in I$,

$$w_0^I(\alpha_i) = -\alpha_{\omega_I(i)}, \qquad w_0^I(\varpi_i^I) = -\varpi_{\omega_I(i)}^I, \qquad w_0^I s_{\omega_I(i)} = s_i w_0^I. \tag{2.1.3}$$

Let $\mathfrak{g}_I^\vee$ denote the (Langlands) dual Lie algebra of $\mathfrak{g}_I$; that is, $\mathfrak{g}_I^\vee$ is the complex finite-dimensional simple Lie algebra of type A_m associated to the transpose ${}^tA_I$ $(= A_I)$ of A_I, with Cartan subalgebra $\mathfrak{h}_I^*$, simple coroots $\alpha_i \in \mathfrak{h}_I^*$, $i \in I$, and simple roots $h_i \in \mathfrak{h}_I$, $i \in I$. Let $U_q(\mathfrak{g}_I^\vee)$ be the quantized universal enveloping algebra over the field $\mathbb{C}(q)$ of rational functions in q associated to the Lie algebra $\mathfrak{g}_I^\vee$, $U_q^-(\mathfrak{g}_I^\vee)$ the negative part of $U_q(\mathfrak{g}_I^\vee)$, and $\mathcal{B}_I(\infty)$ the crystal basis of $U_q^-(\mathfrak{g}_I^\vee)$. Also, for a dominant integral weight $\lambda \in \mathfrak{h}_I$ for $\mathfrak{g}_I^\vee$, $\mathcal{B}_I(\lambda)$ denotes the crystal basis of the finite-dimensional irreducible highest weight $U_q(\mathfrak{g}_I^\vee)$-module of highest weight λ.

2.2 BZ data of type A_m. We set

$$\Gamma_I := \{w\varpi_i^I \mid w \in W_I,\, i \in I\}; \tag{2.2.1}$$

note that by the second equation in (2.1.3), the set Γ_I (of chamber weights) coincides with the set $-\Gamma_I = \{-w\varpi_i^I \mid w \in W_I,\, i \in I\}$. Let $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I}$ be a collection of integers indexed by Γ_I. For each $\gamma \in \Gamma_I$, we call M_γ the γ-component of the collection $\mathbf{M}$, and denote it by $(\mathbf{M})_\gamma$.

Definition 2.2.1. A collection $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ of integers is called a Berenstein-Zelevinsky (BZ for short) datum for $\mathfrak{g}_I$ if it satisfies the following conditions (1) and (2):

(1) (edge inequalities) for all $w \in W_I$ and $i \in I$,

$$M_{w\varpi_i^I} + M_{ws_i\varpi_i^I} + \sum_{j\in I\setminus\{i\}} a_{ji} M_{w\varpi_j^I} \le 0; \tag{2.2.2}$$

(2) (tropical Plücker relations) for all $w \in W_I$ and $i, j \in I$ with $a_{ij} = a_{ji} = -1$ such that $ws_i > w$, $ws_j > w$,

$$M_{ws_i\varpi_i^I} + M_{ws_j\varpi_j^I} = \min\bigl(M_{w\varpi_i^I} + M_{ws_is_j\varpi_j^I},\ M_{w\varpi_j^I} + M_{ws_js_i\varpi_i^I}\bigr). \tag{2.2.3}$$

2.3 Crystal structure on the set of BZ data of type A_m. Let $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ be a BZ datum for $\mathfrak{g}_I$. Following [Kam1, §2.3], we define

$$P(\mathbf{M}) := \bigl\{h \in (\mathfrak{h}_I)_\mathbb{R} \mid \langle h, \gamma\rangle \ge M_\gamma \text{ for all } \gamma \in \Gamma_I\bigr\},$$

where $(\mathfrak{h}_I)_\mathbb{R} := \bigoplus_{i\in I} \mathbb{R}h_i$ is a real form of the Cartan subalgebra $\mathfrak{h}_I$. We know from [Kam1, Proposition 2.2] that $P(\mathbf{M})$ is a convex polytope in $(\mathfrak{h}_I)_\mathbb{R}$ whose set of vertices is given by:

$$\left\{ \mu_w(\mathbf{M}) := \sum_{i\in I} M_{w\varpi_i^I}\, wh_i \;\middle|\; w \in W \right\} \subset (\mathfrak{h}_I)_\mathbb{R}. \tag{2.3.1}$$

The polytope $P(\mathbf{M})$ is called a Mirković-Vilonen (MV) polytope associated to the BZ datum $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$.

We denote by $\mathcal{BZ}_I$ the set of all BZ data $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ for $\mathfrak{g}_I$ satisfying the condition that $M_{w_0^I\varpi_i^I} = 0$ for all $i \in I$, or equivalently, $M_{-\varpi_i^I} = 0$ for all $i \in I$ (by the second equation in (2.1.3)). By [Kam2, §3.3], the set $\mathcal{MV}_I := \bigl\{P(\mathbf{M}) \mid \mathbf{M} \in \mathcal{BZ}_I\bigr\}$ can be endowed with a crystal structure for $U_q(\mathfrak{g}_I^\vee)$, and the resulting crystal $\mathcal{MV}_I$ is isomorphic to the crystal basis $\mathcal{B}_I(\infty)$ of the negative part $U_q^-(\mathfrak{g}_I^\vee)$ of $U_q(\mathfrak{g}_I^\vee)$. Because the map $\mathcal{BZ}_I \to \mathcal{MV}_I$ defined by $\mathbf{M} \mapsto P(\mathbf{M})$ is bijective, we can also endow the set $\mathcal{BZ}_I$ with a crystal structure for $U_q(\mathfrak{g}_I^\vee)$ in such a way that the bijection $\mathcal{BZ}_I \to \mathcal{MV}_I$ is an isomorphism of crystals for $U_q(\mathfrak{g}_I^\vee)$.

Now we recall from [Kam2] the description of the crystal structure on $\mathcal{BZ}_I$. For $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I} \in \mathcal{BZ}_I$, define the weight $\mathrm{wt}(\mathbf{M})$ of $\mathbf{M}$ by:

$$\mathrm{wt}(\mathbf{M}) = \sum_{i\in I} M_{\varpi_i^I}\, h_i. \tag{2.3.2}$$

The raising Kashiwara operators e_p, $p \in I$, on $\mathcal{BZ}_I$ are defined as follows (see [Kam2, Theorem 3.5 (ii)]). Fix $p \in I$. For a BZ datum $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ for $\mathfrak{g}_I$ (not necessarily an element of $\mathcal{BZ}_I$), we set

$$\varepsilon_p(\mathbf{M}) := -\left(M_{\varpi_p^I} + M_{s_p\varpi_p^I} + \sum_{q\in I\setminus\{p\}} a_{qp} M_{\varpi_q^I} \right), \tag{2.3.3}$$

which is nonnegative by condition (1) of Definition 2.2.1. Observe that $\mu_{s_p}(\mathbf{M}) - \mu_e(\mathbf{M}) = \varepsilon_p(\mathbf{M})h_p$, and hence that $\mu_{s_p}(\mathbf{M}) = \mu_e(\mathbf{M})$ if and only if $\varepsilon_p(\mathbf{M}) = 0$. In view of this, we set $e_p\mathbf{M} := \mathbf{0}$ if $\varepsilon_p(\mathbf{M}) = 0$ (cf. [Kam2, Theorem 3.5(ii)]), where $\mathbf{0}$ is an additional element, which is not contained in $\mathcal{BZ}_I$. We know the following fact from [Kam2, Theorem 3.5 (ii)] (see also the comment after [Kam2, Theorem 3.5]).

Fact 2.3.1. *Let $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ be a BZ datum for $\mathfrak{g}_I$ (not necessarily an element of $\mathcal{BZ}_I$). If $\varepsilon_p(\mathbf{M}) > 0$, then there exists a unique BZ datum for $\mathfrak{g}_I$, denoted by $e_p\mathbf{M}$, such that $(e_p\mathbf{M})_{\varpi_p^I} = M_{\varpi_p^I} + 1$, and such that $(e_p\mathbf{M})_\gamma = M_\gamma$ for all $\gamma \in \Gamma_I$ with $\langle h_p, \gamma\rangle \le 0$.*

It is easily verified that if $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I} \in \mathcal{BZ}_I$, then $e_p\mathbf{M} \in \mathcal{BZ}_I \cup \{\mathbf{0}\}$. Indeed, suppose that $\varepsilon_p(\mathbf{M}) > 0$, or equivalently, $e_p\mathbf{M} \neq \mathbf{0}$. Let $i \in I$. Since $\langle h_p, w_0^I\varpi_i^I\rangle \le 0$ by the second equation in (2.1.3), it follows from the definition of $e_p\mathbf{M}$ that $(e_p\mathbf{M})_{w_0^I\varpi_i^I}$ is equal to $M_{w_0^I\varpi_i^I}$, and hence that $(e_p\mathbf{M})_{w_0^I\varpi_i^I} = M_{w_0^I\varpi_i^I} = 0$. Thus, we obtain a map e_p from $\mathcal{BZ}_I$ to $\mathcal{BZ}_I \cup \{\mathbf{0}\}$ sending $\mathbf{M} \in \mathcal{BZ}_I$ to $e_p\mathbf{M} \in \mathcal{BZ}_I \cup \{\mathbf{0}\}$. By convention, we set $e_p\mathbf{0} := \mathbf{0}$.

Similarly, the lowering Kashiwara operators f_p, $p \in I$, on $\mathcal{BZ}_I$ are defined as follows. Fix $p \in I$. Let us recall the following fact from [Kam2, Theorem 3.5 (i)], the comment after [Kam2, Theorem 3.5], and [Kam2, Corollary 5.6].

Fact 2.3.2. *Let $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ be a BZ datum for $\mathfrak{g}_I$ (not necessarily an element of $\mathcal{BZ}_I$). Then, there exists a unique BZ datum for $\mathfrak{g}_I$, denoted by $f_p\mathbf{M}$, such that $(f_p\mathbf{M})_{\varpi_p^I} = M_{\varpi_p^I} - 1$, and such that $(f_p\mathbf{M})_\gamma = M_\gamma$ for all $\gamma \in \Gamma_I$ with $\langle h_p, \gamma\rangle \le 0$. Moreover, for each $\gamma \in \Gamma_I$,*

$$(f_p\mathbf{M})_\gamma = \begin{cases} \min\bigl(M_\gamma,\ M_{s_p\gamma} + c_p(\mathbf{M})\bigr) & \text{if } \langle h_p, \gamma\rangle > 0, \\ M_\gamma & \text{otherwise,} \end{cases} \tag{2.3.4}$$

where $c_p(\mathbf{M}) := M_{\varpi_p^I} - M_{s_p\varpi_p^I} - 1$.

Remark 2.3.3. Keep the notation and assumptions of Fact 2.3.2. By (2.3.4), we have $(f_p\mathbf{M})_\gamma \le M_\gamma$ for all $\gamma \in \Gamma_I$.

In exactly the same way as the case of e_p above, we see that if $\mathbf{M} \in \mathcal{BZ}_I$, then $f_p\mathbf{M} \in \mathcal{BZ}_I$. Thus, we obtain a map f_p from $\mathcal{BZ}_I$ to itself sending $\mathbf{M} \in \mathcal{BZ}_I$ to $f_p\mathbf{M} \in \mathcal{BZ}_I$. By convention, we set $f_p\mathbf{0} := \mathbf{0}$.

Finally, we set $\varphi_p(\mathbf{M}) := \langle \mathrm{wt}(\mathbf{M}), \alpha_p\rangle + \varepsilon_p(\mathbf{M})$ for $\mathbf{M} \in \mathcal{BZ}_I$ and $p \in I$.

Theorem 2.3.4 ([Kam2])**.** *The set $\mathcal{BZ}_I$, equipped with the maps* wt, *e_p, f_p ($p \in I$), and ε_p, φ_p ($p \in I$) above, is a crystal for $U_q(\mathfrak{g}_I^\vee)$ isomorphic to the crystal basis $\mathcal{B}_I(\infty)$ of the negative part $U_q^-(\mathfrak{g}_I^\vee)$ of $U_q(\mathfrak{g}_I^\vee)$.*

Remark 2.3.5. Let $\mathbf{O}$ be the collection of integers indexed by Γ_I whose γ-component is equal to 0 for all $\gamma \in \Gamma_I$. It is obvious that $\mathbf{O}$ is an element of $\mathcal{BZ}_I$ whose weight is equal to 0.

Hence it follows from Theorem 2.3.4 that for each $\mathbf{M} \in \mathcal{BZ}_I$, there exists $p_1, p_2, \ldots, p_N \in I$ such that $\mathbf{M} = f_{p_1} f_{p_2} \cdots f_{p_N} \mathbf{O}$. Therefore, using this fact and Remark 2.3.3, we deduce that if $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I} \in \mathcal{BZ}_I$, then $M_\gamma \in \mathbb{Z}_{\le 0}$ for all $\gamma \in \Gamma_I$.

Let $\lambda \in \mathfrak{h}_I$ be a dominant integral weight for $\mathfrak{g}_I^\vee$. We define $\mathcal{MV}_I(\lambda)$ to be the set of those MV polytopes $P \in \mathcal{MV}_I$ such that $\lambda + P$ is contained in the convex hull $\mathrm{Conv}(W_I\lambda)$ in $(\mathfrak{h}_I)_\mathbb{R}$ of the W_I-orbit $W_I\lambda$ through λ. We see from [Kam2, §3.2] that for $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I} \in \mathcal{BZ}_I$,

$$\lambda + P(\mathbf{M}) = \bigl\{ h \in \mathfrak{h}_\mathbb{R} \mid \langle h,\, \gamma \rangle \ge M'_\gamma \text{ for all } \gamma \in \Gamma_I \bigr\},$$

where $M'_\gamma := M_\gamma + \langle \lambda,\, \gamma \rangle$ for $\gamma \in \Gamma_I$. We know from [Kam1, Theorem 8.5] and [Kam2, §6.2] that $\lambda + P(\mathbf{M}) \subset \mathrm{Conv}(W_I\lambda)$ if and only if $M'_{w_0 s_i \varpi_i^I} \ge \langle w_0\lambda,\, \varpi_i^I \rangle$ for all $i \in I$. A simple computation shows the following lemma.

Lemma 2.3.6. *Let $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I} \in \mathcal{BZ}_I$. Then, the MV polytope $P(\mathbf{M})$ is contained in $\mathcal{MV}_I(\lambda)$ (i.e., $\lambda + P(\mathbf{M}) \subset \mathrm{Conv}(W_I\lambda)$) if and only if*

$$M_{-s_i \varpi_i^I} \ge -\langle \lambda,\, \alpha_i \rangle \quad \text{for all } i \in I. \tag{2.3.5}$$

We denote by $\mathcal{BZ}_I(\lambda)$ the set of all BZ data $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I} \in \mathcal{BZ}_I$ satisfying (2.3.5). By the lemma above, the restriction of the bijection $\mathcal{BZ}_I \to \mathcal{MV}_I$, $\mathbf{M} \mapsto P(\mathbf{M})$, to the subset $\mathcal{BZ}_I(\lambda) \subset \mathcal{BZ}_I$ gives rise to a bijection between $\mathcal{BZ}_I(\lambda)$ and $\mathcal{MV}_I(\lambda)$. By [Kam2, Theorem 6.4], the set $\mathcal{MV}_I(\lambda)$ can be endowed with a crystal structure for $U_q(\mathfrak{g}_I^\vee)$, and the resulting crystal $\mathcal{MV}_I(\lambda)$ is isomorphic to the crystal basis $\mathcal{B}_I(\lambda)$ of the finite-dimensional irreducible highest weight $U_q(\mathfrak{g}_I^\vee)$-module of highest weight λ. Thus, we can also endow the set $\mathcal{BZ}_I(\lambda)$ with a crystal structure for $U_q(\mathfrak{g}_I^\vee)$ in such a way that the bijection $\mathcal{BZ}_I(\lambda) \to \mathcal{MV}_I(\lambda)$ above is an isomorphism of crystals for $U_q(\mathfrak{g}_I^\vee)$.

Now we recall from [Kam2, §6.4] the description of the crystal structure on $\mathcal{BZ}_I(\lambda)$. For $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I} \in \mathcal{BZ}_I(\lambda)$, define the weight $\mathrm{Wt}(\mathbf{M})$ of $\mathbf{M}$ by:

$$\mathrm{Wt}(\mathbf{M}) = \lambda + \mathrm{wt}(\mathbf{M}) = \lambda + \sum_{i \in I} M_{\varpi_i^I} h_i. \tag{2.3.6}$$

The raising Kashiwara operators e_p, $p \in I$, and the maps ε_p, $p \in I$, on $\mathcal{BZ}_I(\lambda)$ are defined by restricting those on $\mathcal{BZ}_I$ to the subset $\mathcal{BZ}_I(\lambda) \subset \mathcal{BZ}_I$. The lowering Kashiwara operators F_p, $p \in I$, on $\mathcal{BZ}_I(\lambda)$ are defined as follows: for $\mathbf{M} \in \mathcal{BZ}_I(\lambda)$ and $p \in I$,

$$F_p\mathbf{M} = \begin{cases} f_p\mathbf{M} & \text{if } f_p\mathbf{M} \text{ is an element of } \mathcal{BZ}_I(\lambda), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Also, we set $\Phi_p(\mathbf{M}) := \langle \mathrm{Wt}(\mathbf{M}),\, \alpha_p \rangle + \varepsilon_p(\mathbf{M})$ for $\mathbf{M} \in \mathcal{BZ}_I(\lambda)$ and $p \in I$. It is easily seen by (2.3.3) and (2.3.6) that if $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I}$, then

$$\Phi_p(\mathbf{M}) = M_{\varpi_p^I} - M_{s_p \varpi_p^I} + \langle \lambda,\, \alpha_p \rangle. \tag{2.3.7}$$

Theorem 2.3.7 ([Kam2, Theorem 6.4]). *Let $\lambda \in \mathfrak{h}_I$ be a dominant integral weight for $\mathfrak{g}_I^\vee$. Then, the set $\mathcal{BZ}_I(\lambda)$, equipped with the maps* wt, *e_p, F_p $(p \in I)$, and ε_p, Φ_p $(p \in I)$ above, is a crystal for $U_q(\mathfrak{g}_I^\vee)$ isomorphic to the crystal basis $\mathcal{B}_I(\lambda)$ of the finite-dimensional irreducible highest weight $U_q(\mathfrak{g}_I^\vee)$-module of highest weight λ.*

2.4 Restriction to subintervals. Let K be a fixed (finite) interval in $\mathbb{Z}$ such that $K \subset I$. The Cartan matrix A_K of the finite-dimensional simple Lie algebra $\mathfrak{g}_K$ equals the principal submatrix of the Cartan matrix A_I of $\mathfrak{g}_I$ corresponding to the subset $K \subset I$. Also, the Weyl group W_K of $\mathfrak{g}_K$ can be identified with the subgroup of the Weyl group W_I of $\mathfrak{g}_I$ generated by the subset $\{s_i \mid i \in K\}$ of $\{s_i \mid i \in I\}$. Moreover, we can (and do) identify the set Γ_K (of chamber weights) for $\mathfrak{g}_K$ (defined by (2.2.1) with I replaced by K) with the subset $\{-w\varpi_i^I \mid w \in W_K,\ i \in K\}$ of the set Γ_I (of chamber weights) through the following bijection of sets:

$$\begin{aligned} \Gamma_K &\ \overset{\sim}{\to}\ \{-w\varpi_i^I \mid w \in W_K,\ i \in K\} \subset \Gamma_I, \\ -w\varpi_i^K &\ \mapsto\ -w\varpi_i^I \quad \text{for } w \in W_K \text{ and } i \in K; \end{aligned} \tag{2.4.1}$$

observe that the map above is well-defined. Indeed, suppose that $w\varpi_i^K = v\varpi_j^K$ for some $w, v \in W_K$ and $i, j \in K$. Since ϖ_i^K and ϖ_j^K are dominant, it follows immediately that $i = j$, and hence $w\varpi_i^K = v\varpi_j^K = v\varpi_i^K$. Since $v^{-1}w\varpi_i^K = \varpi_i^K$ (i.e., $v^{-1}w$ stabilizes ϖ_i^K), we see that $v^{-1}w$ is a product of s_k's for $k \in K \setminus \{i\}$. Therefore, we obtain $v^{-1}w\varpi_i^I = \varpi_i^I$, and hence $w\varpi_i^I = v\varpi_i^I = v\varpi_j^I$, as desired. Also, note that for each $i \in K$, the fundamental weight $\varpi_i^K \in \Gamma_K$ for $\mathfrak{g}_K$ corresponds to $-w_0^K(\varpi_{\omega_K(i)}^I) = w_0^K w_0^I \varpi_{\omega_I\omega_K(i)}^I \in \Gamma_I$ under the bijection (2.4.1), where $\omega_K : K \to K$ denotes the (Dynkin) diagram automorphism for $\mathfrak{g}_K$. For a collection $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I}$ of integers indexed by Γ_I, we set $\mathbf{M}_K := (M_\gamma)_{\gamma \in \Gamma_K}$, regarding the set Γ_K as a subset of the set Γ_I through the bijection (2.4.1).

Lemma 2.4.1. *Keep the notation above. If $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I}$ is an element of $\mathcal{BZ}_I$, then $\mathbf{M}_K = (M_\gamma)_{\gamma \in \Gamma_K}$ is a BZ datum for $\mathfrak{g}_K$ that is an element of $\mathcal{BZ}_K$.*

Proof. First we show that $\mathbf{M}_K$ satisfies condition (1) of Definition 2.2.1 (with I replaced by K), i.e., for $w \in W_K$ and $i \in K$,

$$M_{w\varpi_i^K} + M_{ws_i\varpi_i^K} + \sum_{j \in K \setminus \{i\}} a_{ji} M_{w\varpi_j^K} \le 0. \tag{2.4.2}$$

Observe that under the bijection (2.4.1), we have

$$\begin{aligned} w\varpi_k^K &\ \mapsto\ wv_0\varpi_{\tau(k)}^I \quad (k \in K), \\ ws_i\varpi_i^K &\ \mapsto\ ws_iv_0\varpi_{\tau(i)}^I = wv_0s_{\tau(i)}\varpi_{\tau(i)}^I, \end{aligned} \tag{2.4.3}$$

where we set $v_0 := w_0^K w_0^I$ and $\tau := \omega_I\omega_K$ for simplicity of notation. Since $\mathbf{M}$ is a BZ datum for $\mathfrak{g}_I$, it follows from condition (1) of Definition 2.2.1 for $wv_0 \in W_I$ and $\tau(i) \in I$ that

$$M_{wv_0\varpi_{\tau(i)}^I} + M_{wv_0s_{\tau(i)}\varpi_{\tau(i)}^I} + \sum_{j \in I \setminus \{\tau(i)\}} a_{j,\tau(i)} M_{wv_0\varpi_j^I} \le 0. \tag{2.4.4}$$

Here, using the equality $a_{\omega_I(j),\tau(i)} = a_{j,\omega_K(i)}$ for $j \in I$, we see that

$$\sum_{j \in I \setminus \{\tau(i)\}} a_{j,\tau(i)} M_{wv_0 \varpi_j^I} = \sum_{\omega_I(j) \in I \setminus \{\tau(i)\}} a_{\omega_I(j),\tau(i)} M_{wv_0 \varpi_{\omega_I(j)}^I} = \sum_{j \in I \setminus \{\omega_K(i)\}} a_{j,\omega_K(i)} M_{wv_0 \varpi_{\omega_I(j)}^I}.$$

Also, if $j \in I \setminus K$, then

$$\begin{aligned} M_{wv_0 \varpi_{\omega_I(j)}^I} &= M_{-ww_0^K \varpi_j^I} = M_{-\varpi_j^I} \quad \text{since } ww_0^K \in W_K \\ &= 0 \quad \text{since } \mathbf{M} \in \mathcal{BZ}_I. \end{aligned}$$

Hence it follows that

$$\sum_{j \in I \setminus \{\omega_K(i)\}} a_{j,\omega_K(i)} M_{wv_0 \varpi_{\omega_I(j)}^I} = \sum_{j \in K \setminus \{\omega_K(i)\}} a_{j,\omega_K(i)} M_{wv_0 \varpi_{\omega_I(j)}^I}.$$

Furthermore, using the equality $a_{\omega_K(j),\omega_K(i)} = a_{ji}$ for $j \in K$, we get

$$\begin{aligned} \sum_{j \in K \setminus \{\omega_K(i)\}} a_{j,\omega_K(i)} M_{wv_0 \varpi_{\omega_I(j)}^I} &= \sum_{\omega_K(j) \in K \setminus \{\omega_K(i)\}} a_{\omega_K(j),\omega_K(i)} M_{wv_0 \varpi_{\omega_I(\omega_K(j))}^I} \\ &= \sum_{j \in K \setminus \{i\}} a_{ji} M_{wv_0 \varpi_{\tau(j)}^I}. \end{aligned}$$

Substituting this into (2.4.4), we obtain

$$M_{wv_0 \varpi_{\tau(i)}^I} + M_{wv_0 s_{\tau(i)} \varpi_{\tau(i)}^I} + \sum_{j \in K \setminus \{i\}} a_{ji} M_{wv_0 \varpi_{\tau(j)}^I} \leq 0.$$

The inequality (2.4.2) follows immediately from this inequality and the correspondence (2.4.3).

Next we show that $\mathbf{M}_K$ satisfies condition (2) of Definition 2.2.1 (with I replaced by K), i.e., for $w \in W_K$ and $i, j \in K$ with $a_{ij} = a_{ji} = -1$ such that $ws_i > w$, $ws_j > w$,

$$M_{ws_i \varpi_i^K} + M_{ws_j \varpi_j^K} = \min\bigl(M_{w\varpi_i^K} + M_{ws_i s_j \varpi_j^K},\ M_{w\varpi_j^K} + M_{ws_j s_i \varpi_i^K}\bigr). \tag{2.4.5}$$

Observe that under the bijection (2.4.1), we have

$$\begin{aligned} w\varpi_k^K &\mapsto wv_0 \varpi_{\tau(k)}^I \quad (k \in K), \\ ws_k \varpi_k^K &\mapsto ws_k v_0 \varpi_{\tau(k)}^I = wv_0 s_{\tau(k)} \varpi_{\tau(k)}^I \quad (k \in K), \\ ws_l s_k \varpi_k^K &\mapsto ws_l s_k v_0 \varpi_{\tau(k)}^I = wv_0 s_{\tau(l)} s_{\tau(k)} \varpi_{\tau(k)}^I \quad (k, l \in K). \end{aligned} \tag{2.4.6}$$

Since $a_{\tau(i),\tau(j)} = a_{\tau(j),\tau(i)} = -1$ and $wv_0 s_{\tau(k)} = ws_k v_0 > wv_0$ for $k = i, j$, and since $\mathbf{M}$ is a BZ datum for $\mathfrak{g}_I$, it follows from condition (2) of Definition 2.2.1 for $wv_0 \in W_I$ and $\tau(i), \tau(j) \in I$ that

$$\begin{aligned} &M_{wv_0 s_{\tau(i)} \varpi_{\tau(i)}^I} + M_{wv_0 s_{\tau(j)} \varpi_{\tau(j)}^I} \\ &\quad = \min\bigl(M_{wv_0 \varpi_{\tau(i)}^I} + M_{wv_0 s_{\tau(i)} s_{\tau(j)} \varpi_{\tau(j)}^I},\ M_{wv_0 \varpi_{\tau(j)}^I} + M_{wv_0 s_{\tau(j)} s_{\tau(i)} \varpi_{\tau(i)}^I}\bigr). \end{aligned}$$

The equation (2.4.5) follows immediately from this equation and the correspondence (2.4.6).

Finally, it is obvious that $M_{w_0^K \varpi_i^K} = M_{-\varpi^I_{\omega_K(i)}} = 0$ for all $i \in K$, since $\mathbf{M} \in \mathcal{BZ}_I$. This proves the lemma. $\square$

Now, we set $\Gamma_I^K := \{ w\varpi_i^I \mid w \in W_K,\, i \in K \} \subset \Gamma_I$. Then there exists the following bijection of sets between Γ_K and Γ_I^K:

$$\begin{aligned} \Gamma_K &\stackrel{\sim}{\to} \Gamma_I^K, \\ w\varpi_i^K &\mapsto w\varpi_i^I \quad \text{for } w \in W_K \text{ and } i \in K; \end{aligned} \tag{2.4.7}$$

the argument above for the correspondence (2.4.1) shows that this map is well-defined. For a collection $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I}$ of integers indexed by Γ_I, we define $\mathbf{M}^K := (M_\gamma)_{\gamma \in \Gamma_I^K}$, and regard it as a collection of integers indexed by Γ_K through the bijection (2.4.7) between the index sets.

Lemma 2.4.2. *Keep the notation above. If $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_I}$ is an element of $\mathcal{BZ}_I$, then $\mathbf{M}^K$ is a BZ datum for $\mathfrak{g}_K$.*

Proof. First we show that $\mathbf{M}^K$ satisfies condition (1) of Definition 2.2.1 (with I replaced by K), i.e., for $w \in W_K$ and $i \in K$,

$$M_{w\varpi_i^K} + M_{ws_i\varpi_i^K} + \sum_{j \in K \setminus \{i\}} a_{ji} M_{w\varpi_j^K} \le 0. \tag{2.4.8}$$

Since $\mathbf{M}$ is a BZ datum for $\mathfrak{g}_I$, it follows from condition (1) of Definition 2.2.1 for $w \in W_I$ and $i \in I$ that

$$M_{w\varpi_i^I} + M_{ws_i\varpi_i^I} + \sum_{j \in I \setminus \{i\}} a_{ji} M_{w\varpi_j^I} \le 0,$$

and hence

$$M_{w\varpi_i^I} + M_{ws_i\varpi_i^I} + \sum_{j \in K \setminus \{i\}} a_{ji} M_{w\varpi_j^I} + \sum_{j \in I \setminus K} a_{ji} M_{w\varpi_j^I} \le 0. \tag{2.4.9}$$

Because $M_\gamma \in \mathbb{Z}_{\le 0}$ for all $\gamma \in \Gamma_I$ by Remark 2.3.5, it follows that all terms $a_{ji} M_{w\varpi_j^I}$, $j \in I \setminus K$, of the second sum in (2.4.9) are nonnegative integers. Hence we obtain

$$M_{w\varpi_i^I} + M_{ws_i\varpi_i^I} + \sum_{j \in K \setminus \{i\}} a_{ji} M_{w\varpi_j^I} \le 0.$$

The inequality (2.4.8) follows immediately from this equality and the correspondence (2.4.7).

Next we show that $\mathbf{M}^K$ satisfies condition (2) of Definition 2.2.1 (with I replaced by K), i.e., for $w \in W_K$ and $i, j \in K$ with $a_{ij} = a_{ji} = -1$ such that $ws_i > w$, $ws_j > w$,

$$M_{ws_i\varpi_i^K} + M_{ws_j\varpi_j^K} = \min\big(M_{w\varpi_i^K} + M_{ws_is_j\varpi_j^K},\ M_{w\varpi_j^K} + M_{ws_js_i\varpi_i^K} \big). \tag{2.4.10}$$

Since $\mathbf{M}$ is a BZ datum for $\mathfrak{g}_I$, it follows from condition (2) of Definition 2.2.1 for $w \in W_I$ and $i,\, j \in I$ that

$$M_{ws_i\varpi_i^I} + M_{ws_j\varpi_j^I} = \min\bigl(M_{w\varpi_i^I} + M_{ws_is_j\varpi_j^I},\ M_{w\varpi_j^I} + M_{ws_js_i\varpi_i^I}\bigr).$$

The equation (2.4.10) follows immediately from this equation and the correspondence (2.4.7). This proves the lemma. □

3 Berenstein-Zelevinsky data of type A_∞.

3.1 Basic notation in type A_∞. Let $A_{\mathbb{Z}} = (a_{ij})_{i,j\in\mathbb{Z}}$ denote the generalized Cartan matrix of type A_∞ with index set $\mathbb{Z}$; the entries a_{ij} are given by:

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1.1}$$

for $i,\, j \in \mathbb{Z}$. Let

$$\bigl(A_{\mathbb{Z}},\, \Pi := \bigl\{\alpha_i\bigr\}_{i\in\mathbb{Z}},\, \Pi^\vee := \bigl\{h_i\bigr\}_{i\in\mathbb{Z}},\, \mathfrak{h}^*,\, \mathfrak{h}\bigr)$$

be the root datum of type A_∞. Namely, $\mathfrak{h}$ is a complex infinite-dimensional vector space, with $\Pi^\vee$ a basis of $\mathfrak{h}$, and Π is a linearly independent subset of the (full) dual space $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h},\, \mathbb{C})$ of $\mathfrak{h}$ such that $\langle h_i,\, \alpha_j\rangle = a_{ij}$ for $i,\, j \in \mathbb{Z}$, where $\langle \cdot,\, \cdot\rangle$ is the canonical pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. For each $i \in \mathbb{Z}$, define $\Lambda_i \in \mathfrak{h}^*$ by: $\langle h_j,\, \Lambda_i\rangle = \delta_{ij}$ for $j \in \mathbb{Z}$. Let $W_{\mathbb{Z}} := \langle s_i \mid i \in \mathbb{Z}\rangle$ ($\subset GL(\mathfrak{h}^*)$) be the Weyl group of type A_∞, where s_i is the simple reflection for $i \in \mathbb{Z}$. Also, we denote by $\le$ the (strong) Bruhat order on $W_{\mathbb{Z}}$ (cf. [BjB, §8.3]).

Set

$$\Gamma_{\mathbb{Z}} := \bigl\{-w\Lambda_i \mid w \in W_{\mathbb{Z}},\ i \in \mathbb{Z}\bigr\}, \quad \text{and} \quad \Xi_{\mathbb{Z}} := -\Gamma_{\mathbb{Z}}. \tag{3.1.2}$$

We should note that $\Gamma_{\mathbb{Z}} \cap \Xi_{\mathbb{Z}} = \emptyset$. Indeed, suppose that $\gamma \in \Gamma_{\mathbb{Z}} \cap \Xi_{\mathbb{Z}}$. Since $\gamma \in \Gamma_{\mathbb{Z}}$ (resp., $\gamma \in \Xi_{\mathbb{Z}}$), it can be written as: $\gamma = -w\Lambda_i$ (resp., $\gamma = v\Lambda_j$) for some $w \in W_{\mathbb{Z}}$ and $i \in \mathbb{Z}$ (resp., $v \in W_{\mathbb{Z}}$ and $j \in \mathbb{Z}$). Then we have $\gamma = -w\Lambda_i = v\Lambda_j$, and hence $-\Lambda_i = w^{-1}v\Lambda_j$. Since Λ_j is a dominant integral weight, we see that $w^{-1}v\Lambda_j$ is of the form:

$$w^{-1}v\Lambda_j = \Lambda_j - (m_1\alpha_{i_1} + m_2\alpha_{i_2} + \cdots + m_p\alpha_{i_p})$$

for some $m_1,\, m_2,\, \ldots,\, m_p \in \mathbb{Z}_{>0}$ and $i_1,\, i_2,\, \ldots,\, i_p \in \mathbb{Z}$ with $i_1 < i_2 < \cdots < i_p$. If we set $k := i_p + 1$, then we see that

$$\langle h_k,\, w^{-1}v\Lambda_j\rangle = \langle h_k,\, \Lambda_j\rangle - m_p\langle h_k,\, \alpha_{i_p}\rangle = \langle h_k,\, \Lambda_j\rangle + m_p > 0.$$

However, we have

$$0 < \langle h_k,\, w^{-1}v\Lambda_j\rangle = \langle h_k,\, -\Lambda_i\rangle \le 0,$$

which is a contradiction. Thus we have shown that $\Gamma_{\mathbb{Z}} \cap \Xi_{\mathbb{Z}} = \emptyset$.

Let $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$ (resp., $\mathbf{M} = (M_\xi)_{\xi\in\Xi_{\mathbb{Z}}}$) be a collection of integers indexed by $\Gamma_{\mathbb{Z}}$ (resp., $\Xi_{\mathbb{Z}}$). For each $\gamma \in \Gamma_{\mathbb{Z}}$ (resp., $\xi \in \Xi_{\mathbb{Z}}$), we call M_γ (resp., M_ξ) the γ-component (resp. the ξ-component) of $\mathbf{M}$, and denote it by $(\mathbf{M})_\gamma$ (resp., $(\mathbf{M})_\xi$).

Let I be a (finite) interval in $\mathbb{Z}$. Then the Cartan matrix A_I of the finite-dimensional simple Lie algebra $\mathfrak{g}_I$ (see §2.1) equals the principal submatrix of $A_{\mathbb{Z}}$ corresponding to $I \subset \mathbb{Z}$. Also, the Weyl group W_I of $\mathfrak{g}_I$ can be identified with the subgroup of the Weyl group $W_{\mathbb{Z}}$ generated by the subset $\{s_i \mid i \in I\}$ of $\{s_i \mid i \in \mathbb{Z}\}$. Moreover, we can (and do) identify the set Γ_I (of chamber weights) for $\mathfrak{g}_I$, defined by (2.2.1), with the subset $\{-w\Lambda_i \mid w \in W_I,\, i \in I\}$ of the set $\Gamma_{\mathbb{Z}}$ (of chamber weights) through the following bijection of sets:

$$\begin{aligned} \Gamma_I \quad &\stackrel{\sim}{\to} \quad \{-w\Lambda_i \mid w \in W_I,\, i \in I\} \subset \Gamma_{\mathbb{Z}}, \\ -w\varpi_i^I \quad &\mapsto \quad -w\Lambda_i \quad \text{for } w \in W_I \text{ and } i \in I; \end{aligned} \tag{3.1.3}$$

the same argument as for the correspondence (2.4.1) shows that this map is well-defined. Note that for each $i \in I$, the fundamental weight $\varpi_i^I \in \Gamma_I$ for $\mathfrak{g}_I$ corresponds to $-w_0^I(\Lambda_{\omega_I(i)}) \in \Gamma_{\mathbb{Z}}$ under the bijection (3.1.3), where $\omega_I : I \to I$ denotes the (Dynkin) diagram automorphism for $\mathfrak{g}_I$.

Remark 3.1.1. Let I be an interval in $\mathbb{Z}$, and fix $i \in I$. The element $\varpi_i^I = -w_0^I(\Lambda_{\omega_I(i)}) \in \Gamma_{\mathbb{Z}}$ satisfies the following property: for $j \in \mathbb{Z}$,

$$\langle h_j,\, \varpi_i^I\rangle = \begin{cases} \delta_{ij} & \text{if } j \in I, \\ -1 & \text{if } j = (\min I) - 1 \text{ or } j = (\max I) + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.4}$$

Indeed, it is easily seen that $\langle h_j,\, \varpi_i^I\rangle = \delta_{ij}$ for $j \in I$. Also, if $j < (\min I)-1$ or $j > (\max I)+1$, then $(w_0^I)^{-1}h_j = h_j$ since $w_0^I \in W_I = \langle s_i \mid i \in I\rangle$. Hence

$$\langle h_j,\, \varpi_i^I\rangle = \langle h_j,\, -w_0^I(\Lambda_{\omega_I(i)})\rangle = -\langle (w_0^I)^{-1}h_j,\, \Lambda_{\omega_I(i)}\rangle = -\langle h_j,\, \Lambda_{\omega_I(i)}\rangle = 0.$$

It remains to show that $\langle h_j,\, \varpi_i^I\rangle = -1$ if $j = (\min I) - 1$ or $j = (\max I) + 1$. For simplicity of notation, suppose that $I = \{1, 2 \ldots, m\}$ and $j = 0$. Then, by using the reduced expression $w_0^I = (s_1 s_2 \cdots s_m)(s_1 s_2 \cdots s_{m-1}) \cdots (s_1 s_2) s_1$ of the longest element $w_0^I \in W_I$, we deduce that $(w_0^I)^{-1}h_0 = h_0 + h_1 + \cdots + h_m$. Therefore,

$$\begin{aligned} \langle h_0,\, \varpi_i^I\rangle &= \langle h_0,\, -w_0^I(\Lambda_{\omega_I(i)})\rangle = -\langle (w_0^I)^{-1}h_0,\, \Lambda_{\omega_I(i)}\rangle \\ &= -\langle h_0 + h_1 + \cdots + h_m,\, \Lambda_{\omega_I(i)}\rangle = -1, \end{aligned}$$

as desired.

For a collection $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$ of integers indexed by $\Gamma_{\mathbb{Z}}$, we set $\mathbf{M}_I := (M_\gamma)_{\gamma\in\Gamma_I}$, regarding the set Γ_I as a subset of the set $\Gamma_{\mathbb{Z}}$ through the bijection (3.1.3). Note that if K is an interval in $\mathbb{Z}$ such that $K \subset I$, then $(\mathbf{M}_I)_K = \mathbf{M}_K$ (for the notation, see §2.4).

3.2 BZ data of type A_∞.

Definition 3.2.1. A collection $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$ of integers indexed by $\Gamma_\mathbb{Z}$ is called a BZ datum of type A_∞ if it satisfies the following conditions:

(a) For each interval K in $\mathbb{Z}$, $\mathbf{M}_K = (M_\gamma)_{\gamma \in \Gamma_K}$ is a BZ datum for $\mathfrak{g}_K$, and is an element of $\mathcal{BZ}_K$ (cf. Lemma 2.4.1).

(b) For each $w \in W_\mathbb{Z}$ and $i \in \mathbb{Z}$, there exists an interval I in $\mathbb{Z}$ such that $i \in I$, $w \in W_I$, and $M_{w\varpi_i^J} = M_{w\varpi_i^I}$ for all intervals J in $\mathbb{Z}$ containing I.

Example 3.2.2. Let $\mathbf{O}$ be a collection of integers indexed by $\Gamma_\mathbb{Z}$ whose γ-component is equal to 0 for each $\gamma \in \Gamma_\mathbb{Z}$. Then it is obvious that $\mathbf{O}$ is a BZ datum of type A_∞ (cf. Remark 2.3.5).

Let $\mathcal{BZ}_\mathbb{Z}$ denote the set of all BZ data of type A_∞. For $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$, and for each $w \in W$ and $i \in \mathbb{Z}$, we denote by $\mathrm{Int}(\mathbf{M}; w, i)$ the set of all intervals I in $\mathbb{Z}$ satisfying condition (b) of Definition 3.2.1 for the w and i.

Remark 3.2.3. (1) Let $\mathbf{M}$ be a BZ datum of type A_∞, i.e., $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, and let $w \in W_\mathbb{Z}$ and $i \in \mathbb{Z}$. It is obvious that if $I \in \mathrm{Int}(\mathbf{M}; w, i)$, then $J \in \mathrm{Int}(\mathbf{M}; w, i)$ for every interval J in $\mathbb{Z}$ containing I.

(2) Let $\mathbf{M}_b$ ($1 \le b \le a$) be BZ data of type A_∞, and let $w_b \in W_\mathbb{Z}$ ($1 \le b \le a$) and $i_b \in \mathbb{Z}$ ($1 \le b \le a$). Then the intersection

$$\mathrm{Int}(\mathbf{M}_1; w_1, i_1) \cap \mathrm{Int}(\mathbf{M}_2; w_2, i_2) \cap \cdots \cap \mathrm{Int}(\mathbf{M}_a; w_a, i_a)$$

is nonempty. Indeed, we first take $I_b \in \mathrm{Int}(\mathbf{M}_b; w_b, i_b)$ arbitrarily for each $1 \le b \le a$, and then take an interval J in $\mathbb{Z}$ such that $J \supset I_b$ for all $1 \le b \le a$ (i.e., $J \supset I_1 \cup I_2 \cup \cdots \cup I_a$). Then, it follows immediately from part (1) that $J \in \mathrm{Int}(\mathbf{M}_b; w_b, i_b)$ for all $1 \le b \le a$, and hence that $J \in \mathrm{Int}(\mathbf{M}_1; w_1, i_1) \cap \mathrm{Int}(\mathbf{M}_2; w_2, i_2) \cap \cdots \cap \mathrm{Int}(\mathbf{M}_a; w_a, i_a)$.

For each $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$, we define a collection $\Theta(\mathbf{M}) = (M_\xi)_{\xi \in \Xi_\mathbb{Z}}$ of integers indexed by $\Xi_\mathbb{Z} = -\Gamma_\mathbb{Z}$ as follows. Fix $\xi \in \Xi_\mathbb{Z}$, and write it as $\xi = w\Lambda_i$ for some $w \in W_\mathbb{Z}$ and $i \in \mathbb{Z}$. Here we note that if I_1, $I_2 \in \mathrm{Int}(\mathbf{M}; w, i)$, then $M_{w\varpi_i^{I_1}} = M_{w\varpi_i^{I_2}}$. Indeed, take an interval J in $\mathbb{Z}$ such that $J \supset I_1 \cup I_2$. Then we have $M_{w\varpi_i^{I_1}} = M_{w\varpi_i^J} = M_{w\varpi_i^{I_2}}$, and hence $M_{w\varpi_i^{I_1}} = M_{w\varpi_i^{I_2}}$. We now define $M_\xi = M_{w\Lambda_i} := M_{w\varpi_i^I}$ for $I \in \mathrm{Int}(\mathbf{M}; w, i)$. Let us check that this definition of M_ξ does not depend on the choice of an expression $\xi = w\Lambda_i$. Suppose that $\xi = w\Lambda_i = v\Lambda_j$ for some w, $v \in W_\mathbb{Z}$ and i, $j \in \mathbb{Z}$; it is obvious that $i = j$ since Λ_i and Λ_j are dominant integral weights. Take an interval I in $\mathbb{Z}$ such that $I \in \mathrm{Int}(\mathbf{M}; w, i) \cap \mathrm{Int}(\mathbf{M}; v, j)$ (see Remark 3.2.3 (2)). Then, since w, $v \in W_I$ and $w\Lambda_i = v\Lambda_j$, the same argument as for the correspondence (2.4.1) shows that $w\varpi_i^I = v\varpi_j^I$. Therefore, we obtain $M_{w\Lambda_i} = M_{w\varpi_i^I} = M_{v\varpi_j^I} = M_{v\Lambda_j}$, as desired.

3.3 Kashiwara operators on the set of BZ data of type A_∞. Let $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$, and fix $p \in \mathbb{Z}$. We define $f_p\mathbf{M} = (M'_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$ as follows. For each $\gamma \in \Gamma_\mathbb{Z}$, take an interval I in $\mathbb{Z}$ such that

$$\gamma \in \Gamma_I \quad \text{and} \quad I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p); \tag{3.3.1}$$

since $\mathbf{M}_I \in \mathcal{BZ}_I$ by condition (a) of Definition 3.2.1, we can apply the lowering Kashiwara operator f_p on $\mathcal{BZ}_I$ to $\mathbf{M}_I$. We define $(f_p\mathbf{M})_\gamma = M'_\gamma$ to be $(f_p\mathbf{M}_I)_\gamma$. It follows from (2.3.4) that

$$M'_\gamma = \begin{cases} \min\bigl(M_\gamma,\ M_{s_p\gamma} + c_p(\mathbf{M}_I)\bigr) & \text{if } \langle h_p, \gamma\rangle > 0, \\ M_\gamma & \text{otherwise,} \end{cases}$$

where $c_p(\mathbf{M}_I) = M_{\varpi_p^I} - M_{s_p\varpi_p^I} - 1$. Since $I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p)$, we have

$$c_p(\mathbf{M}_I) = M_{\varpi_p^I} - M_{s_p\varpi_p^I} - 1 = M_{\Lambda_p} - M_{s_p\Lambda_p} - 1 =: c_p(\mathbf{M}),$$

where $M_{\Lambda_p} := \Theta(\mathbf{M})_{\Lambda_p}$, and $M_{s_p\Lambda_p} := \Theta(\mathbf{M})_{s_p\Lambda_p}$. Thus,

$$M'_\gamma = \begin{cases} \min\bigl(M_\gamma,\ M_{s_p\gamma} + c_p(\mathbf{M})\bigr) & \text{if } \langle h_p, \gamma\rangle > 0, \\ M_\gamma & \text{otherwise.} \end{cases} \tag{3.3.2}$$

From this description, we see that the definition of M'_γ does not depend on the choice of an interval I satisfying (3.3.1).

Remark 3.3.1. (1) Keep the notation and assumptions above. It follows from (3.3.2) that $M'_\gamma = (f_p\mathbf{M})_\gamma \le M_\gamma$ for all $\gamma \in \Gamma_\mathbb{Z}$.

(2) For $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$ and $p \in I$, there holds

$$(f_p\mathbf{M})_I = f_p\mathbf{M}_I \quad \text{if} \quad I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p). \tag{3.3.3}$$

Proposition 3.3.2. *Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, and $p \in \mathbb{Z}$. Then, $f_p\mathbf{M}$ is an element of $\mathcal{BZ}_\mathbb{Z}$.*

By this proposition, for each $p \in \mathbb{Z}$, we obtain a map f_p from $\mathcal{BZ}_\mathbb{Z}$ to itself sending $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$ to $f_p\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, which we call the lowering Kashiwara operator on $\mathcal{BZ}_\mathbb{Z}$.

Proof of Proposition 3.3.2. First we show that $f_p\mathbf{M}$ satisfies condition (a) of Definition 3.2.1. Let K be an interval in $\mathbb{Z}$. Take an interval I in $\mathbb{Z}$ such that $K \subset I$ and $I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p)$. Then, by (3.3.3), we have $(f_p\mathbf{M})_I = f_p\mathbf{M}_I \in \mathcal{BZ}_I$. Also, it follows from Lemma 2.4.1 that $\bigl((f_p\mathbf{M})_I\bigr)_K = (f_p\mathbf{M}_I)_K \in \mathcal{BZ}_K$. Since $\bigl((f_p\mathbf{M})_I\bigr)_K = (f_p\mathbf{M})_K$, we conclude that $(f_p\mathbf{M})_K \in \mathcal{BZ}_K$, as desired.

Next we show that $f_p\mathbf{M}$ satisfies condition (b) of Definition 3.2.1. Write $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$ and $f_p\mathbf{M}$ as: $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$ and $f_p\mathbf{M} = (M'_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$. Fix $w \in W_\mathbb{Z}$ and $i \in \mathbb{Z}$, and take an interval I in $\mathbb{Z}$ such that

$$I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; w, i) \cap \mathrm{Int}(\mathbf{M}; s_pw, i). \tag{3.3.4}$$

Then, by (3.3.2), we have

$$M'_{w\varpi_i^I} = \begin{cases} \min\bigl(M_{w\varpi_i^I},\ M_{s_p w\varpi_i^I} + c_p(\mathbf{M})\bigr) & \text{if } \langle h_p,\, w\varpi_i^I\rangle > 0, \\ M_{w\varpi_i^I} & \text{otherwise.} \end{cases}$$

Now, let J be an interval in $\mathbb{Z}$ containing I. Then, J is also an element of the intersection in (3.3.4) (see Remark 3.2.3 (1)). Therefore, again by (3.3.2),

$$M'_{w\varpi_i^J} = \begin{cases} \min\bigl(M_{w\varpi_i^J},\ M_{s_p w\varpi_i^J} + c_p(\mathbf{M})\bigr) & \text{if } \langle h_p,\, w\varpi_i^J\rangle > 0, \\ M_{w\varpi_i^J} & \text{otherwise.} \end{cases}$$

Since $I \in \mathrm{Int}(\mathbf{M}; w, i)$ (resp., $I \in \mathrm{Int}(\mathbf{M}; s_p w, i)$) and $J \supset I$, it follows from the definition that $M_{w\varpi_i^J} = M_{w\varpi_i^I}$ (resp., $M_{s_p w\varpi_i^J} = M_{s_p w\varpi_i^I}$). Also, since $w \in W_I$ and $p \in I$, we see that $w^{-1}h_p \in \bigoplus_{j\in I} \mathbb{Z}h_j \subset \bigoplus_{j\in J} \mathbb{Z}h_j$. Hence it follows from (3.1.4) that

$$\langle h_p,\, w\varpi_i^I\rangle = \langle w^{-1}h_p,\, \varpi_i^I\rangle = \langle w^{-1}h_p,\, \varpi_i^J\rangle = \langle h_p,\, w\varpi_i^J\rangle.$$

In particular, $\langle h_p,\, w\varpi_i^I\rangle > 0$ if and only if $\langle h_p,\, w\varpi_i^J\rangle > 0$. Consequently, we obtain $M'_{w\varpi_i^J} = M'_{w\varpi_i^I}$, which shows that $f_p\mathbf{M} = (M'_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$ satisfies condition (b) of Definition 3.2.1, as desired. Thus, we have proved that $f_p\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, thereby completing the proof of the proposition. □

Remark 3.3.3. Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, and fix $p \in \mathbb{Z}$. Also, let $w \in W_\mathbb{Z}$ and $i \in \mathbb{Z}$. The proof of Proposition 3.3.2 shows that if an interval I in $\mathbb{Z}$ is an element of the intersection

$$\mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; w, i) \cap \mathrm{Int}(\mathbf{M}; s_p w, i),$$

then I is an element of $\mathrm{Int}(f_p\mathbf{M}; w, i)$.

For intervals I, K in $\mathbb{Z}$ such that $I \supset K$, let $\mathcal{BZ}_\mathbb{Z}(I,\, K)$ denote the subset of $\mathcal{BZ}_\mathbb{Z}$ consisting of all elements $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$ such that $I \in \mathrm{Int}(\mathbf{M}; v, k)$ for every $v \in W_K$ and $k \in K$; note that $\mathcal{BZ}_\mathbb{Z}(I,\, K)$ is nonempty since $\mathbf{O} \in \mathcal{BZ}_\mathbb{Z}(I,\, K)$ (for the definition of $\mathbf{O}$, see Example 3.2.2).

Lemma 3.3.4. *Keep the notation above.*

(1) *The set $\mathcal{BZ}_\mathbb{Z}(I,\, K)$ is stable under the lowering Kashiwara operators f_p for $p \in K$.*

(2) *Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}(I,\, K)$, and $p_1, p_2, \ldots, p_a \in K$. Then,*

$$(f_{p_a} f_{p_{a-1}} \cdots f_{p_1}\mathbf{M})_I = f_{p_a} f_{p_{a-1}} \cdots f_{p_1}\mathbf{M}_I. \tag{3.3.5}$$

Proof. (1) Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}(I,\, K)$, and $p \in K$. We show that $I \in \mathrm{Int}(f_p\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$. Fix $v \in W_K$ and $k \in K$. Since the interval I is an element of the intersection

$$\mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; v, k) \cap \mathrm{Int}(\mathbf{M}; s_p v, k),$$

it follows from Remark 3.3.3 that $I \in \mathrm{Int}(f_p\mathbf{M}; v, k)$. This proves part (1).

(2) We show formula (3.3.5) by induction on a. Assume first that $a = 1$. Since $I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p)$ for all $p \in K$, it follows from (3.3.3) that $(f_{p_1}\mathbf{M})_I = f_{p_1}\mathbf{M}_I$. Assume next that $a > 1$. We set $\mathbf{M}' := f_{p_{a-1}} \cdots f_{p_1}\mathbf{M}$. Because $\mathbf{M}' \in \mathcal{BZ}_{\mathbb{Z}}(I,\, K)$ by part (1), we see by the same argument as above that $(f_{p_a} f_{p_{a-1}} \cdots f_{p_1}\mathbf{M})_I = (f_{p_a}\mathbf{M}')_I = f_{p_a}\mathbf{M}'_I$. Also, by the induction hypothesis, $\mathbf{M}'_I = (f_{p_{a-1}} \cdots f_{p_1}\mathbf{M})_I = f_{p_{a-1}} \cdots f_{p_1}\mathbf{M}_I$. Combining these, we obtain $(f_{p_a} f_{p_{a-1}} \cdots f_{p_1}\mathbf{M})_I = f_{p_a} f_{p_{a-1}} \cdots f_{p_1}\mathbf{M}_I$, as desired. This proves part (2). $\square$

For $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, we set

$$\varepsilon_p(\mathbf{M}) := -\left(M_{\Lambda_p} + M_{s_p\Lambda_p} + \sum_{q \in \mathbb{Z} \setminus \{p\}} a_{qp} M_{\Lambda_q} \right), \tag{3.3.6}$$

where $M_{\Lambda_i} := \Theta(\mathbf{M})_{\Lambda_i}$ for $i \in \mathbb{Z}$, and $M_{s_p\Lambda_p} := \Theta(\mathbf{M})_{s_p\Lambda_p}$. Note that $\varepsilon_p(\mathbf{M})$ is a nonnegative integer. Indeed, let I be an interval in $\mathbb{Z}$ such that

$$I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; e, p+1) \cap \mathrm{Int}(\mathbf{M}; e, p-1).$$

Then, we have

$$\begin{aligned} \varepsilon_p(\mathbf{M}) &= -\left(M_{\Lambda_p} + M_{s_p\Lambda_p} - M_{\Lambda_{p-1}} - M_{\Lambda_{p+1}} \right) \\ &= -\left(M_{\varpi_p^I} + M_{s_p\varpi_p^I} - M_{\varpi_{p-1}^I} - M_{\varpi_{p+1}^I} \right) \\ &= -\left(M_{\varpi_p^I} + M_{s_p\varpi_p^I} + \sum_{q \in I \setminus \{p\}} a_{qp} M_{\varpi_q^I} \right) = \varepsilon_p(\mathbf{M}_I). \end{aligned} \tag{3.3.7}$$

Hence it follows from condition (a) of Definition 3.2.1 and the comment following (2.3.3) that $\varepsilon_p(\mathbf{M}) = \varepsilon_p(\mathbf{M}_I)$ is a nonnegative integer.

Now, for $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, we define $e_p\mathbf{M}$ as follows. If $\varepsilon_p(\mathbf{M}) = 0$, then we set $e_p\mathbf{M} := \mathbf{0}$, where $\mathbf{0}$ is an additional element, which is not contained in $\mathcal{BZ}_{\mathbb{Z}}$. If $\varepsilon_p(\mathbf{M}) > 0$, then we define $e_p\mathbf{M} = (M'_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}}$ as follows. For each $\gamma \in \Gamma_{\mathbb{Z}}$, take an interval I in $\mathbb{Z}$ such that

$$\begin{aligned} &\gamma \in \Gamma_I \quad \text{and} \\ &I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; e, p-1) \cap \mathrm{Int}(\mathbf{M}; e, p+1); \end{aligned} \tag{3.3.8}$$

note that $\min I < p < \max I$, since $p-1,\, p+1 \in I$. Consider $\mathbf{M}_I \in \mathcal{BZ}_I$ (see condition (a) of Definition 3.2.1); since $\varepsilon_p(\mathbf{M}) = \varepsilon_p(\mathbf{M}_I)$ by (3.3.7), we have $\varepsilon_p(\mathbf{M}_I) > 0$, which implies that $e_p\mathbf{M}_I \neq \mathbf{0}$. We define $(e_p\mathbf{M})_\gamma = M'_\gamma$ to be $(e_p\mathbf{M}_I)_\gamma$. By virtue of the following lemma, this definition of M'_γ does not depend on the choice of an interval I satisfying (3.3.8).

Lemma 3.3.5. *Keep the notation and assumptions above. Assume that an interval J in $\mathbb{Z}$ satisfies the condition* (3.3.8) *with I replaced by J. Then, we have $(e_p\mathbf{M}_J)_\gamma = (e_p\mathbf{M}_I)_\gamma$.*

Proof. We may assume from the beginning that $J \supset I$. Indeed, let K be an interval in $\mathbb{Z}$ containing both of the intervals J and I. Then we see from Remark 3.2.3 (1) that K satisfies the condition (3.3.8) with I replaced by K. If the assertion is true for K, then we have $(e_p\mathbf{M}_K)_\gamma = (e_p\mathbf{M}_I)_\gamma$ and $(e_p\mathbf{M}_K)_\gamma = (e_p\mathbf{M}_J)_\gamma$, and hence $(e_p\mathbf{M}_J)_\gamma = (e_p\mathbf{M}_I)_\gamma$.

We may further assume that $J = I \cup \{\max I + 1\}$ or $J = I \cup \{\min I - 1\}$; for simplicity of notation, suppose that $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, m, m+1\}$. Note that $1 = \min I < p < \max I = m$ (see the comment preceding this proposition).

We write $e_p\mathbf{M}_I \in \mathcal{BZ}_I$ and $e_p\mathbf{M}_J \in \mathcal{BZ}_J$ as: $e_p\mathbf{M}_I = (M'_\gamma)_{\gamma\in\Gamma_I}$ and $e_p\mathbf{M}_J = (M''_\gamma)_{\gamma\in\Gamma_J}$, respectively; we need to show that $M''_\gamma = M'_\gamma$ for all $\gamma \in \Gamma_I$. Recall that $e_p\mathbf{M}_I = (M'_\gamma)_{\gamma\in\Gamma_I}$ is defined to be the unique BZ datum for $\mathfrak{g}_I$ such that $M'_{\varpi_p^I} = M_{\varpi_p^I} + 1$, and such that $M'_\gamma = M_\gamma$ for all $\gamma \in \Gamma_I$ with $\langle h_p, \gamma\rangle \le 0$ (see Fact 2.3.1). It follows from Lemma 2.4.1 that $(e_p\mathbf{M}_J)_I = (M''_\gamma)_{\gamma\in\Gamma_I}$ is a BZ datum for $\mathfrak{g}_I$. Also, we see from the definition of $e_p\mathbf{M}_J$ that $M''_\gamma = M_\gamma$ for all $\gamma \in \Gamma_I \subset \Gamma_J$ with $\langle h_p, \gamma\rangle \le 0$. Therefore, if we can show the equality $M''_{\varpi_p^I} = M_{\varpi_p^I} + 1$, then it follows from the uniqueness that $(e_p\mathbf{M}_J)_I = (M''_\gamma)_{\gamma\in\Gamma_I}$ is equal to $e_p\mathbf{M}_I = (M'_\gamma)_{\gamma\in\Gamma_I}$, and hence $M''_\gamma = M'_\gamma$ for all $\gamma \in \Gamma_I$, as desired. We will show that $M''_{\varpi_p^I} = M_{\varpi_p^I} + 1$.

First, let us verify the following formula:

$$\varpi_k^I = s_{m+1} \cdots s_{k+2}s_{k+1}(\varpi_{k+1}^J) \quad \text{for } 1 \le k \le m. \tag{3.3.9}$$

Indeed, we have

$$\begin{aligned}\varpi_k^I &= -w_0^I(\Lambda_{\omega_I(k)}) = -w_0^I(\Lambda_{m-k+1}) \\ &= -w_0^I w_0^J w_0^J(\Lambda_{m-k+1}) = w_0^I w_0^J(\varpi^J_{\omega_J(m-k+1)}) = w_0^I w_0^J(\varpi_{k+1}^J).\end{aligned}$$

Consequently, by using the reduced expressions

$$\begin{aligned}w_0^J &= s_1(s_2s_1)(s_3s_2s_1)\cdots(s_m\cdots s_2s_1)(s_{m+1}\cdots s_2s_1), \\ w_0^I &= (s_m\cdots s_2s_1)\cdots(s_1s_2s_3)(s_1s_2)s_1,\end{aligned}$$

we see that $\varpi_k^I = s_{m+1}\cdots s_2s_1(\varpi_{k+1}^J) = s_{m+1}\cdots s_{k+2}s_{k+1}(\varpi_{k+1}^J)$, as desired.

Now, let us show that $M''_{\varpi_p^I} = M_{\varpi_p^I} + 1$. We set $w := s_{m+1}\cdots s_{p+3}s_{p+2} \in W_J$. Then, $a_{p,p+1} = a_{p+1,p} = -1$ and $ws_{p+1} > w$, $ws_p > w$. Therefore, since $e_p\mathbf{M}_J = (M''_\gamma)_{\gamma\in\Gamma_J} \in \mathcal{BZ}_J$, it follows from condition (2) of Definition 2.2.1 that

$$M''_{ws_{p+1}\varpi_{p+1}^J} + M''_{ws_p\varpi_p^J} = \min\bigl(M''_{w\varpi_{p+1}^J} + M''_{ws_{p+1}s_p\varpi_p^J},\ M''_{w\varpi_p^J} + M''_{ws_ps_{p+1}\varpi_{p+1}^J}\bigr). \tag{3.3.10}$$

Also, by using (3.3.9) and the facts that $s_q\varpi_p^J = \varpi_p^J$, $s_q\varpi_{p+1}^J = \varpi_{p+1}^J$ for $p+2 \le q \le m+1$

and that $s_q s_p = s_p s_q$ for $p+2 \le q \le m+1$, we get

$$\begin{aligned}
& w s_{p+1} \varpi_{p+1}^J = s_{m+1} \cdots s_{p+2} s_{p+1} \varpi_{p+1}^J = \varpi_p^I, \\
& w s_p \varpi_p^J = s_{m+1} \cdots s_{p+2} s_p \varpi_p^J = s_p s_{m+1} \cdots s_{p+2} \varpi_p^J = s_p \varpi_p^J, \\
& w \varpi_{p+1}^J = s_{m+1} \cdots s_{p+2} \varpi_{p+1}^J = \varpi_{p+1}^J, \\
& w s_{p+1} s_p \varpi_p^J = s_{m+1} \cdots s_{p+2} s_{p+1} s_p \varpi_p^J = \varpi_{p-1}^I, \\
& w \varpi_p^J = s_{m+1} \cdots s_{p+2} \varpi_p^J = \varpi_p^J, \\
& w s_p s_{p+1} \varpi_{p+1}^J = s_{m+1} \cdots s_{p+2} s_p s_{p+1} \varpi_{p+1}^J = s_p s_{m+1} \cdots s_{p+2} s_{p+1} \varpi_{p+1}^J = s_p \varpi_p^I.
\end{aligned}$$

Hence the equation (3.3.10) can be rewritten as:

$$M''_{\varpi_p^I} + M''_{s_p \varpi_p^J} = \min\big(M''_{\varpi_{p+1}^J} + M''_{\varpi_{p-1}^I},\ M''_{\varpi_p^J} + M''_{s_p \varpi_p^I}\big). \tag{3.3.11}$$

Since $\langle h_p,\, s_p \varpi_p^J \rangle = -1 < 0$, it follows from the definition of $e_p \mathbf{M}_J$ that $M''_{s_p \varpi_p^J} = M_{s_p \varpi_p^J}$. Similarly, $M''_{\varpi_{p+1}^J} = M_{\varpi_{p+1}^J}$, $M''_{\varpi_{p-1}^J} = M_{\varpi_{p-1}^J}$, and $M''_{s_p \varpi_p^I} = M_{s_p \varpi_p^I}$. In addition, it follows from the definition of $e_p \mathbf{M}_J$ that $M''_{\varpi_p^J} = M_{\varpi_p^J} + 1$. Substituting these into (3.3.11), we obtain

$$M''_{\varpi_p^I} + M_{s_p \varpi_p^J} = \min\big(M_{\varpi_{p+1}^J} + M_{\varpi_{p-1}^I},\ M_{\varpi_p^J} + 1 + M_{s_p \varpi_p^I}\big). \tag{3.3.12}$$

Here, observe that $M_{\varpi_{p-1}^I} = M_{\varpi_{p-1}^J}$ (resp., $M_{s_p \varpi_p^I} = M_{s_p \varpi_p^J}$) since $I \in \mathrm{Int}(\mathbf{M}; e, p-1)$ (resp., $I \in \mathrm{Int}(\mathbf{M}; s_p, p)$) and $J \supset I$. As a result, we get

$$M''_{\varpi_p^I} + M_{s_p \varpi_p^J} = \min\big(M_{\varpi_{p+1}^J} + M_{\varpi_{p-1}^J},\ M_{\varpi_p^J} + 1 + M_{s_p \varpi_p^J}\big). \tag{3.3.13}$$

Moreover, since $\varepsilon_p(\mathbf{M}) > 0$ by assumption, we see from (3.3.7) with I replaced by J that $M_{\varpi_p^J} + M_{s_p \varpi_p^J} < M_{\varpi_{p+1}^J} + M_{\varpi_{p-1}^J}$, which implies that

$$\min\big(M_{\varpi_{p+1}^J} + M_{\varpi_{p-1}^J},\ M_{\varpi_p^J} + 1 + M_{s_p \varpi_p^J}\big) = M_{\varpi_p^J} + 1 + M_{s_p \varpi_p^J}.$$

Combining this and (3.3.13), we obtain $M''_{\varpi_p^I} = M_{\varpi_p^J} + 1$. Noting that $M_{\varpi_p^J} = M_{\varpi_p^I}$ since $I \in \mathrm{Int}(\mathbf{M}; e, p)$ and $J \supset I$, we conclude that $M''_{\varpi_p^I} = M_{\varpi_p^I} + 1$, as desired. This completes the proof of the lemma. □

Remark 3.3.6. (1) Let $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$ be such that $e_p \mathbf{M} \ne \mathbf{0}$. Then,

$$(e_p \mathbf{M})_\gamma = M_\gamma \quad \text{for all } \gamma \in \Gamma_{\mathbb{Z}} \text{ with } \langle h_p,\, \gamma \rangle \le 0. \tag{3.3.14}$$

Indeed, let $\gamma \in \Gamma_{\mathbb{Z}}$ be such that $\langle h_p,\, \gamma \rangle \le 0$. Take an interval I in $\mathbb{Z}$ satisfying the condition (3.3.8). Then, by the definition, $(e_p \mathbf{M})_\gamma = (e_p \mathbf{M}_I)_\gamma$. Also, we see from the definition of e_p on $\mathcal{BZ}_I$ (see Fact 2.3.1) that $(e_p \mathbf{M}_I)_\gamma = M_\gamma$. Hence we get $(e_p \mathbf{M})_\gamma = (e_p \mathbf{M}_I)_\gamma = M_\gamma$, as desired.

(2) For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, there holds

$$\begin{aligned}
& (e_p \mathbf{M})_I = e_p \mathbf{M}_I \\
& \text{if } I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; e, p-1) \cap \mathrm{Int}(\mathbf{M}; e, p+1).
\end{aligned} \tag{3.3.15}$$

Proposition 3.3.7. *Let* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$*, and* $p \in \mathbb{Z}$*. Then,* $e_p\mathbf{M}$ *is an element of* $\mathcal{BZ}_{\mathbb{Z}} \cup \{\mathbf{0}\}$.

By this proposition, for each $p \in \mathbb{Z}$, we obtain a map e_p from $\mathcal{BZ}_{\mathbb{Z}}$ to $\mathcal{BZ}_{\mathbb{Z}} \cup \{\mathbf{0}\}$ sending $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ to $e_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}} \cup \{\mathbf{0}\}$, which we call the raising Kashiwara operator on $\mathcal{BZ}_{\mathbb{Z}}$. By convention, we set $e_p\mathbf{0} := \mathbf{0}$ for all $p \in \mathbb{Z}$, and $f_p\mathbf{0} := \mathbf{0}$ for all $p \in \mathbb{Z}$.

Proof of Proposition 3.3.7. Assume that $e_p\mathbf{M} \neq \mathbf{0}$. Using (3.3.15) instead of (3.3.3), we can show by an argument (for $f_p\mathbf{M}$) in the proof of Proposition 3.3.2 that $e_p\mathbf{M}$ satisfies condition (a) of Definition 3.2.1. We will, therefore, show that $e_p\mathbf{M}$ satisfies condition (b) of Definition 3.2.1. We write $\mathbf{M}$ and $e_p\mathbf{M}$ as: $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$ and $e_p\mathbf{M} = (M'_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$, respectively. Fix $w \in W$ and $i \in \mathbb{Z}$, and then fix an interval K in $\mathbb{Z}$ such that $w \in W_K$ and i, $p-1$, p, $p+1 \in K$. Now, take an interval I in $\mathbb{Z}$ such that $I \in \mathrm{Int}(\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$ (see Remark 3.2.3 (2)); note that I is an element of the intersection

$$\mathrm{Int}(\mathbf{M}; e,\, p) \cap \mathrm{Int}(\mathbf{M}; s_p,\, p) \cap \mathrm{Int}(\mathbf{M}; e,\, p-1) \cap \mathrm{Int}(\mathbf{M}; e,\, p+1), \tag{3.3.16}$$

since $p-1,\, p,\, p+1 \in K$. We need to show that $M'_{w\varpi_i^J} = M'_{w\varpi_i^I}$ for all intervals J in $\mathbb{Z}$ containing I.

Before we proceed further, we make some remarks: Through the bijections (2.4.7) and (3.1.3), we can (and do) identify the set Γ_K (of chamber weights) for $\mathfrak{g}_K$ with the subset $\Gamma_I^K = \{v\varpi_k^I \mid v \in W_K,\, k \in K\} \subset \Gamma_I \subset \Gamma_{\mathbb{Z}}$; note that $v\varpi_k^K \in \Gamma_K$ corresponds to $v\varpi_k^I \in \Gamma_I^K$ for $v \in W_K$ and $k \in K$. Let J be an interval in $\mathbb{Z}$ containing I. As above, we can (and do) identify the set Γ_K (of chamber weights) for $\mathfrak{g}_K$ with the subset $\Gamma_J^K = \{v\varpi_k^J \mid v \in W_K,\, k \in K\} \subset \Gamma_J \subset \Gamma_{\mathbb{Z}}$; note that $v\varpi_k^K \in \Gamma_K$ corresponds to $v\varpi_k^J \in \Gamma_J^K$ for $v \in W_K$ and $k \in K$. Thus, the three sets Γ_J^K $(\subset \Gamma_J \subset \Gamma_{\mathbb{Z}})$, Γ_I^K $(\subset \Gamma_I \subset \Gamma_{\mathbb{Z}})$, and Γ_K can be identified as follows:

$$\begin{array}{ccccc} \Gamma_K & \stackrel{\sim}{\to} & \Gamma_J^K & \stackrel{\sim}{\to} & \Gamma_I^K, \\ v\varpi_k^K & \mapsto & v\varpi_k^J & \mapsto & v\varpi_k^I \quad \text{for } v \in W_K \text{ and } k \in K. \end{array} \tag{3.3.17}$$

Also, it follows from the definition of $\mathcal{BZ}_{\mathbb{Z}}$ that $\mathbf{M}_I = (M_\gamma)_{\gamma\in\Gamma_I} \in \mathcal{BZ}_I$ and $\mathbf{M}_J = (M_\gamma)_{\gamma\in\Gamma_J} \in \mathcal{BZ}_J$. Therefore, by Lemma 2.4.2, $(\mathbf{M}_I)^K = (M_\gamma)_{\gamma\in\Gamma_I^K}$ and $(\mathbf{M}_J)^K = (M_\gamma)_{\gamma\in\Gamma_J^K}$ are BZ data for $\mathfrak{g}_K$ if we identify the sets Γ_I^K and Γ_J^K with the set Γ_K through the bijection (3.3.17). Since $M_{v\varpi_k^J} = M_{v\varpi_k^I}$ for all $v \in W_K$ and $k \in K$ by our assumption, we deduce that $(\mathbf{M}_J)^K = (\mathbf{M}_I)^K$ if we identify the three sets Γ_J^K, Γ_I^K, and Γ_K as in (3.3.17).

Now we are ready to show that $M'_{w\varpi_i^J} = M'_{w\varpi_i^I}$. By our assumption that $e_p\mathbf{M} \neq \mathbf{0}$ and (3.3.16), it follows that $e_p\mathbf{M}_I \neq \mathbf{0}$, and hence $e_p\mathbf{M}_I$ is an element of $\mathcal{BZ}_I$; we see from (3.3.15) that $e_p\mathbf{M}_I = (e_p\mathbf{M})_I = (M'_\gamma)_{\gamma\in\Gamma_I}$. Hence, by Lemma 2.4.2, $(e_p\mathbf{M}_I)^K = (M'_\gamma)_{\gamma\in\Gamma_I^K}$ is a BZ datum for $\mathfrak{g}_K$ if we identify the set Γ_I^K with the set Γ_K through the bijection (3.3.17). Also, by the definition of $e_p\mathbf{M}_I$, we see that $M'_{\varpi_p^I} = M_{\varpi_p^I} + 1$, and $M'_{v\varpi_k^I} = M_{v\varpi_k^I}$ for all $v \in W_K$ and $k \in K$ with $\langle h_p,\, v\varpi_k^I\rangle \leq 0$. Here we observe that for $v \in W_K$ and $k \in K$, the inequality $\langle h_p,\, v\varpi_k^I\rangle \leq 0$ holds if and only if the inequality $\langle h_p,\, v\varpi_k^K\rangle \leq 0$ holds. Indeed, let $v \in W_K$, and $k \in K$. Note that $v^{-1}h_p \in \bigoplus_{j\in K} \mathbb{Z}h_j \subset \bigoplus_{j\in I} \mathbb{Z}h_j$ since $p \in K$ by our assumption. Hence it follows from (3.1.4) that

$$\langle h_p,\, v\varpi_k^I \rangle = \langle v^{-1}h_p,\, \varpi_k^I \rangle = \langle v^{-1}h_p,\, \varpi_k^K \rangle = \langle h_p,\, v\varpi_k^K \rangle,$$

which implies that $\langle h_p,\, v\varpi_k^I \rangle \le 0$ if and only if $\langle h_p,\, v\varpi_k^K \rangle \le 0$. Therefore, we deduce from Fact 2.3.1 that $(e_p\mathbf{M}_I)^K = (M'_\gamma)_{\gamma\in\Gamma_I^K}$ is equal to $e_p\big((\mathbf{M}_I)^K\big)$ if we identify Γ_I^K and Γ_K by (3.3.17). Furthermore, we see from Remark 3.2.3 (1) that the interval $J \supset I$ is also an element of $\mathrm{Int}(\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$. In exactly the same way as above (with I replaced by J), we can show that $(e_p\mathbf{M}_J)^K = (M'_\gamma)_{\gamma\in\Gamma_J^K}$ is a BZ datum for $\mathfrak{g}_K$, and is equal to $e_p\big((\mathbf{M}_J)^K\big)$ if we identify Γ_J^K and Γ_K by (3.3.17). Since $(\mathbf{M}_I)^K = (\mathbf{M}_J)^K$ as seen above, we obtain $e_p\big((\mathbf{M}_I)^K\big) = e_p\big((\mathbf{M}_J)^K\big)$. Consequently, we infer that $(e_p\mathbf{M}_J)^K = (M'_\gamma)_{\gamma\in\Gamma_J^K}$ is equal to $(e_p\mathbf{M}_I)^K = (M'_\gamma)_{\gamma\in\Gamma_I^K}$ if we identify Γ_J^K and Γ_I^K by (3.3.17). Because $w\varpi_i^J \in \Gamma_J^K$ corresponds to $w\varpi_i^I \in \Gamma_I^K$ through the bijection (3.3.17), we finally obtain $M'_{w\varpi_i^J} = M'_{w\varpi_i^I}$, as desired. This completes the proof of the proposition. $\square$

Remark 3.3.8. Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$ and $p \in \mathbb{Z}$ be such that $e_p\mathbf{M} \neq \mathbf{0}$. Let K be an interval in $\mathbb{Z}$ such that $p-1,\, p,\, p+1 \in K$. The proof of Proposition 3.3.7 shows that if an interval I in $\mathbb{Z}$ is an element of $\mathrm{Int}(\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$, then $I \in \mathrm{Int}(e_p\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$.

Lemma 3.3.9. *Let I and K be intervals in $\mathbb{Z}$ such that $I \supset K$ and $\#K \ge 3$.*

(1) *The set $\mathcal{BZ}_\mathbb{Z}(I,\, K) \cup \{\mathbf{0}\}$ is stable under the raising Kashiwara operators e_p for $p \in K$ with $\min K < p < \max K$.*

(2) *Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}(I,\, K)$, and let $p_1,\, p_2,\, \dots,\, p_a \in K$ be such that $\min K < p_1,\, p_2,\, \dots,\, p_a < \max K$. Then, $e_{p_a}e_{p_{a-1}}\cdots e_{p_1}\mathbf{M} \neq \mathbf{0}$ if and only if $e_{p_a}e_{p_{a-1}}\cdots e_{p_1}\mathbf{M}_I \neq \mathbf{0}$. Moreover, if $e_{p_a}e_{p_{a-1}}\cdots e_{p_1}\mathbf{M} \neq \mathbf{0}$ (or equivalently, $e_{p_a}e_{p_{a-1}}\cdots e_{p_1}\mathbf{M}_I \neq \mathbf{0}$), then*

$$(e_{p_a}e_{p_{a-1}}\cdots e_{p_1}\mathbf{M})_I = e_{p_a}e_{p_{a-1}}\cdots e_{p_1}\mathbf{M}_I. \tag{3.3.18}$$

Proof. Part (1) follows immediately from Remark 3.3.8. We will show part (2) by induction on a. Assume first that $a = 1$. Since $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}(I,\, K)$, it follows immediately that

$$I \in \mathrm{Int}(\mathbf{M}; e, p_1) \cap \mathrm{Int}(\mathbf{M}; s_{p_1}, p_1) \cap \mathrm{Int}(\mathbf{M}; e, p_1+1) \cap \mathrm{Int}(\mathbf{M}; e, p_1-1).$$

Therefore, we have $\varepsilon_{p_1}(\mathbf{M}) = \varepsilon_{p_1}(\mathbf{M}_I)$ by (3.3.7), which implies that $e_{p_1}\mathbf{M} \neq \mathbf{0}$ if and only if $e_{p_1}\mathbf{M}_I \neq \mathbf{0}$. Also, it follows from (3.3.15) that if $e_{p_1}\mathbf{M} \neq \mathbf{0}$, then $(e_{p_1}\mathbf{M})_I = e_{p_1}\mathbf{M}_I$.

Assume next that $a > 1$. For simplicity of notation, we set

$$\mathbf{M}' := e_{p_{a-1}}\cdots e_{p_1}\mathbf{M} \quad \text{and} \quad \mathbf{M}'' := e_{p_{a-1}}\cdots e_{p_1}\mathbf{M}_I.$$

Let us show that $e_{p_a}\mathbf{M}' \neq \mathbf{0}$ if and only if $e_{p_a}\mathbf{M}'' \neq \mathbf{0}$. By the induction hypothesis, we may assume that $\mathbf{M}' \neq \mathbf{0}$, $\mathbf{M}'' \neq \mathbf{0}$, and $\mathbf{M}'_I = \mathbf{M}''$. It follows from part (1) that $\mathbf{M}' \in \mathcal{BZ}_\mathbb{Z}(I,\, K)$.

Hence, by the same argument as above (the case $a = 1$), we deduce that $e_{p_a}\mathbf{M}' \neq \mathbf{0}$ if and only if $e_{p_a}\mathbf{M}'_I \neq \mathbf{0}$, which implies that $e_{p_a}\mathbf{M}' \neq \mathbf{0}$ if and only if $e_{p_a}\mathbf{M}'' \neq \mathbf{0}$. Furthermore, it follows from (3.3.15) that if $e_{p_a}\mathbf{M}' \neq \mathbf{0}$, then $(e_{p_a}\mathbf{M}')_I = e_{p_a}\mathbf{M}'_I = e_{p_a}\mathbf{M}''$. This proves the lemma. □

3.4 Some properties of Kashiwara operators on $\mathcal{BZ}_{\mathbb{Z}}$.

Lemma 3.4.1. (1) *Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, and $p \in \mathbb{Z}$. Then, $e_p f_p \mathbf{M} = \mathbf{M}$. Also, if $e_p\mathbf{M} \neq \mathbf{0}$, then $f_p e_p \mathbf{M} = \mathbf{M}$.*

(2) *Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, and let $p, q \in \mathbb{Z}$ be such that $|p-q| \geq 2$. Then, $\varepsilon_p(f_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) + 1$ and $\varepsilon_q(f_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$. Also, if $e_p\mathbf{M} \neq \mathbf{0}$, then $\varepsilon_p(e_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) - 1$ and $\varepsilon_q(e_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$.*

(3) *Let $p, q \in \mathbb{Z}$ be such that $|p-q| \geq 2$. Then, $f_pf_q = f_qf_p$, $e_pe_q = e_qe_p$, and $e_pf_q = f_qe_p$ on $\mathcal{BZ}_{\mathbb{Z}} \cup \{\mathbf{0}\}$.*

Proof. (1) We prove that $e_pf_p\mathbf{M} = \mathbf{M}$; by a similar argument, we can prove that $f_pe_p\mathbf{M} = \mathbf{M}$ if $e_p\mathbf{M} \neq \mathbf{0}$. We need to show that $e_pf_p\mathbf{M} \neq \mathbf{0}$, and that the γ-component of $e_pf_p\mathbf{M}$ is equal to that of $\mathbf{M}$ for each $\gamma \in \Gamma_{\mathbb{Z}}$. We fix $\gamma \in \Gamma_{\mathbb{Z}}$. Set $K := \{p-1, p, p+1\}$, and take an interval I in $\mathbb{Z}$ such that $\gamma \in \Gamma_I$, and such that $I \in \mathrm{Int}(\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$. Then, we have $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}(I, K)$, and hence we see from Lemma 3.3.4 that $f_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}(I, K)$ and $(f_p\mathbf{M})_I = f_p\mathbf{M}_I$. Because $e_p(f_p\mathbf{M})_I = e_p(f_p\mathbf{M}_I) = \mathbf{M}_I \neq \mathbf{0}$ by condition (a) of Definition 3.2.1 and Theorem 2.3.4, it follows from Lemma 3.3.9 (2) that $e_pf_p\mathbf{M} \neq \mathbf{0}$. Also, we deduce from Lemmas 3.3.4 (2) and 3.3.9 (2) that $(e_pf_p\mathbf{M})_I = e_pf_p\mathbf{M}_I = \mathbf{M}_I$. Since $\gamma \in \Gamma_I$ by our assumption on I, we infer that the γ-component of $e_pf_p\mathbf{M}$ is equal to that of $\mathbf{M}$. This proves part (1).

(2) We give a proof only for the equalities $\varepsilon_p(f_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) + 1$ and $\varepsilon_q(f_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$; by a similar argument, we can prove that $\varepsilon_p(e_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) - 1$ and $\varepsilon_q(e_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$ if $e_p\mathbf{M} \neq \mathbf{0}$. Write $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ and $f_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ as: $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$ and $f_p\mathbf{M} = (M'_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$, respectively. Also, write $\Theta(\mathbf{M})$ and $\Theta(f_p\mathbf{M})$ as: $\Theta(\mathbf{M}) = (M_\xi)_{\xi\in\Xi_{\mathbb{Z}}}$ and $\Theta(f_p\mathbf{M}) = (M'_\xi)_{\xi\in\Xi_{\mathbb{Z}}}$, respectively. First we show that for $i \in \mathbb{Z}$,

$$M'_{\Lambda_i} = \begin{cases} M_{\Lambda_p} - 1 & \text{if } i = p, \\ M_{\Lambda_i} & \text{otherwise.} \end{cases} \tag{3.4.1}$$

Fix $i \in \mathbb{Z}$, and take an interval I in $\mathbb{Z}$ such that

$$I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; e, i) \cap \mathrm{Int}(\mathbf{M}; s_p, i).$$

We see from Remark 3.3.3 that $I \in \mathrm{Int}(f_p\mathbf{M}; e, i)$, and hence that $M'_{\Lambda_i} = M'_{\varpi_i^I}$ by the definition. Assume now that $i \neq p$. Since $\langle h_p, \varpi_i^I\rangle \leq 0$ by (3.1.4), it follows from (3.3.2) that $M'_{\varpi_i^I} = (f_p\mathbf{M})_{\varpi_i^I} = M_{\varpi_i^I}$. Also, since $I \in \mathrm{Int}(\mathbf{M}; e, i)$, we have $M_{\varpi_i^I} = M_{\Lambda_i}$ by the definition. Therefore, we obtain

$$M'_{\Lambda_i} = M'_{\varpi_i^I} = M_{\varpi_i^I} = M_{\Lambda_i} \quad \text{if } i \neq p.$$

Assume then that $i = p$. Since $\langle h_p, \varpi_p^I \rangle = 1$, it follows from (3.3.2) that

$$M'_{\varpi_p^I} = (f_p\mathbf{M})_{\varpi_p^I} = \min\bigl(M_{\varpi_p^I},\ M_{s_p\varpi_p^I} + c_p(\mathbf{M})\bigr), \tag{3.4.2}$$

where $c_p(\mathbf{M}) = M_{\Lambda_p} - M_{s_p\Lambda_p} - 1$. Note that $M_{\varpi_p^I} = M_{\Lambda_p}$ (resp., $M_{s_p\varpi_p^I} = M_{s_p\Lambda_p}$) since $I \in \mathrm{Int}(\mathbf{M}; e, p)$ (resp., $I \in \mathrm{Int}(\mathbf{M}; s_p, p)$). Substituting these into (3.4.2), we conclude that $M'_{\Lambda_p} = M'_{\varpi_p^I} = M_{\Lambda_p} - 1$, as desired.

Next we show that

$$M'_{s_i\Lambda_i} = M_{s_i\Lambda_i} \quad \text{for } i \in \mathbb{Z} \text{ with } i \neq p-1,\, p+1. \tag{3.4.3}$$

Take an interval I in $\mathbb{Z}$ such that

$$I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; s_i, i) \cap \mathrm{Int}(\mathbf{M}; s_p s_i, i).$$

We see from Remark 3.3.3 that $I \in \mathrm{Int}(f_p\mathbf{M}; s_i, i)$, and hence that $M'_{s_i\Lambda_i} = M'_{s_i\varpi_i^I}$ by the definition. Since $i \neq p-1,\, p+1$, we deduce from (3.1.4) that $\langle h_p, s_i\varpi_i^I \rangle \leq 0$. Hence it follows from (3.3.2) that $M'_{s_i\varpi_i^I} = (f_p\mathbf{M})_{s_i\varpi_i^I} = M_{s_i\varpi_i^I}$. Also, since $I \in \mathrm{Int}(\mathbf{M}; s_i, i)$, we have $M_{s_i\varpi_i^I} = M_{s_i\Lambda_i}$. Thus we obtain $M'_{s_i\Lambda_i} = M'_{s_i\varpi_i^I} = M_{s_i\varpi_i^I} = M_{s_i\Lambda_i}$, as desired.

Now, recall from (3.3.6) that

$$\varepsilon_p(f_p\mathbf{M}) = -\left(M'_{\Lambda_p} + M'_{s_p\Lambda_p} + \sum_{r \in \mathbb{Z}\setminus\{p\}} a_{rp} M'_{\Lambda_r} \right).$$

Here, by (3.4.1) and (3.4.3), we have $M'_{\Lambda_p} = M_{\Lambda_p} - 1$, $M'_{s_p\Lambda_p} = M_{s_p\Lambda_p}$, and

$$\sum_{r \in \mathbb{Z}\setminus\{p\}} a_{rp} M'_{\Lambda_r} = \sum_{r \in \mathbb{Z}\setminus\{p\}} a_{rp} M_{\Lambda_r}.$$

Therefore, by (3.3.6), we conclude that

$$\varepsilon_p(f_p\mathbf{M}) = -\left((M_{\Lambda_p} - 1) + M_{s_p\Lambda_p} + \sum_{r \in \mathbb{Z}\setminus\{p\}} a_{rp} M_{\Lambda_r} \right) = \varepsilon_p(\mathbf{M}) + 1.$$

Arguing in the same manner, we can prove that $\varepsilon_q(f_p\mathbf{M}) = \varepsilon_q(\mathbf{M})$. This proves part (2).

(3) We prove that $e_p f_q = f_q e_p$; the proofs of the other equalities are similar. Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$. Assume first that $e_p\mathbf{M} = \mathbf{0}$, or equivalently, $\varepsilon_p(\mathbf{M}) = 0$. Then we have $f_q e_p \mathbf{M} = \mathbf{0}$. Also, it follows from part (2) that $\varepsilon_p(f_q\mathbf{M}) = \varepsilon_p(\mathbf{M}) = 0$, which implies that $e_p(f_q\mathbf{M}) = \mathbf{0}$. Thus we get $e_p f_q \mathbf{M} = f_q e_p \mathbf{M} = \mathbf{0}$.

Assume next that $e_p\mathbf{M} \neq \mathbf{0}$, or equivalently, $\varepsilon_p(\mathbf{M}) > 0$. Then we have $f_q e_p \mathbf{M} \neq \mathbf{0}$. Also, it follows from part (2) that $\varepsilon_p(f_q\mathbf{M}) = \varepsilon_p(\mathbf{M}) > 0$, which implies that $e_p(f_q\mathbf{M}) \neq \mathbf{0}$. We need to show that $(e_p f_q \mathbf{M})_\gamma = (f_q e_p \mathbf{M})_\gamma$ for all $\gamma \in \Gamma_{\mathbb{Z}}$. Fix $\gamma \in \Gamma_{\mathbb{Z}}$, and take an interval I in $\mathbb{Z}$ satisfying the following conditions:

(i) $\gamma \in \Gamma_I$;

(ii) $I \in \mathrm{Int}(f_q\mathbf{M}; e, p) \cap \mathrm{Int}(f_q\mathbf{M}; s_p, p) \cap \mathrm{Int}(f_q\mathbf{M}; e, p-1) \cap \mathrm{Int}(f_q\mathbf{M}; e, p+1)$;

(iii) $I \in \mathrm{Int}(\mathbf{M}; e, q) \cap \mathrm{Int}(\mathbf{M}; s_q, q)$;

(iv) $I \in \mathrm{Int}(e_p\mathbf{M}; e, q) \cap \mathrm{Int}(e_p\mathbf{M}; s_q, q)$;

(v) $I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p) \cap \mathrm{Int}(\mathbf{M}; e, p-1) \cap \mathrm{Int}(\mathbf{M}; e, p+1)$.

Then, we have

$$\begin{aligned}(e_pf_q\mathbf{M})_I &= e_p(f_q\mathbf{M})_I && \text{by (3.3.15) and condition (ii)} \\ &= e_p(f_q\mathbf{M}_I) && \text{by (3.3.3) and condition (iii)} \\ &= e_pf_q\mathbf{M}_I, \end{aligned}$$

and

$$\begin{aligned}(f_qe_p\mathbf{M})_I &= f_q(e_p\mathbf{M})_I && \text{by (3.3.3) and condition (iv)} \\ &= f_q(e_p\mathbf{M}_I) && \text{by (3.3.15) and condition (v)} \\ &= f_qe_p\mathbf{M}_I. \end{aligned}$$

Hence we see from condition (a) of Definition 3.2.1 and Theorem 2.3.4 that $e_pf_q\mathbf{M}_I = f_qe_p\mathbf{M}_I$, and hence $(e_pf_q\mathbf{M})_I = (f_qe_p\mathbf{M})_I$. Therefore, we obtain $(e_pf_q\mathbf{M})_\gamma = (f_qe_p\mathbf{M})_\gamma$ since $\gamma \in \Gamma_I$ by condition (i). This proves part (3), thereby completing the proof of the lemma. ☐

Remark 3.4.2. Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, and $p \in I$. From the definition, it follows that $\varepsilon_p(\mathbf{M}) = 0$ if and only if $e_p\mathbf{M} = \mathbf{0}$, and that $\varepsilon_p(\mathbf{M}) \in \mathbb{Z}_{\geq 0}$. In addition, $\varepsilon_p(e_p\mathbf{M}) = \varepsilon_p(\mathbf{M}) - 1$ by Lemma 3.4.1 (2). Consequently, we deduce that $\varepsilon_p(\mathbf{M}) = \max\{N \geq 0 \mid e_p^N\mathbf{M} \neq \mathbf{0}\}$.

4 Berenstein-Zelevinsky data of type $A_\ell^{(1)}$.

Throughout this section, we take and fix $\ell \in \mathbb{Z}_{\geq 2}$ arbitrarily.

4.1 Basic notation in type $A_\ell^{(1)}$. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra of type $A_\ell^{(1)}$ over $\mathbb{C}$. Let $\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}}$ denote the Cartan matrix of $\widehat{\mathfrak{g}}$ with index set $\widehat{I} := \{0, 1, \ldots, \ell\}$; the entries $\widehat{a}_{ij}$ are given by:

$$\widehat{a}_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ or } \ell, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.1}$$

for $i, j \in \widehat{I}$. Denote by $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, by $\widehat{h}_i \in \widehat{\mathfrak{h}}$, $i \in \widehat{I}$, the simple coroots of $\widehat{\mathfrak{g}}$, and by $\widehat{\alpha}_i \in \widehat{\mathfrak{h}}^* := \mathrm{Hom}_\mathbb{C}(\widehat{\mathfrak{h}}, \mathbb{C})$, $i \in \widehat{I}$, the simple roots of $\widehat{\mathfrak{g}}$; note that $\langle \widehat{h}_i, \widehat{\alpha}_j \rangle = \widehat{a}_{ij}$ for $i, j \in \widehat{I}$, where $\langle \cdot\,, \cdot \rangle$ is the canonical pairing between $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$.

Also, let $\widehat{\mathfrak{g}}^\vee$ denote the (Langlands) dual Lie algebra of $\widehat{\mathfrak{g}}$; that is, $\widehat{\mathfrak{g}}^\vee$ is the affine Lie algebra of type $A_\ell^{(1)}$ over $\mathbb{C}$ associated to the transpose ${}^t\widehat{A}\ (= \widehat{A})$ of $\widehat{A}$, with Cartan subalgebra

$\widehat{\mathfrak{h}}^*$, simple coroots $\widehat{\alpha}_i \in \widehat{\mathfrak{h}}^*$, $i \in \widehat{I}$, and simple roots $\widehat{h}_i \in \widehat{\mathfrak{h}}$, $i \in \widehat{I}$. Let $U_q(\widehat{\mathfrak{g}}^\vee)$ be the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to the Lie algebra $\widehat{\mathfrak{g}}^\vee$, $U_q^-(\widehat{\mathfrak{g}}^\vee)$ the negative part of $U_q(\widehat{\mathfrak{g}}^\vee)$, and $\widehat{\mathcal{B}}(\infty)$ the crystal basis of $U_q^-(\widehat{\mathfrak{g}}^\vee)$. For a dominant integral weight $\widehat{\lambda} \in \widehat{\mathfrak{h}}$ for $\widehat{\mathfrak{g}}^\vee$, $\widehat{\mathcal{B}}(\widehat{\lambda})$ denotes the crystal basis of the irreducible highest weight $U_q(\widehat{\mathfrak{g}}^\vee)$-module of highest weight $\widehat{\lambda}$.

4.2 Dynkin diagram automorphism in type A_∞ and its action on $\mathcal{BZ}_\mathbb{Z}$. For the fixed $\ell \in \mathbb{Z}_{\geq 2}$, the (Dynkin) diagram automorphism in type A_∞ is a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}$ given by: $\sigma(i) = i + \ell + 1$ for $i \in \mathbb{Z}$. This induces a $\mathbb{C}$-linear automorphism $\sigma : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}$ of $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}h_i$ by: $\sigma(h_i) = h_{\sigma(i)}$ for $i \in \mathbb{Z}$, and also a $\mathbb{C}$-linear automorphism $\sigma : \mathfrak{h}^*_{\mathrm{res}} \xrightarrow{\sim} \mathfrak{h}^*_{\mathrm{res}}$ of the restricted dual space $\mathfrak{h}^*_{\mathrm{res}} := \bigoplus_{i\in\mathbb{Z}} \mathbb{C}\Lambda_i$ of $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}h_i$ by: $\sigma(\Lambda_i) = \Lambda_{\sigma(i)}$ for $i \in \mathbb{Z}$. Observe that $\langle \sigma(h), \sigma(\Lambda)\rangle = \langle h, \Lambda\rangle$ for all $h \in \mathfrak{h}$ and $\Lambda \in \mathfrak{h}^*_{\mathrm{res}}$, and $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ for $i \in \mathbb{Z}$; note also that $\alpha_i \in \mathfrak{h}^*_{\mathrm{res}}$ for all $i \in \mathbb{Z}$, since $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$. Moreover, this $\sigma : \mathbb{Z} \to \mathbb{Z}$ naturally induces a group automorphism $\sigma : W_\mathbb{Z} \xrightarrow{\sim} W_\mathbb{Z}$ of the Weyl group $W_\mathbb{Z}$ by: $\sigma(s_i) = s_{\sigma(i)}$ for $i \in \mathbb{Z}$.

It is easily seen that $-w\Lambda_i \in \mathfrak{h}^*_{\mathrm{res}}$ for all $w \in W_\mathbb{Z}$ and $i \in \mathbb{Z}$, and hence the set $\Gamma_\mathbb{Z}$ (of chamber weights) is a subset of $\mathfrak{h}^*_{\mathrm{res}}$. In addition,

$$\sigma(-w\Lambda_i) = -\sigma(w)\Lambda_{\sigma(i)} \quad \text{for } w \in W_\mathbb{Z} \text{ and } i \in \mathbb{Z}. \tag{4.2.1}$$

Therefore, the restriction of $\sigma : \mathfrak{h}^*_{\mathrm{res}} \xrightarrow{\sim} \mathfrak{h}^*_{\mathrm{res}}$ to the subset $\Gamma_\mathbb{Z}$ gives rise to a bijection $\sigma : \Gamma_\mathbb{Z} \xrightarrow{\sim} \Gamma_\mathbb{Z}$.

Remark 4.2.1. Let I be an interval in $\mathbb{Z}$, and $i \in I$; note that $\sigma(i)$ is contained in $\sigma(I)$. Because $\varpi_i^I \in \Gamma_\mathbb{Z}$ can be written as: $\varpi_i^I = \Lambda_i - \Lambda_{(\min I)-1} - \Lambda_{(\max I)+1}$ (see (3.1.4)), we deduce that $\sigma(\varpi_i^I) = \varpi_{\sigma(i)}^{\sigma(I)}$.

Let $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$ be a collection of integers indexed by $\Gamma_\mathbb{Z}$. We define collections $\sigma(\mathbf{M})$ and $\sigma^{-1}(\mathbf{M})$ of integers indexed by $\Gamma_\mathbb{Z}$ by: $\sigma(\mathbf{M})_\gamma = M_{\sigma^{-1}(\gamma)}$ and $\sigma^{-1}(\mathbf{M})_\gamma = M_{\sigma(\gamma)}$ for each $\gamma \in \Gamma_\mathbb{Z}$, respectively.

Lemma 4.2.2. *If $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, then $\sigma(\mathbf{M}) \in \mathcal{BZ}_\mathbb{Z}$ and $\sigma^{-1}(\mathbf{M}) \in \mathcal{BZ}_\mathbb{Z}$.*

Proof. We prove that $\sigma(\mathbf{M}) \in \mathcal{BZ}_\mathbb{Z}$; we can prove that $\sigma^{-1}(\mathbf{M}) \in \mathcal{BZ}_\mathbb{Z}$ similarly. Write $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$ and $\sigma(\mathbf{M})$ as: $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$ and $\sigma(\mathbf{M}) = (M'_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$, respectively. First we prove that $\sigma(\mathbf{M}) = (M'_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$ satisfies condition (a) of Definition 3.2.1. Let K be an interval in $\mathbb{Z}$. We need to show that $\sigma(\mathbf{M})_K = (M'_\gamma)_{\gamma\in\Gamma_K}$ satisfies condition (1) of Definition 2.2.1 (with I replaced by K). Fix $w \in W_K$, and $i \in K$. For simplicity of notation, we set $w_1 := \sigma^{-1}(w)$, $i_1 := \sigma^{-1}(i)$, and $K_1 := \sigma^{-1}(K)$; note that $w_1 \in W_{K_1}$, and $i_1 \in K_1$. Since $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$, it follows from condition (a) of Definition 3.2.1 that $\mathbf{M}_{K_1} = (M_\gamma)_{\gamma\in\Gamma_{K_1}} \in \mathcal{BZ}_{K_1}$. Hence we see from condition (1) of Definition 2.2.1 that

$$M_{w_1\varpi_{i_1}^{K_1}} + M_{w_1 s_{i_1}\varpi_{i_1}^{K_1}} + \sum_{j\in K_1\setminus\{i_1\}} a_{j,i_1} M_{w_1\varpi_j^{K_1}} \leq 0.$$

Here, by the equality $a_{\sigma^{-1}(j),i_1} = a_{j,\sigma(i_1)}$,

$$\sum_{j \in K_1 \setminus \{i_1\}} a_{j,i_1} M_{w_1 \varpi_j^{K_1}} = \sum_{j \in K \setminus \{i\}} a_{\sigma^{-1}(j),i_1} M_{w_1 \varpi_{\sigma^{-1}(j)}^{K_1}} = \sum_{j \in K \setminus \{i\}} a_{ji} M_{w_1 \varpi_{\sigma^{-1}(j)}^{K_1}}.$$

Also, we see from (4.2.1) and Remark 4.2.1 that

$$\begin{aligned} M'_{w\varpi_i^K} &= M_{\sigma^{-1}(w\varpi_i^K)} = M_{w_1 \varpi_{i_1}^{K_1}}, \\ M'_{ws_i\varpi_i^K} &= M_{\sigma^{-1}(ws_i\varpi_i^K)} = M_{w_1 s_{i_1} \varpi_{i_1}^{K_1}}, \\ M'_{w\varpi_j^K} &= M_{\sigma^{-1}(w\varpi_j^K)} = M_{w_1 \varpi_{\sigma^{-1}(j)}^{K_1}} \quad \text{for } j \in K \setminus \{i\}. \end{aligned}$$

Combining these, we obtain

$$M'_{w\varpi_i^K} + M'_{ws_i\varpi_i^K} + \sum_{j \in K \setminus \{i\}} a_{ji} M'_{w\varpi_j^K} \leq 0,$$

as desired. Similarly, we can show that $\sigma(\mathbf{M})_K = (M'_\gamma)_{\gamma \in \Gamma_K}$ satisfies condition (2) of Definition 2.2.1 (with I replaced by K); use the fact that if i, $j \in K$ and $w \in W_K$ are such that $a_{ij} = a_{ji} = -1$, and $ws_i > w$, $ws_j > w$, then $a_{i_1,j_1} = a_{j_1,i_1} = -1$, and $w_1 s_{i_1} > w_1$, $w_1 s_{j_1} > w_1$, where $i_1 := \sigma^{-1}(i)$, $j_1 := \sigma^{-1}(j) \in K_1 = \sigma^{-1}(K)$, and $w_1 := \sigma^{-1}(w) \in W_{K_1}$. It remains to show that $M'_{w_0^K \varpi_i^K} = 0$ for all $i \in K$. Let $i \in K$, and set $i_1 := \sigma^{-1}(i) \in K_1 = \sigma^{-1}(K)$. Then, by (4.2.1) and Remark 4.2.1, we have

$$M'_{w_0^K \varpi_i^K} = M_{\sigma^{-1}(w_0^K \varpi_i^K)} = M_{w_0^{K_1} \varpi_{i_1}^{K_1}},$$

which is equal to zero since $\mathbf{M}_{K_1} \in \mathcal{BZ}_{K_1}$. This proves that $\sigma(\mathbf{M})_K \in \mathcal{BZ}_K$, as desired.

Next we prove that $\sigma(\mathbf{M}) = (M'_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$ satisfies condition (b) of Definition 3.2.1. Fix $w \in W_\mathbb{Z}$, and $i \in \mathbb{Z}$. Take an interval I in $\mathbb{Z}$ such that $I_1 := \sigma^{-1}(I)$ is an element of $\mathrm{Int}(\mathbf{M}; w_1, i_1)$, where $w_1 := \sigma^{-1}(w)$ and $i_1 := \sigma^{-1}(i)$. Let J be an arbitrary interval in $\mathbb{Z}$ containing I, and set $J_1 := \sigma^{-1}(J)$; note that $J_1 \supset I_1$. Then, we have

$$\begin{aligned} M'_{w\varpi_i^J} &= M_{\sigma^{-1}(w\varpi_i^J)} = M_{w_1 \varpi_{i_1}^{J_1}} \quad \text{by (4.2.1) and Remark 4.2.1} \\ &= M_{w_1 \varpi_{i_1}^{I_1}} \quad \text{since } I_1 \in \mathrm{Int}(\mathbf{M}; w_1, i_1) \text{ and } J_1 \supset I_1 \\ &= M_{\sigma^{-1}(w\varpi_i^I)} \quad \text{by (4.2.1) and Remark 4.2.1} \\ &= M'_{w\varpi_i^I}. \end{aligned}$$

This proves that $\sigma(\mathbf{M}) = (M'_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$ satisfies condition (b) of Definition 3.2.1, thereby completing the proof of the lemma. □

Remark 4.2.3. Let $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$, and write $\sigma(\mathbf{M}) \in \mathcal{BZ}_\mathbb{Z}$ as: $\sigma(\mathbf{M}) = (M'_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$. Fix $w \in W_\mathbb{Z}$, and $i \in \mathbb{Z}$. Set $w_1 := \sigma^{-1}(w)$, and $i_1 := \sigma^{-1}(i)$. We see from the proof of

Lemma 4.2.2 that if we take an interval I in $\mathbb{Z}$ such that $I_1 := \sigma^{-1}(I)$ is an element of $\mathrm{Int}(\mathbf{M}; w_1, i_1)$, then the interval I is an element of $\mathrm{Int}(\sigma(\mathbf{M}); w, i)$. Moreover, since $M'_{w\varpi_i^I} = M_{w_1\varpi_{i_1}^{I_1}}$, we have

$$M'_{w\Lambda_i} = M'_{w\varpi_i^I} = M_{w_1\varpi_{i_1}^{I_1}} = M_{w_1\Lambda_{i_1}} = M_{\sigma^{-1}(w\Lambda_i)},$$

where $M'_{w\Lambda_i} := \Theta(\sigma(\mathbf{M}))_{w\Lambda_i}$, and $M_{w_1\Lambda_{i_1}} := \Theta(\mathbf{M})_{w_1\Lambda_{i_1}}$.

By Lemma 4.2.2, we obtain maps $\sigma : \mathcal{BZ}_\mathbb{Z} \to \mathcal{BZ}_\mathbb{Z}$, $\mathbf{M} \mapsto \sigma(\mathbf{M})$, and $\sigma^{-1} : \mathcal{BZ}_\mathbb{Z} \to \mathcal{BZ}_\mathbb{Z}$, $\mathbf{M} \mapsto \sigma^{-1}(\mathbf{M})$; since both of the composite maps $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$ are the identity map on $\mathcal{BZ}_\mathbb{Z}$, it follows that $\sigma : \mathcal{BZ}_\mathbb{Z} \to \mathcal{BZ}_\mathbb{Z}$ and $\sigma^{-1} : \mathcal{BZ}_\mathbb{Z} \to \mathcal{BZ}_\mathbb{Z}$ are bijective.

Lemma 4.2.4. (1) *Let* $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, *and* $p \in \mathbb{Z}$. *Then,* $\varepsilon_p(\sigma(\mathbf{M})) = \varepsilon_{\sigma^{-1}(p)}(\mathbf{M})$.

(2) *There hold* $\sigma \circ e_p = e_{\sigma(p)} \circ \sigma$ *and* $\sigma \circ f_p = f_{\sigma(p)} \circ \sigma$ *on* $\mathcal{BZ}_\mathbb{Z} \cup \{\mathbf{0}\}$ *for all* $p \in \mathbb{Z}$. *Here it is understood that* $\sigma(\mathbf{0}) := \mathbf{0}$.

Proof. Part (1) follows immediately from (3.3.6) by using Remark 4.2.3. We will prove part (2). Let $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}$, and $p \in \mathbb{Z}$. First we show that $\sigma(f_p\mathbf{M}) = f_{\sigma(p)}(\sigma(\mathbf{M}))$, i.e., $\big(\sigma(f_p\mathbf{M})\big)_\gamma = \big(f_{\sigma(p)}(\sigma(\mathbf{M}))\big)_\gamma$ for all $\gamma \in \Gamma_\mathbb{Z}$. We write $\mathbf{M}$ and $\sigma(\mathbf{M})$ as: $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$ and $\sigma(\mathbf{M}) = (M'_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$, respectively. It follows from (3.3.2) that

$$\begin{aligned}\big(\sigma(f_p\mathbf{M})\big)_\gamma &= (f_p\mathbf{M})_{\sigma^{-1}(\gamma)} \\ &= \begin{cases} \min\big(M_{\sigma^{-1}(\gamma)},\ M_{s_p\sigma^{-1}(\gamma)} + c_p(\mathbf{M})\big) & \text{if } \langle h_p,\, \sigma^{-1}(\gamma)\rangle > 0, \\ M_{\sigma^{-1}(\gamma)} & \text{otherwise,} \end{cases}\end{aligned} \tag{4.2.2}$$

where $c_p(\mathbf{M}) = M_{\Lambda_p} - M_{s_p\Lambda_p} - 1$ with $M_{\Lambda_p} := \Theta(\mathbf{M})_{\Lambda_p}$ and $M_{s_p\Lambda_p} := \Theta(\mathbf{M})_{s_p\Lambda_p}$. Also, it follows from (3.3.2) that

$$\big(f_{\sigma(p)}(\sigma(\mathbf{M}))\big)_\gamma = \begin{cases} \min\big(M'_\gamma,\ M'_{s_{\sigma(p)}\gamma} + c_{\sigma(p)}(\sigma(\mathbf{M}))\big) & \text{if } \langle h_{\sigma(p)},\, \gamma\rangle > 0, \\ M'_\gamma & \text{otherwise,} \end{cases} \tag{4.2.3}$$

where $c_{\sigma(p)}(\sigma(\mathbf{M})) = M'_{\Lambda_{\sigma(p)}} - M'_{s_{\sigma(p)}\Lambda_{\sigma(p)}} - 1$ with $M'_{\Lambda_{\sigma(p)}} := \Theta(\sigma(\mathbf{M}))_{\Lambda_{\sigma(p)}}$ and $M'_{s_{\sigma(p)}\Lambda_{\sigma(p)}} := \Theta(\sigma(\mathbf{M}))_{s_{\sigma(p)}\Lambda_{\sigma(p)}}$. Here we see from Remark 4.2.3 that

$$M'_{\Lambda_{\sigma(p)}} = M_{\sigma^{-1}(\Lambda_{\sigma(p)})} = M_{\Lambda_p} \quad \text{and} \quad M'_{s_{\sigma(p)}\Lambda_{\sigma(p)}} = M_{\sigma^{-1}(s_{\sigma(p)}\Lambda_{\sigma(p)})} = M_{s_p\Lambda_p},$$

and hence that $c_{\sigma(p)}(\sigma(\mathbf{M})) = c_p(\mathbf{M})$. In addition,

$$M'_\gamma = M_{\sigma^{-1}(\gamma)} \quad \text{and} \quad M'_{s_{\sigma(p)}\gamma} = M_{\sigma^{-1}(s_{\sigma(p)}\gamma)} = M_{s_p\sigma^{-1}(\gamma)}$$

by the definitions. Observe that $\langle h_{\sigma(p)},\, \gamma\rangle = \langle \sigma(h_p),\, \gamma\rangle = \langle h_p,\, \sigma^{-1}(\gamma)\rangle$, and hence that $\langle h_{\sigma(p)},\, \gamma\rangle > 0$ if and only if $\langle h_p,\, \sigma^{-1}(\gamma)\rangle > 0$. Substituting these into (4.2.3), we obtain

$$\begin{aligned}\big(f_{\sigma(p)}(\sigma(\mathbf{M}))\big)_\gamma &= \begin{cases} \min\big(M_{\sigma^{-1}(\gamma)},\ M_{s_p\sigma^{-1}(\gamma)} + c_p(\mathbf{M})\big) & \text{if } \langle h_p,\, \sigma^{-1}(\gamma)\rangle > 0, \\ M_{\sigma^{-1}(\gamma)} & \text{otherwise,} \end{cases} \\ &= \big(\sigma(f_p\mathbf{M})\big)_\gamma,\end{aligned}$$

as desired.

Next we show that $\sigma(e_p\mathbf{M}) = e_{\sigma(p)}(\sigma(\mathbf{M}))$. If $e_p\mathbf{M} = \mathbf{0}$, or equivalently, $\varepsilon_p(\mathbf{M}) = 0$, then it follows from part (1) that $\varepsilon_{\sigma(p)}(\sigma(\mathbf{M})) = \varepsilon_p(\mathbf{M}) = 0$, and hence $e_{\sigma(p)}(\sigma(\mathbf{M})) = \mathbf{0}$, which implies that $\sigma(e_p\mathbf{M}) = e_{\sigma(p)}(\sigma(\mathbf{M})) = \mathbf{0}$. Assume, therefore, that $e_p\mathbf{M} \neq \mathbf{0}$, or equivalently, $\varepsilon_p(\mathbf{M}) > 0$. Then, it follows from part (1) that $\varepsilon_{\sigma(p)}(\sigma(\mathbf{M})) = \varepsilon_p(\mathbf{M}) > 0$, and hence $e_{\sigma(p)}(\sigma(\mathbf{M})) \neq \mathbf{0}$. Consequently, we see from Lemma 3.4.1 (1) that $f_{\sigma(p)}e_{\sigma(p)}(\sigma(\mathbf{M})) = \sigma(\mathbf{M})$. Also,

$$\begin{aligned} f_{\sigma(p)}(\sigma(e_p\mathbf{M})) &= \sigma(f_pe_p\mathbf{M}) \quad \text{since } f_{\sigma(p)} \circ \sigma = \sigma \circ f_p \\ &= \sigma(\mathbf{M}) \quad \text{by Lemma 3.4.1 (1).} \end{aligned}$$

Thus, we have $f_{\sigma(p)}e_{\sigma(p)}(\sigma(\mathbf{M})) = \sigma(\mathbf{M}) = f_{\sigma(p)}(\sigma(e_p\mathbf{M}))$. Applying $e_{\sigma(p)}$ to both sides of this equation, we obtain $e_{\sigma(p)}(\sigma(\mathbf{M})) = \sigma(e_p\mathbf{M})$ by Lemma 3.4.1 (1), as desired. This completes the proof of the lemma. ☐

4.3 BZ data of type $A_\ell^{(1)}$ and a crystal structure on them.

Definition 4.3.1. A BZ datum of type $A_\ell^{(1)}$ is a BZ datum $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}$ of type A_∞ such that $\sigma(\mathbf{M}) = \mathbf{M}$, or equivalently, $M_{\sigma^{-1}(\gamma)} = M_\gamma$ for all $\gamma \in \Gamma_\mathbb{Z}$.

Remark 4.3.2. Keep the notation of Remark 4.2.3. In addition, we assume that $\sigma(\mathbf{M}) = \mathbf{M}$. Because $I \in \mathrm{Int}(\sigma(\mathbf{M}); w, i) = \mathrm{Int}(\mathbf{M}; w, i)$ and $M'_{w\varpi_i^I} = M_{w\varpi_i^I}$ by the assumption that $\sigma(\mathbf{M}) = \mathbf{M}$, it follows that $M'_{w\Lambda_i} = M'_{w\varpi_i^I} = M_{w\varpi_i^I} = M_{w\Lambda_i}$. Since $M'_{w\Lambda_i} = M_{\sigma^{-1}(w\Lambda_i)}$ as shown in Remark 4.2.3, we obtain $M_{\sigma^{-1}(w\Lambda_i)} = M_{w\Lambda_i}$.

Denote by $\mathcal{BZ}_\mathbb{Z}^\sigma$ the set of all BZ data of type $A_\ell^{(1)}$; that is,

$$\mathcal{BZ}_\mathbb{Z}^\sigma := \bigl\{\mathbf{M} \in \mathcal{BZ}_\mathbb{Z} \mid \sigma(\mathbf{M}) = \mathbf{M}\bigr\}. \tag{4.3.1}$$

Let us define a crystal structure for $U_q(\widehat{\mathfrak{g}}^\vee)$ on the set $\mathcal{BZ}_\mathbb{Z}^\sigma$ (see Proposition 4.3.8 below).

For $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma$, we set

$$\mathrm{wt}(\mathbf{M}) := \sum_{i\in\widehat{I}} M_{\Lambda_i}\widehat{h}_i, \tag{4.3.2}$$

where $M_{\Lambda_i} := \Theta(\mathbf{M})_{\Lambda_i}$ for $i \in \mathbb{Z}$.

In what follows, we need the following notation. Let L be a finite subset of $\mathbb{Z}$ such that $|q - q'| \geq 2$ for all $q, q' \in L$ with $q \neq q'$. Then, it follows from Lemma 3.4.1 (3) that $f_qf_{q'} = f_{q'}f_q$ and $e_qe_{q'} = e_{q'}e_q$ for all $q, q' \in L$. Hence we can define the following operator on $\mathcal{BZ}_\mathbb{Z} \cup \{\mathbf{0}\}$:

$$f_L := \prod_{q\in L} f_q \quad \text{and} \quad e_L := \prod_{q\in L} e_q.$$

For $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma$ and $p \in \mathbb{Z}$, we define $\widehat{f}_p\mathbf{M} = (M'_\gamma)_{\gamma\in\Gamma_\mathbb{Z}}$ by

$$(\widehat{f}_p\mathbf{M})_\gamma = M'_\gamma := (f_{L(\gamma,p)}\mathbf{M})_\gamma \quad \text{for } \gamma \in \Gamma_\mathbb{Z}, \tag{4.3.3}$$

where we set

$$L(\gamma, p) := \{q \in p + (\ell+1)\mathbb{Z} \mid \langle h_q, \gamma\rangle > 0\}$$

for $\gamma \in \Gamma_{\mathbb{Z}}$ and $p \in \widehat{I}$; note that $L(\gamma, p)$ is a finite subset of $p + (\ell+1)\mathbb{Z}$. It is obvious that if $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ are congruent modulo $\ell + 1$, then

$$\widehat{f}_p\mathbf{M} = \widehat{f}_q\mathbf{M} \quad \text{for all } \mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}. \tag{4.3.4}$$

Remark 4.3.3. Let $\mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}$, and $p \in \mathbb{Z}$. For each $\gamma \in \Gamma_{\mathbb{Z}}$, take an arbitrary finite subset L of $p + (\ell+1)\mathbb{Z}$ containing $L(\gamma, p)$. Then we have

$$(f_L\mathbf{M})_\gamma = (f_{L(\gamma,p)}\mathbf{M})_\gamma = (\widehat{f}_p\mathbf{M})_\gamma. \tag{4.3.5}$$

Indeed, we have $(f_L\mathbf{M})_\gamma = (f_{L(\gamma,p)}f_{L\setminus L(\gamma,p)}\mathbf{M})_\gamma$. Since $\langle h_q, \gamma\rangle \le 0$ for all $q \in L \setminus L(\gamma, p)$ by the definition of $L(\gamma, p)$, we deduce, using (3.3.2) repeatedly, that $(f_{L(\gamma,p)}f_{L\setminus L(\gamma,p)}\mathbf{M})_\gamma = (f_{L(\gamma,p)}\mathbf{M})_\gamma$.

Proposition 4.3.4. *Let* $\mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}$*, and* $p \in \mathbb{Z}$*. Then,* $\widehat{f}_p\mathbf{M}$ *is an element of* $\mathcal{BZ}^{\sigma}_{\mathbb{Z}}$*.*

By this proposition, for each $p \in \mathbb{Z}$, we obtain a map $\widehat{f}_p$ from $\mathcal{BZ}^{\sigma}_{\mathbb{Z}}$ to itself sending $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ to $\widehat{f}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, which we call the lowering Kashiwara operator on $\mathcal{BZ}^{\sigma}_{\mathbb{Z}}$. By convention, we set $\widehat{f}_p\mathbf{0} := \mathbf{0}$ for all $p \in \mathbb{Z}$.

Proof of Proposition 4.3.4. First we show that $\widehat{f}_p\mathbf{M}$ satisfies condition (a) of Definition 3.2.1. Let K be an interval in $\mathbb{Z}$. Take a finite subset L of $p+(\ell+1)\mathbb{Z}$ such that $L \supset L(\gamma, p)$ for all $\gamma \in \Gamma_K$. Then, we see from Remark 4.3.3 that $(\widehat{f}_p\mathbf{M})_\gamma = (f_L\mathbf{M})_\gamma$ for all $\gamma \in \Gamma_K$, and hence that $(\widehat{f}_p\mathbf{M})_K = (f_L\mathbf{M})_K$. Since $f_L\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ by Proposition 3.3.2, it follows from condition (a) of Definition 3.2.1 that $(f_L\mathbf{M})_K \in \mathcal{BZ}_K$, and hence $(\widehat{f}_p\mathbf{M})_K \in \mathcal{BZ}_K$.

Next we show that $\widehat{f}_p\mathbf{M}$ satisfies condition (b) of Definition 3.2.1. Fix $w \in W_{\mathbb{Z}}$ and $i \in \mathbb{Z}$. We set

$$L := \begin{cases} \{q \in p + (\ell+1)\mathbb{Z} \mid w^{-1}h_q \neq h_q\} & \text{if } i \notin p + (\ell+1)\mathbb{Z}, \\ \{q \in p + (\ell+1)\mathbb{Z} \mid w^{-1}h_q \neq h_q\} \cup \{i\} & \text{otherwise.} \end{cases} \tag{4.3.6}$$

It is easily checked that L is a finite subset of $p + (\ell+1)\mathbb{Z}$. Furthermore, we can verify that $L \supset L(w\varpi^I_i, p)$ for all intervals I in $\mathbb{Z}$ such that $w \in W_I$ and $i \in I$. Indeed, suppose that $q \in p + (\ell+1)\mathbb{Z}$ is not contained in L; note that $q \neq i$ and $w^{-1}h_q = h_q$. We see that

$$\langle h_q, w\varpi^I_i\rangle = \langle w^{-1}h_q, \varpi^I_i\rangle = \langle h_q, \varpi^I_i\rangle,$$

and that $\langle h_q, \varpi^I_i\rangle \le 0$ by (3.1.4) since $q \neq i$. This implies that q is not contained in $L(w\varpi^I_i, p)$.

Now, let us take $I \in \mathrm{Int}(f_L\mathbf{M}; w, i)$, and let J be an arbitrary interval in $\mathbb{Z}$ containing I. We claim that $(\widehat{f}_p\mathbf{M})_{w\varpi^J_i} = (\widehat{f}_p\mathbf{M})_{w\varpi^I_i}$. Since $I \in \mathrm{Int}(f_L\mathbf{M}; w, i)$, it follows that $(f_L\mathbf{M})_{w\varpi^J_i} =$

$(f_L\mathbf{M})_{w\varpi_i^I}$. Also, because $L \supset L(w\varpi_i^J, p)$ and $L \supset L(w\varpi_i^I, p)$ as seen above, we see from Remark 4.3.3 that $(\widehat{f}_p\mathbf{M})_{w\varpi_i^J} = (f_L\mathbf{M})_{w\varpi_i^J}$ and $(\widehat{f}_p\mathbf{M})_{w\varpi_i^I} = (f_L\mathbf{M})_{w\varpi_i^I}$. Combining these, we obtain $(\widehat{f}_p\mathbf{M})_{w\varpi_i^J} = (f_L\mathbf{M})_{w\varpi_i^J} = (f_L\mathbf{M})_{w\varpi_i^I} = (\widehat{f}_p\mathbf{M})_{w\varpi_i^I}$, as desired. Thus, we have shown that $\widehat{f}_p\mathbf{M}$ satisfies condition (b) of Definition 3.2.1, and hence $\widehat{f}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$.

Finally, we show that $\sigma(\widehat{f}_p\mathbf{M}) = \widehat{f}_p\mathbf{M}$, or equivalently, $(\widehat{f}_p\mathbf{M})_{\sigma^{-1}(\gamma)} = (\widehat{f}_p\mathbf{M})_\gamma$ for all $\gamma \in \Gamma_{\mathbb{Z}}$. Fix $\gamma \in \Gamma_{\mathbb{Z}}$. Observe that $\sigma(L(\sigma^{-1}(\gamma), p)) = L(\gamma, p)$ since $\langle h_{\sigma(q)}, \gamma\rangle = \langle \sigma(h_q), \gamma\rangle = \langle h_q, \sigma^{-1}(\gamma)\rangle$. Therefore, we have

$$\begin{aligned}
(\widehat{f}_p\mathbf{M})_{\sigma^{-1}(\gamma)} &= (f_{L(\sigma^{-1}(\gamma),p)}\mathbf{M})_{\sigma^{-1}(\gamma)} = \big(\sigma(f_{L(\sigma^{-1}(\gamma),p)}\mathbf{M})\big)_\gamma \\
&= \big(f_{\sigma(L(\sigma^{-1}(\gamma),p))}\sigma(\mathbf{M})\big)_\gamma \quad \text{by Lemma 4.2.4 (2)} \\
&= \big(f_{\sigma(L(\sigma^{-1}(\gamma),p))}\mathbf{M}\big)_\gamma \quad \text{by the assumption that } \sigma(\mathbf{M}) = \mathbf{M} \\
&= \big(f_{L(\gamma,p)}\mathbf{M}\big)_\gamma \quad \text{since } \sigma(L(\sigma^{-1}(\gamma), p)) = L(\gamma, p) \\
&= (\widehat{f}_p\mathbf{M})_\gamma,
\end{aligned}$$

as desired. This completes the proof of the proposition. □

Now, for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^\sigma$ and $p \in \mathbb{Z}$, we set

$$\widehat{\varepsilon}_p(\mathbf{M}) := -\left(M_{\Lambda_p} + M_{s_p\Lambda_p} + \sum_{q\in\mathbb{Z}\setminus\{p\}} a_{qp}M_{\Lambda_q}\right) = \varepsilon_p(\mathbf{M}), \tag{4.3.7}$$

where $M_{\Lambda_i} := \Theta(\mathbf{M})_{\Lambda_i}$ for $i \in \mathbb{Z}$, and $M_{s_p\Lambda_p} := \Theta(\mathbf{M})_{s_p\Lambda_p}$. It follows from (3.3.7) that $\widehat{\varepsilon}_p(\mathbf{M}) = \varepsilon_p(\mathbf{M})$ is a nonnegative integer. Also, using Lemma 4.2.4 (1) repeatedly, we can easily verify that if $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ are congruent modulo $\ell + 1$, then

$$\widehat{\varepsilon}_p(\mathbf{M}) = \varepsilon_p(\mathbf{M}) = \varepsilon_q(\mathbf{M}) = \widehat{\varepsilon}_q(\mathbf{M}) \quad \text{for all } \mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^\sigma. \tag{4.3.8}$$

Lemma 4.3.5. *Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^\sigma$, and $p \in \mathbb{Z}$. Suppose that $\widehat{\varepsilon}_p(\mathbf{M}) > 0$. Then, $e_L\mathbf{M} \neq \mathbf{0}$ for every finite subset L of $p + (\ell + 1)\mathbb{Z}$.*

Proof. We show by induction on the cardinality $|L|$ of L that $e_L\mathbf{M} \neq \mathbf{0}$, and $\varepsilon_q(e_L\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) > 0$ for all $q \in p + (\ell+1)\mathbb{Z}$ with $q \notin L$. Assume first that $|L| = 1$. Then, $L = \{q'\}$ for some $q' \in p + (\ell+1)\mathbb{Z}$, and $e_L = e_{q'}$. It follows from (4.3.8) that $\varepsilon_{q'}(\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) > 0$, which implies that $e_{q'}\mathbf{M} \neq \mathbf{0}$. Also, for $q \in p+(\ell+1)\mathbb{Z}$ with $q \neq q'$, it follows from Lemma 3.4.1 (2) and (4.3.8) that $\varepsilon_q(e_{q'}\mathbf{M}) = \varepsilon_q(\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M})$.

Assume next that $|L| > 1$. Take an arbitrary $q' \in L$, and set $L' := L \setminus \{q'\}$. Then, by the induction hypothesis, we have $e_{L'}\mathbf{M} \neq \mathbf{0}$, and $\varepsilon_{q'}(e_{L'}\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) > 0$; note that $q' \notin L'$. This implies that $e_L\mathbf{M} = e_{q'}(e_{L'}\mathbf{M}) \neq \mathbf{0}$. Also, for $q \in p + (\ell + 1)\mathbb{Z}$ with $q \notin L$, we see from Lemma 3.4.1 (2) and the induction hypothesis that $\varepsilon_q(e_L\mathbf{M}) = \varepsilon_q(e_{q'}e_{L'}\mathbf{M}) = \varepsilon_q(e_{L'}\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M})$. This proves the lemma. □

For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \mathbb{Z}$, we define $\widehat{e}_p\mathbf{M}$ as follows. If $\widehat{\varepsilon}_p(\mathbf{M}) = 0$, then we set $\widehat{e}_p\mathbf{M} := \mathbf{0}$. If $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, then we define $\widehat{e}_p\mathbf{M} = (M'_{\gamma})_{\gamma\in\Gamma_{\mathbb{Z}}}$ by

$$(\widehat{e}_p\mathbf{M})_{\gamma} = M'_{\gamma} := (e_{L(\gamma,p)}\mathbf{M})_{\gamma} \quad \text{for each } \gamma \in \Gamma_{\mathbb{Z}}; \tag{4.3.9}$$

note that $e_{L(\gamma,p)}\mathbf{M} \neq \mathbf{0}$ by Lemma 4.3.5. It is easily seen by (4.3.8) that if $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ are congruent modulo $\ell + 1$, then

$$\widehat{e}_p\mathbf{M} = \widehat{e}_q\mathbf{M} \quad \text{for all } \mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}. \tag{4.3.10}$$

Remark 4.3.6. Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$, and $p \in \mathbb{Z}$. Assume that $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, or equivalently, $\widehat{e}_p\mathbf{M} \neq \mathbf{0}$. For each $\gamma \in \Gamma_{\mathbb{Z}}$, take an arbitrary finite subset L of $p + (\ell+1)\mathbb{Z}$ containing $L(\gamma, p)$. Then we see by Lemma 4.3.5 that $e_L\mathbf{M} \neq \mathbf{0}$. Moreover, by the same argument as for (4.3.5) (using (3.3.14) instead of (3.3.2)), we derive

$$(e_L\mathbf{M})_{\gamma} = (e_{L(\gamma,p)}\mathbf{M})_{\gamma} = (\widehat{e}_p\mathbf{M})_{\gamma}. \tag{4.3.11}$$

Proposition 4.3.7. *Let* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$*, and* $p \in \mathbb{Z}$*. Then,* $\widehat{e}_p\mathbf{M}$ *is contained in* $\mathcal{BZ}_{\mathbb{Z}}^{\sigma} \cup \{\mathbf{0}\}$*.*

Because the proof of this proposition is similar to that of Proposition 4.3.4, we omit it. By this proposition, for each $p \in \mathbb{Z}$, we obtain a map $\widehat{e}_p$ from $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ to $\mathcal{BZ}_{\mathbb{Z}}^{\sigma} \cup \{\mathbf{0}\}$ sending $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$ to $\widehat{e}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}} \cup \{\mathbf{0}\}$, which we call the raising Kashiwara operator on $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$. By convention, we set $\widehat{e}_p\mathbf{0} := \mathbf{0}$ for all $p \in \mathbb{Z}$.

Finally, we set

$$\widehat{\varphi}_p(\mathbf{M}) := \langle \mathrm{wt}(\mathbf{M}), \widehat{\alpha}_{\overline{p}} \rangle + \widehat{\varepsilon}_p(\mathbf{M}) \quad \text{for } \mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma} \text{ and } p \in \mathbb{Z}, \tag{4.3.12}$$

where $\overline{p}$ denotes a unique element in $\widehat{I} = \{0, 1, \ldots, \ell\}$ to which $p \in \mathbb{Z}$ is congruent modulo $\ell + 1$.

Proposition 4.3.8. *The set* $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$*, equipped with the maps* wt, $\widehat{e}_p$, $\widehat{f}_p$ $(p \in \widehat{I})$*, and* $\widehat{\varepsilon}_p$, $\widehat{\varphi}_p$ $(p \in \widehat{I})$ *above, is a crystal for* $U_q(\widehat{\mathfrak{g}}^{\vee})$*.*

Proof. It is obvious from (4.3.12) that $\widehat{\varphi}_p(\mathbf{M}) = \langle \mathrm{wt}(\mathbf{M}), \widehat{\alpha}_p \rangle + \widehat{\varepsilon}_p(\mathbf{M})$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$ (see condition (1) of [HK, Definition 4.5.1]).

We show that $\mathrm{wt}(\widehat{f}_p\mathbf{M}) = \mathrm{wt}(\mathbf{M}) - \widehat{h}_p$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$ (see condition (3) of [HK, Definition 4.5.1]). Write $\mathbf{M}$, $f_p\mathbf{M}$, and $\widehat{f}_p\mathbf{M}$ as: $\mathbf{M} = (M_{\gamma})_{\gamma\in\Gamma_{\mathbb{Z}}}$, $f_p\mathbf{M} = (M'_{\gamma})_{\gamma\in\Gamma_{\mathbb{Z}}}$, and $\widehat{f}_p\mathbf{M} = (M''_{\gamma})_{\gamma\in\Gamma_{\mathbb{Z}}}$, respectively; write $\Theta(\mathbf{M})$, $\Theta(f_p\mathbf{M})$, and $\Theta(\widehat{f}_p\mathbf{M})$ as: $\Theta(\mathbf{M}) = (M_{\xi})_{\xi\in\Xi_{\mathbb{Z}}}$, $\Theta(f_p\mathbf{M}) = (M'_{\xi})_{\xi\in\Xi_{\mathbb{Z}}}$, and $\Theta(\widehat{f}_p\mathbf{M}) = (M''_{\xi})_{\xi\in\Xi_{\mathbb{Z}}}$, respectively. We claim that $M''_{\Lambda_i} = M'_{\Lambda_i}$ for all $i \in \mathbb{Z}$. Fix $i \in \mathbb{Z}$, and take an interval I in $\mathbb{Z}$ such that $I \in \mathrm{Int}(\widehat{f}_p\mathbf{M}; e, i) \cap \mathrm{Int}(f_p\mathbf{M}; e, i)$. Then, we have $M''_{\Lambda_i} = M''_{\varpi_i^I} = (\widehat{f}_p\mathbf{M})_{\varpi_i^I}$, and $M'_{\Lambda_i} = M'_{\varpi_i^I}$ by the definitions. Also, since

$L(\varpi_i^I, p) \subset \{p\}$ by (3.1.4), it follows from Remark 4.3.3 that $(\widehat{f}_p\mathbf{M})_{\varpi_i^I} = (f_p\mathbf{M})_{\varpi_i^I} = M'_{\varpi_i^I}$. Combining these, we infer that $M''_{\Lambda_i} = M'_{\Lambda_i}$, as desired. Therefore, we see from (3.4.1) that

$$M''_{\Lambda_i} = M'_{\Lambda_i} = \begin{cases} M_{\Lambda_p} - 1 & \text{if } i = p, \\ M_{\Lambda_i} & \text{otherwise.} \end{cases} \tag{4.3.13}$$

The equation $\mathrm{wt}(\widehat{f}_p\mathbf{M}) = \mathrm{wt}(\mathbf{M}) - \widehat{h}_p$ follows immediately from (4.3.13) and the definition (4.3.2) of the map wt.

Similarly, we can show that $\mathrm{wt}(\widehat{e}_p\mathbf{M}) = \mathrm{wt}(\mathbf{M}) + \widehat{h}_p$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$ if $\widehat{e}_p\mathbf{M} \neq \mathbf{0}$ (see condition (2) of [HK, Definition 4.5.1]).

Let us show that $\widehat{\varepsilon}_p(\widehat{f}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) + 1$ and $\widehat{\varphi}_p(\widehat{f}_p\mathbf{M}) = \widehat{\varphi}_p(\mathbf{M}) - 1$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$ (see condition (5) of [HK, Definition 4.5.1]). The second equation follows immediately from the first one and the definition (4.3.12) of the map $\widehat{\varphi}$, since $\mathrm{wt}(\widehat{f}_p\mathbf{M}) = \mathrm{wt}(\mathbf{M}) - \widehat{h}_p$ as shown above. It, therefore, suffices to show the first equation; to do this, we use the notation above. We claim that $M''_{s_p\Lambda_p} = M'_{s_p\Lambda_p} = M_{s_p\Lambda_p}$. Indeed, let I be an interval in $\mathbb{Z}$ such that $I \in \mathrm{Int}(\widehat{f}_p\mathbf{M}; s_p, p) \cap \mathrm{Int}(f_p\mathbf{M}; s_p, p)$. Then, in exactly the same way as above, we see that

$$\begin{aligned} M''_{s_p\Lambda_p} &= M''_{s_p\varpi_p^I} = (\widehat{f}_p\mathbf{M})_{s_p\varpi_p^I} \\ &= (f_p\mathbf{M})_{s_p\varpi_p^I} \quad \text{by Remark 4.3.3 (note that } L(s_p\varpi_p^I,\, p) = \emptyset \text{ by (3.1.4))} \\ &= M'_{s_p\varpi_p^I} = M'_{s_p\Lambda_p}. \end{aligned}$$

In addition, the equality $M'_{s_p\Lambda_p} = M_{s_p\Lambda_p}$ follows from (3.4.3). Hence we get $M''_{s_p\Lambda_p} = M_{s_p\Lambda_p}$, as desired. Using this and (4.3.13), we deduce from the definition (4.3.7) of the map $\widehat{\varepsilon}_p$ that $\widehat{\varepsilon}_p(\widehat{f}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) + 1$.

Similarly, we can show that $\widehat{\varepsilon}_p(\widehat{e}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) - 1$ and $\widehat{\varphi}_p(\widehat{e}_p\mathbf{M}) = \widehat{\varphi}_p(\mathbf{M}) + 1$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$ if $\widehat{e}_p\mathbf{M} \neq \mathbf{0}$ (see condition (4) of [HK, Definition 4.5.1]).

Finally, we show that $\widehat{e}_p\widehat{f}_p\mathbf{M} = \mathbf{M}$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$, and that $\widehat{f}_p\widehat{e}_p\mathbf{M} = \mathbf{M}$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$ if $\widehat{e}_p\mathbf{M} \neq \mathbf{0}$ (see condition (6) of [HK, Definition 4.5.1]). We give a proof only for the first equation, since the proof of the second one is similar. Write $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ as: $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}}$. Note that $\widehat{e}_p\widehat{f}_p\mathbf{M} \neq \mathbf{0}$, since $\widehat{\varepsilon}_p(\widehat{f}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) + 1 > 0$. We need to show that $(\widehat{e}_p\widehat{f}_p\mathbf{M})_\gamma = M_\gamma$ for all $\gamma \in \Gamma_{\mathbb{Z}}$. Fix $\gamma \in \Gamma_{\mathbb{Z}}$. We deduce from Lemma 4.3.11 below that

$$(\widehat{e}_p\widehat{f}_p\mathbf{M})_\gamma = (e_{L(\gamma,p)}f_{L(\gamma,p)}\mathbf{M})_\gamma.$$

Therefore, it follows from Lemma 3.4.1 (1) and (3) that $e_{L(\gamma,p)}f_{L(\gamma,p)}\mathbf{M} = \mathbf{M}$. Hence we obtain $(\widehat{e}_p\widehat{f}_p\mathbf{M})_\gamma = M_\gamma$. Thus, we have shown that $\widehat{e}_p\widehat{f}_p\mathbf{M} = \mathbf{M}$, thereby completing the proof of the proposition. □

Remark 4.3.9. Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$, and $p \in \widehat{I}$. From the definition, it follows that $\widehat{\varepsilon}_p(\mathbf{M}) = 0$ if and only if $\widehat{e}_p\mathbf{M} = \mathbf{0}$, and that $\widehat{\varepsilon}_p(\mathbf{M}) \in \mathbb{Z}_{\geq 0}$. In addition, $\widehat{\varepsilon}_p(\widehat{e}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) - 1$. Consequently, we deduce that $\widehat{\varepsilon}_p(\mathbf{M}) = \max\{N \geq 0 \mid \widehat{e}_p^N\mathbf{M} \neq \mathbf{0}\}$. Moreover, by (4.3.8) and (4.3.10), the same is true for all $p \in \mathbb{Z}$.

The following lemma will be needed in the proof of Lemma 4.3.11 below.

Lemma 4.3.10. *Let K be an interval in $\mathbb{Z}$, and let X be a product of Kashiwara operators of the form: $X = x_1 x_2 \cdots x_a$, where $x_b \in \{f_q, e_q \mid \min K < q < \max K\}$ for each $1 \le b \le a$. If $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $X\widehat{y}_p\mathbf{M} \neq \mathbf{0}$ for some $p \in \mathbb{Z}$, where $\widehat{y}_p = \widehat{e}_p$ or $\widehat{f}_p$, then there exists a finite subset L_0 of $p + (\ell+1)\mathbb{Z}$ such that $Xy_L\mathbf{M} \neq \mathbf{0}$ and $(X\widehat{y}_p\mathbf{M})_K = (Xy_L\mathbf{M})_K$ for every finite subset L of $p + (\ell+1)\mathbb{Z}$ containing L_0, where $y_L = e_L$ if $\widehat{y}_p = \widehat{e}_p$, and $y_L = f_L$ if $\widehat{y}_p = \widehat{f}_p$.*

Proof. Note that $\widehat{y}_p\mathbf{M} \neq \mathbf{0}$ since $X\widehat{y}_p\mathbf{M} \neq \mathbf{0}$ by our assumption. Let I be an interval in $\mathbb{Z}$ containing K such that $I \in \mathrm{Int}(\widehat{y}_p\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$, and such that $\min I < \min K \le \max K < \max I$. Then, we have $\widehat{y}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}(I, K)$ (for the definition of $\mathcal{BZ}_{\mathbb{Z}}(I, K)$, see the paragraph following Remark 3.3.3). Because X is a product of those Kashiwara operators which are taken from the set $\{f_q, e_q \mid \min K < q < \max K\}$, it follows from Lemmas 3.3.4 (2) and 3.3.9 (2) that

$$X(\widehat{y}_p\mathbf{M})_I \neq \mathbf{0} \quad \text{and} \quad (X\widehat{y}_p\mathbf{M})_I = X(\widehat{y}_p\mathbf{M})_I. \tag{4.3.14}$$

Now, we set $L_0 := \bigcup_{\zeta \in \Gamma_I} L(\zeta, p)$, and take an arbitrary finite subset L of $p + (\ell+1)\mathbb{Z}$ containing L_0. Then, we see from Remark 4.3.3 (if $\widehat{y}_p = \widehat{f}_p$) or Remark 4.3.6 (if $\widehat{y}_p = \widehat{e}_p$) that

$$(\widehat{y}_p\mathbf{M})_\zeta = (y_L\mathbf{M})_\zeta \quad \text{for all } \zeta \in \Gamma_I, \tag{4.3.15}$$

which implies that $(\widehat{y}_p\mathbf{M})_I = (y_L\mathbf{M})_I$. Combining this and (4.3.14), we obtain

$$X(y_L\mathbf{M})_I \neq \mathbf{0} \quad \text{and} \quad (X\widehat{y}_p\mathbf{M})_I = X(y_L\mathbf{M})_I. \tag{4.3.16}$$

We show that $I \in \mathrm{Int}(y_L\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$. To do this, we need the following claim.

Claim. *Keep the notation above. If J is an interval in $\mathbb{Z}$ containing I, then $L(v\varpi_k^J, p) = L(v\varpi_k^I, p)$ for all $v \in W_K$ and $k \in K$.*

Proof of Claim. Fix $v \in W_K$ and $k \in K$. First, let us show that if $q \in p + (\ell+1)\mathbb{Z}$ is not contained in I, then q is contained neither in $L(v\varpi_k^J, p)$ nor in $L(v\varpi_k^I, p)$. Because $\min I < \min K$ and $\max I > \max K$, we have $q < (\min K) - 1$ or $q > (\max K) + 1$. Hence it follows that $v^{-1}h_q = h_q$ since $v \in W_K$. Also, note that $q \neq k$ since $k \in K \subset I$. Therefore, we see that $\langle h_q, v\varpi_k^J \rangle = \langle h_q, \varpi_k^J \rangle \le 0$ and $\langle h_q, v\varpi_k^I \rangle = \langle h_q, \varpi_k^I \rangle \le 0$ by (3.1.4), which implies that $q \notin L(v\varpi_k^J, p)$ and $q \notin L(v\varpi_k^I, p)$.

Next, let us consider the case that $q \in p + (\ell+1)\mathbb{Z}$ is contained in I. In this case, we have $v^{-1}h_q \in \bigoplus_{i \in I} \mathbb{Z}h_i \subset \bigoplus_{i \in J} \mathbb{Z}h_i$, and hence $\langle h_q, v\varpi_k^J \rangle = \langle v^{-1}h_q, \varpi_k^J \rangle = \langle v^{-1}h_q, \varpi_k^I \rangle = \langle h_q, v\varpi_k^I \rangle$ by (3.1.4). In particular, $\langle h_q, v\varpi_k^J \rangle > 0$ if and only if $\langle h_q, v\varpi_k^I \rangle > 0$. Therefore, $q \in L(v\varpi_k^J, p)$ if and only if $q \in L(v\varpi_k^I, p)$. This proves the claim. ▮

Fix $v \in W_K$ and $k \in K$, and let J be an arbitrary interval in $\mathbb{Z}$ containing I. We verify that $(y_L\mathbf{M})_{v\varpi_k^J} = (y_L\mathbf{M})_{v\varpi_k^I}$. Since $I \in \mathrm{Int}(\widehat{y}_p\mathbf{M}; v, k)$ by assumption, it follows that $(\widehat{y}_p\mathbf{M})_{v\varpi_k^J} = (\widehat{y}_p\mathbf{M})_{v\varpi_k^I}$. Note that $(\widehat{y}_p\mathbf{M})_{v\varpi_k^I} = (y_L\mathbf{M})_{v\varpi_k^I}$ by (4.3.15) since $v\varpi_k^I \in \Gamma_I$. Also, it follows from the claim above that $L(v\varpi_k^J, p) = L(v\varpi_k^I, p) \subset L_0 \subset L$. Hence we see again from Remark 4.3.3 (if $\widehat{y}_p = \widehat{f}_p$) or Remark 4.3.6 (if $\widehat{y}_p = \widehat{e}_p$) that $(\widehat{y}_p\mathbf{M})_{v\varpi_k^J} = (y_L\mathbf{M})_{v\varpi_k^J}$. Combining these, we obtain $(y_L\mathbf{M})_{v\varpi_k^J} = (\widehat{y}_p\mathbf{M})_{v\varpi_k^J} = (\widehat{y}_p\mathbf{M})_{v\varpi_k^I} = (y_L\mathbf{M})_{v\varpi_k^I}$, as desired. Thus we have shown that $I \in \mathrm{Int}(y_L\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$, which implies that $y_L\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}(I, K)$.

Here we recall that X is a product of those Kashiwara operators which are taken from the set $\{f_q, e_q \mid \min K < q < \max K\}$ by assumption, and that $X(y_L\mathbf{M})_I \neq \mathbf{0}$ by (4.3.16). Therefore, we deduce again from Lemmas 3.3.4 (2) and 3.3.9 (2) that $Xy_L\mathbf{M} \neq \mathbf{0}$, and $X(y_L\mathbf{M})_I = (Xy_L\mathbf{M})_I$. Combining this and (4.3.16), we obtain $(X\widehat{y}_p\mathbf{M})_I = (Xy_L\mathbf{M})_I$. Since $K \subset I$ (recall the correspondences (2.4.1) and (3.1.3)), it follows that

$$(X\widehat{y}_p\mathbf{M})_K = \big((X\widehat{y}_p\mathbf{M})_I\big)_K = \big((Xy_L\mathbf{M})_I\big)_K = (Xy_L\mathbf{M})_K.$$

This completes the proof of the lemma. □

We used the following lemma in the proof of Proposition 4.3.8 above; we will also use this lemma in the proof of Theorem 4.4.5 below.

Lemma 4.3.11. *Let $p, q \in \mathbb{Z}$ be such that $0 < |p-q| < \ell$, and let $\widehat{X}$ be a product of Kashiwara operators of the form: $\widehat{X} = \widehat{x}_1\widehat{x}_2\cdots\widehat{x}_a$, where $\widehat{x}_b \in \{\widehat{e}_p, \widehat{f}_p, \widehat{e}_q, \widehat{f}_q\}$ for each $1 \le b \le a$. If $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and $\widehat{X}\mathbf{M} \neq \mathbf{0}$, then $X\mathbf{M} \neq \mathbf{0}$, and $(\widehat{X}\mathbf{M})_\gamma = (X\mathbf{M})_\gamma$ for each $\gamma \in \Gamma_{\mathbb{Z}}$, where X is a product of Kashiwara operators of the form $X := x_1x_2\cdots x_a$, with*

$$x_b = \begin{cases} e_{L_p} & \text{if } \widehat{x}_b = \widehat{e}_p, \\ f_{L_p} & \text{if } \widehat{x}_b = \widehat{f}_p, \\ e_{L_q} & \text{if } \widehat{x}_b = \widehat{e}_q, \\ f_{L_q} & \text{if } \widehat{x}_b = \widehat{f}_q, \end{cases} \tag{4.3.17}$$

for each $1 \le b \le a$. Here, L_p is an arbitrary finite subset of $p+(\ell+1)\mathbb{Z}$ such that $L_p \supset L(\gamma, p)$ and such that $L_q := \{t + (q-p) \mid t \in L_p\} \supset L(\gamma, q)$.

Remark 4.3.12. Keep the notation and assumptions of Lemma 4.3.11. If $r \in p + (\ell+1)\mathbb{Z}$ is not contained in L_p, then $|r - t| \ge 2$ for all $t \in L_p \cup L_q$. Indeed, if $t \in L_p$, then it is obvious that $|r - t| \ge \ell + 1 > 2$. If $t \in L_q$, then

$$|r - t| = \big|r - \{t + (p-q)\} + (p-q)\big| \ge \big|r - \{t + (p-q)\}\big| - |p-q|.$$

Here note that $\big|r - \{t + (p-q)\}\big| \ge \ell + 1$ since $t + (p-q) \in L_p$, and that $|p-q| < \ell$ by assumption. Therefore, we get $|r - t| \ge 2$. Similarly, we can show that if $r \in q + (\ell+1)\mathbb{Z}$ is not contained in L_q, then $|r - t| \ge 2$ for all $t \in L_p \cup L_q$.

Proof of Lemma 4.3.11. For each $1 \le b \le a$, we set $\widehat{X}_b := \widehat{x}_{b+1}\widehat{x}_{b+2}\cdots\widehat{x}_a$ and $X_b := x_1 x_2 \cdots x_b$. We prove by induction on b the claim that $X_b\widehat{X}_b\mathbf{M} \ne \mathbf{0}$ and $(\widehat{X}\mathbf{M})_\gamma = (X_b\widehat{X}_b\mathbf{M})_\gamma$ for all $1 \le b \le a$; the assertion of the lemma follows from the case $b = a$. We see easily from Remark 4.3.3 (if $\widehat{x}_1 = \widehat{f}_p$ or $\widehat{f}_q$) or Remark 4.3.6 (if $\widehat{x}_1 = \widehat{e}_p$ or $\widehat{e}_q$) that the claim above holds if $b = 1$. Assume, therefore, that $b > 1$. By the induction hypothesis, we have

$$X_{b-1}\widehat{X}_{b-1}\mathbf{M} = X_{b-1}\widehat{x}_b\widehat{X}_b\mathbf{M} \ne \mathbf{0} \quad \text{and} \quad (\widehat{X}\mathbf{M})_\gamma = (X_{b-1}\widehat{x}_b\widehat{X}_b\mathbf{M})_\gamma. \tag{4.3.18}$$

Take an interval K in $\mathbb{Z}$ such that $\gamma \in \Gamma_K$, and such that $\min K < t < \max K$ for all $t \in L_p \cup L_q$. Define $r \in \{p, q\}$ by: $r = p$ if $\widehat{x}_b = \widehat{e}_p$ or $\widehat{f}_p$, and $r = q$ if $\widehat{x}_b = \widehat{e}_q$ or $\widehat{f}_q$. Then we deduce from Lemma 4.3.10 that there exists a finite subset L of $r + (\ell+1)\mathbb{Z}$ such that

$$X_{b-1}x'_b\widehat{X}_b\mathbf{M} \ne \mathbf{0} \quad \text{and} \quad (X_{b-1}\widehat{x}_b\widehat{X}_b\mathbf{M})_K = (X_{b-1}x'_b\widehat{X}_b\mathbf{M})_K,$$

where x'_b is defined by the formula (4.3.17), with L_p and L_q replaced by $L \cup L_p$ and $L \cup L_q$, respectively. Also, it follows from Remark 4.3.12 and Lemma 3.4.1 (3) that

$$(\mathbf{0} \ne) \quad X_{b-1}x'_b\widehat{X}_b\mathbf{M} = X_{b-1}x''_b x_b\widehat{X}_b\mathbf{M} = x''_b X_{b-1}x_b\widehat{X}_b\mathbf{M} = x''_b X_b\widehat{X}_b\mathbf{M},$$

where x''_b is defined by the formula (4.3.17), with L_p and L_q replaced by $L \setminus L_p$ and $L \setminus L_q$, respectively. In particular, we obtain $X_b\widehat{X}_b\mathbf{M} \ne \mathbf{0}$. Moreover, since $\gamma \in \Gamma_K$, we have

$$(X_{b-1}\widehat{x}_b\widehat{X}_b\mathbf{M})_\gamma = (X_{b-1}x'_b\widehat{X}_b\mathbf{M})_\gamma = (x''_b X_b\widehat{X}_b\mathbf{M})_\gamma.$$

Since $L_r \supset L(\gamma, r)$, the intersection of $L \setminus L_r$ and $L(\gamma, r)$ is empty, and hence $\langle h_t, \gamma\rangle \le 0$ for all $t \in L \setminus L_r$. Therefore, we see from (3.3.2) (if $\widehat{x}_1 = \widehat{f}_p$ or $\widehat{f}_q$) or (3.3.14) (if $\widehat{x}_1 = \widehat{e}_p$ or $\widehat{e}_q$) that $(x''_b X_b\widehat{X}_b\mathbf{M})_\gamma = (X_b\widehat{X}_b\mathbf{M})_\gamma$. Combining these with (4.3.18), we conclude that $(\widehat{X}\mathbf{M})_\gamma = (X_b\widehat{X}_b\mathbf{M})_\gamma$, as desired. This proves the lemma. $\square$

4.4 Main results. Recall the BZ datum $\mathbf{O}$ of type A_∞ whose γ-component is equal to 0 for each $\gamma \in \Gamma_\mathbb{Z}$ (see Example 3.2.2). It is obvious that $\sigma(\mathbf{O}) = \mathbf{O}$, and hence $\mathbf{O} \in \mathcal{BZ}^\sigma_\mathbb{Z}$. Also, $\widehat{\varepsilon}_p(\mathbf{O}) = 0$ for all $p \in \widehat{I}$, which implies that $\widehat{e}_p\mathbf{O} = \mathbf{0}$ for all $p \in \widehat{I}$. Let $\mathcal{BZ}^\sigma_\mathbb{Z}(\mathbf{O})$ denote the connected component of (the crystal graph of) the crystal $\mathcal{BZ}^\sigma_\mathbb{Z}$ containing $\mathbf{O}$. The following theorem is the first main result of this paper; the proof will be given in the next section.

Theorem 4.4.1. *The crystal $\mathcal{BZ}^\sigma_\mathbb{Z}(\mathbf{O})$ is isomorphic, as a crystal for $U_q(\widehat{\mathfrak{g}}^\vee)$, to the crystal basis $\widehat{\mathcal{B}}(\infty)$ of the negative part $U_q^-(\widehat{\mathfrak{g}}^\vee)$ of $U_q(\widehat{\mathfrak{g}}^\vee)$.*

For each dominant integral weight $\widehat{\lambda} \in \widehat{\mathfrak{h}}$ for $\widehat{\mathfrak{g}}^\vee$, let $\mathcal{BZ}^\sigma_\mathbb{Z}(\mathbf{O}; \widehat{\lambda})$ denote the subset of $\mathcal{BZ}^\sigma_\mathbb{Z}(\mathbf{O})$ consisting of all elements $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_\mathbb{Z}} \in \mathcal{BZ}^\sigma_\mathbb{Z}(\mathbf{O})$ satisfying the condition (cf. (2.3.5)) that

$$M_{-s_i\Lambda_i} \ge -\langle \widehat{\lambda}, \widehat{\alpha}_{\bar{i}}\rangle \quad \text{for all } i \in \mathbb{Z}; \tag{4.4.1}$$

recall that $\bar{i}$ denotes a unique element in $\widehat{I} = \{0, 1, \ldots, \ell\}$ to which $i \in \mathbb{Z}$ is congruent modulo $\ell+1$. Let us define a crystal structure for $U_q(\widehat{\mathfrak{g}}^\vee)$ on the set $\mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$ (see Proposition 4.4.4 below).

Lemma 4.4.2. *The set $\mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda}) \cup \{\mathbf{0}\}$ is stable under the raising Kashiwara operators $\widehat{e}_p$ on $\mathcal{BZ}_\mathbb{Z}^\sigma$ for $p \in \mathbb{Z}$.*

Proof. Let $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_\mathbb{Z}} \in \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$, and $p \in \mathbb{Z}$. Suppose that $\mathbf{M}' := \widehat{e}_p \mathbf{M} \neq \mathbf{0}$, and write it as: $\mathbf{M}' = \widehat{e}_p \mathbf{M} = (M'_\gamma)_{\gamma \in \Gamma_\mathbb{Z}}$. In order to prove that $\widehat{e}_p \mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$, it suffices to show that $M_\gamma \leq M'_\gamma$ for all $\gamma \in \Gamma_\mathbb{Z}$. Fix $\gamma \in \Gamma_\mathbb{Z}$. We know from Proposition 4.3.8 that $\widehat{f}_p \mathbf{M}' = \widehat{f}_p \widehat{e}_p \mathbf{M} = \mathbf{M}$. Also, it follows from the definition of $\widehat{f}_p$ that $M_\gamma = (\widehat{f}_p \mathbf{M}')_\gamma = (f_{L(\gamma,p)} \mathbf{M}')_\gamma$. Therefore, we deduce from Remark 3.3.1 (1) that $(f_{L(\gamma,p)} \mathbf{M}')_\gamma \leq M'_\gamma$, and hence $M_\gamma \leq M'_\gamma$. This proves the lemma. □

Remark 4.4.3. In contrast to the situation in Lemma 4.4.2, the set $\mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$ is not stable under the lowering Kashiwara operators $\widehat{f}_p$ on $\mathcal{BZ}_\mathbb{Z}^\sigma$ for $p \in \mathbb{Z}$.

For each $p \in \mathbb{Z}$, we define a map $\widehat{F}_p : \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda}) \to \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda}) \cup \{\mathbf{0}\}$ by:

$$\widehat{F}_p \mathbf{M} = \begin{cases} \widehat{f}_p \mathbf{M} & \text{if } \widehat{f}_p \mathbf{M} \text{ is contained in } \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda}), \\ \mathbf{0} & \text{otherwise,} \end{cases} \tag{4.4.2}$$

for $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$; by convention, we set $\widehat{F}_p \mathbf{0} := \mathbf{0}$ for all $p \in \mathbb{Z}$. We define the weight $\mathrm{Wt}(\mathbf{M})$ of $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$ by:

$$\mathrm{Wt}(\mathbf{M}) = \widehat{\lambda} + \mathrm{wt}(\mathbf{M}) = \widehat{\lambda} + \sum_{i \in \widehat{I}} M_{\Lambda_i} \widehat{h}_i, \tag{4.4.3}$$

where $M_{\Lambda_i} := \Theta(\mathbf{M})_{\Lambda_i}$ for $i \in \widehat{I}$. Also, we set

$$\widehat{\Phi}_p(\mathbf{M}) := \langle \mathrm{Wt}(\mathbf{M}), \widehat{\alpha}_{\overline{p}} \rangle + \widehat{\varepsilon}_p(\mathbf{M}) \quad \text{for } \mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda}) \text{ and } p \in \mathbb{Z}. \tag{4.4.4}$$

Then, it is easily seen from the definition (4.3.7) of the map $\widehat{\varepsilon}_p$ and Remark 4.3.2 that

$$\widehat{\Phi}_p(\mathbf{M}) = M_{\Lambda_p} - M_{s_p \Lambda_p} + \langle \widehat{\lambda}, \widehat{\alpha}_{\overline{p}} \rangle, \tag{4.4.5}$$

where $M_{\Lambda_p} := \Theta(\mathbf{M})_{\Lambda_p}$ and $M_{s_p \Lambda_p} := \Theta(\mathbf{M})_{s_p \Lambda_p}$ (cf. (2.3.7)).

Proposition 4.4.4. (1) *The set $\mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$, equipped with the maps* Wt, *$\widehat{e}_p$, $\widehat{F}_p$ ($p \in \widehat{I}$), and $\widehat{\varepsilon}_p$, $\widehat{\Phi}_p$ ($p \in \widehat{I}$) above, is a crystal for $U_q(\widehat{\mathfrak{g}}^\vee)$.*

(2) *For $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}^\sigma(\mathbf{O}; \widehat{\lambda})$ and $p \in \widehat{I}$, there hold*

$$\widehat{\varepsilon}_p(\mathbf{M}) = \max\{N \geq 0 \mid \widehat{e}_p^N \mathbf{M} \neq \mathbf{0}\}, \qquad \widehat{\Phi}_p(\mathbf{M}) = \max\{N \geq 0 \mid \widehat{F}_p^N \mathbf{M} \neq \mathbf{0}\}.$$

Proof. (1) This follows easily from Proposition 4.3.8. As examples, we show that

$$\mathrm{Wt}(\widehat{F}_p\mathbf{M}) = \mathrm{Wt}(\mathbf{M}) - \widehat{h}_p, \tag{4.4.6}$$

$$\widehat{\varepsilon}_p(\widehat{F}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) + 1 \quad \text{and} \quad \widehat{\Phi}_p(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_p(\mathbf{M}) - 1, \tag{4.4.7}$$

for $\mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}(\mathbf{O}; \widehat{\lambda})$ and $p \in \widehat{I}$ if $\widehat{F}_p\mathbf{M} \neq \mathbf{0}$. Note that in this case, $\widehat{F}_p\mathbf{M} = \widehat{f}_p\mathbf{M}$ by the definition of $\widehat{F}_p$. First we show (4.4.6). It follows from the definition of Wt that

$$\mathrm{Wt}(\widehat{F}_p\mathbf{M}) = \mathrm{Wt}(\widehat{f}_p\mathbf{M}) = \widehat{\lambda} + \mathrm{wt}(\widehat{f}_p\mathbf{M}).$$

Since $\mathrm{wt}(\widehat{f}_p\mathbf{M}) = \mathrm{wt}(\mathbf{M}) - \widehat{h}_p$ by Proposition 4.3.8, we have

$$\mathrm{Wt}(\widehat{F}_p\mathbf{M}) = \widehat{\lambda} + \mathrm{wt}(\widehat{f}_p\mathbf{M}) = \widehat{\lambda} + \mathrm{wt}(\mathbf{M}) - \widehat{h}_p = \mathrm{Wt}(\mathbf{M}) - \widehat{h}_p,$$

as desired. Next we show (4.4.7). It follows from (the proof of) Proposition 4.3.8 that $\widehat{\varepsilon}_p(\widehat{F}_p\mathbf{M}) = \widehat{\varepsilon}_p(\widehat{f}_p\mathbf{M}) = \widehat{\varepsilon}_p(\mathbf{M}) + 1$. Also, we compute:

$$\begin{aligned}
\widehat{\Phi}_p(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_p(\widehat{f}_p\mathbf{M}) &= \langle \mathrm{Wt}(\widehat{f}_p\mathbf{M}),\, \widehat{\alpha}_p \rangle + \widehat{\varepsilon}_p(\widehat{f}_p\mathbf{M}) \quad \text{by the definition of } \widehat{\Phi}_p \\
&= \langle \mathrm{Wt}(\mathbf{M}) - \widehat{h}_p,\, \widehat{\alpha}_p \rangle + \widehat{\varepsilon}_p(\mathbf{M}) + 1 \quad \text{by (4.4.6) and Proposition 4.3.8} \\
&= \langle \mathrm{Wt}(\mathbf{M}),\, \widehat{\alpha}_p \rangle + \widehat{\varepsilon}_p(\mathbf{M}) - 1 = \widehat{\Phi}_p(\mathbf{M}) - 1 \quad \text{by the definition of } \widehat{\Phi}_p,
\end{aligned}$$

as desired.

(2) The first equation follows immediately from Remark 4.3.9 together with Lemma 4.4.2. We will prove the second equation. Fix $p \in \widehat{I}$. We first show that

$$\widehat{\Phi}_p(\mathbf{M}) \geq 0 \quad \text{for all } \mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}(\mathbf{O}; \widehat{\lambda}). \tag{4.4.8}$$

Fix $\mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}(\mathbf{O}; \widehat{\lambda})$, and take an interval I in $\mathbb{Z}$ such that $I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p)$. Then we see from (4.4.5) that

$$\widehat{\Phi}_p(\mathbf{M}) = M_{\Lambda_p} - M_{s_p\Lambda_p} + \langle \widehat{\lambda},\, \widehat{\alpha}_p \rangle = M_{\varpi_p^I} - M_{s_p\varpi_p^I} + \langle \widehat{\lambda},\, \widehat{\alpha}_p \rangle. \tag{4.4.9}$$

Now we define a dominant integral weight $\lambda \in \mathfrak{h}_I$ for $\mathfrak{g}_I^{\vee}$ by: $\langle \lambda,\, \alpha_i \rangle = \langle \widehat{\lambda},\, \widehat{\alpha}_{\bar{i}} \rangle$ for $i \in I$. Then, we deduce from (2.3.5), (4.4.1), and (3.1.3) that $\mathbf{M}_I \in \mathcal{BZ}_I$ is contained in $\mathcal{BZ}_I(\lambda) \subset \mathcal{BZ}_I$. Because $\mathcal{BZ}_I(\lambda)$ is isomorphic, as a crystal for $U_q(\mathfrak{g}_I^{\vee})$, to the crystal basis $\mathcal{B}_I(\lambda)$ (see Theorem 2.3.7), it follows that $\Phi_p(\mathbf{M}_I) \geq 0$. Also, we see from (2.3.7) that

$$\Phi_p(\mathbf{M}_I) = M_{\varpi_p^I} - M_{s_p\varpi_p^I} + \langle \lambda,\, \alpha_p \rangle. \tag{4.4.10}$$

Since $\langle \lambda,\, \alpha_p \rangle = \langle \widehat{\lambda},\, \widehat{\alpha}_p \rangle$ by the definition of $\lambda \in \mathfrak{h}_I$, we conclude from (4.4.9) and (4.4.10) that $\widehat{\Phi}_p(\mathbf{M}) = \Phi_p(\mathbf{M}_I) \geq 0$, as desired.

Next we show that for $\mathbf{M} \in \mathcal{BZ}^{\sigma}_{\mathbb{Z}}(\mathbf{O}; \widehat{\lambda})$,

$$\widehat{F}_p\mathbf{M} = \mathbf{0} \quad \text{if and only if} \quad \widehat{\Phi}_p(\mathbf{M}) = 0. \tag{4.4.11}$$

Fix $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$. Suppose that $\widehat{\Phi}_p(\mathbf{M}) = 0$, and $\widehat{F}_p\mathbf{M} \neq \mathbf{0}$. Then, since $\widehat{\Phi}_p(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_p(\mathbf{M}) - 1$ by (4.4.7), we have $\widehat{\Phi}_p(\widehat{F}_p\mathbf{M}) = -1$, which contradicts (4.4.8). Hence, if $\widehat{\Phi}_p(\mathbf{M}) = 0$, then $\widehat{F}_p\mathbf{M} = \mathbf{0}$. To show the converse, assume that $\widehat{F}_p\mathbf{M} = \mathbf{0}$, or equivalently, $\widehat{f}_p\mathbf{M} \notin \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$. Let us write $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ and $\widehat{f}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ as: $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$ and $\widehat{f}_p\mathbf{M} = (M'_\gamma)_{\gamma\in\Gamma_{\mathbb{Z}}}$, respectively. From the assumption that $\widehat{f}_p\mathbf{M} \notin \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$, it follows that $M'_{-s_q\Lambda_q} < -\langle\widehat{\lambda},\, \widehat{\alpha}_{\overline{q}}\rangle$ for some $q \in \mathbb{Z}$. Note that since $M'_\gamma = M'_{\sigma^{-1}(\gamma)}$ for all $\gamma \in \Gamma_{\mathbb{Z}}$, we may assume $q \in \widehat{I}$. Then, we infer that this q is equal to p. Indeed, for each $i \in \widehat{I} \setminus \{p\}$, we have $L(-s_i\Lambda_i, p) = \emptyset$, since $\langle h_i,\, s_i\Lambda_i\rangle = -1$ and $\langle h_j,\, s_i\Lambda_i\rangle \geq 0$ for all $j \in \mathbb{Z}$ with $j \neq i$. Therefore, by the definition of $\widehat{f}_p$,

$$M'_{-s_i\Lambda_i} = (\widehat{f}_p\mathbf{M})_{-s_i\Lambda_i} = (f_\emptyset\mathbf{M})_{-s_i\Lambda_i} = M_{-s_i\Lambda_i}.$$

Hence it follows that $M'_{-s_i\Lambda_i} = M_{-s_i\Lambda_i} \geq -\langle\widehat{\lambda},\, \widehat{\alpha}_{\overline{i}}\rangle$ since $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$. Consequently, $q \in \widehat{I}$ is not equal to any $i \in \widehat{I} \setminus \{p\}$, that is, $q = p$.

Now, as in the proof of (4.4.8) above, take an interval I in $\mathbb{Z}$ such that $I \in \mathrm{Int}(\mathbf{M}; e, p) \cap \mathrm{Int}(\mathbf{M}; s_p, p)$, and then define a dominant integral weight $\lambda \in \mathfrak{h}_I$ for $\mathfrak{g}_I^\vee$ by: $\langle\lambda,\, \alpha_i\rangle = \langle\widehat{\lambda},\, \widehat{\alpha}_{\overline{i}}\rangle$ for $i \in I$; we know from the argument above that $\mathbf{M}_I \in \mathcal{BZ}_I(\lambda)$, and $\widehat{\Phi}_p(\mathbf{M}) = \Phi_p(\mathbf{M}_I)$. Therefore, in order to show that $\widehat{\Phi}_p(\mathbf{M}) = 0$, it suffices to show that $\Phi_p(\mathbf{M}_I) = 0$, which is equivalent to $F_p\mathbf{M}_I = \mathbf{0}$ by Theorem 2.3.7. Recall from the above that $M'_{-s_p\Lambda_p} < -\langle\widehat{\lambda},\, \widehat{\alpha}_p\rangle = -\langle\lambda,\, \alpha_p\rangle$. Also, it follows from the definition of $\widehat{f}_p$ on $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ and the definition of f_p on $\mathcal{BZ}_{\mathbb{Z}}$ that

$$\begin{aligned} M'_{-s_p\Lambda_p} &= (\widehat{f}_p\mathbf{M})_{-s_p\Lambda_p} = (f_p\mathbf{M})_{-s_p\Lambda_p} \quad \text{since } L(-s_p\Lambda_p, p) = \{p\} \\ &= (f_p\mathbf{M}_I)_{-s_p\Lambda_p}. \end{aligned}$$

Combining these, we obtain $(f_p\mathbf{M}_I)_{-s_p\Lambda_p} < -\langle\lambda,\, \alpha_p\rangle$, which implies that $f_p\mathbf{M}_I \notin \mathcal{BZ}_I(\lambda)$, and hence $F_p\mathbf{M}_I = \mathbf{0}$ by the definition. Thus we have shown (4.4.11).

From (4.4.8), (4.4.11), and the second equation of (4.4.7), we deduce that $\widehat{\Phi}_p(\mathbf{M}) = \max\{N \geq 0 \mid \widehat{F}_p^N\mathbf{M} \neq \mathbf{0}\}$ for $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ and $p \in \widehat{I}$, as desired. This completes the proof of the proposition. □

The following theorem is the second main result of this paper; the proof will be given in the next section.

Theorem 4.4.5. *Let $\widehat{\lambda} \in \mathfrak{h}$ be a dominant integral weight for $\widehat{\mathfrak{g}}^\vee$. The crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ is isomorphic, as a crystal for $U_q(\widehat{\mathfrak{g}}^\vee)$, to the crystal basis $\widehat{\mathcal{B}}(\widehat{\lambda})$ of the irreducible highest weight $U_q(\widehat{\mathfrak{g}}^\vee)$-module of highest weight $\widehat{\lambda}$.*

4.5 Proofs of Theorems 4.4.1 and 4.4.5.

We first prove Theorem 4.4.5; Theorem 4.4.1 is obtained as a corollary of Theorem 4.4.5.

Proof of Theorem 4.4.5. By Proposition 4.4.4 and Theorem A.1.1 in the Appendix, it suffices to prove that the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ satisfies conditions (C1)–(C6) of Theorem A.1.1. First we prove that the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ satisfies condition (C6). Note that $\mathbf{O} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$. It follows from the definition of the raising Kashiwara operators $\widehat{e}_p$, $p \in \widehat{I}$, on $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ (see also the beginning of §4.4) that $\widehat{e}_p\mathbf{O} = \mathbf{0}$ for all $p \in \widehat{I}$. Also, $\Theta(\mathbf{O})_{\Lambda_p}$ and $\Theta(\mathbf{O})_{s_p\Lambda_p}$ are equal to 0 by the definitions. Therefore, it follows from (4.4.3) and (4.4.5) that $\mathrm{Wt}(\mathbf{O}) = \widehat{\lambda}$ and $\widehat{\Phi}_p(\mathbf{O}) = \langle \widehat{\lambda}, \widehat{\alpha}_p \rangle$ for all $p \in \widehat{I}$, as desired.

We also need to prove that the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ satisfies conditions (C1)–(C5) of Theorem A.1.1. We will prove that $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$ satisfies condition (C5); the proofs for the other conditions are similar. Namely, we will prove the following assertion: Let $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O};\widehat{\lambda})$, and $p, q \in \widehat{I}$. Assume that $\widehat{F}_p\mathbf{M} \neq \mathbf{0}$ and $\widehat{F}_q\mathbf{M} \neq \mathbf{0}$, and that $\widehat{\Phi}_q(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_q(\mathbf{M}) + 1$ and $\widehat{\Phi}_p(\widehat{F}_q\mathbf{M}) = \widehat{\Phi}_p(\mathbf{M}) + 1$. Then,

$$\widehat{F}_p\widehat{F}_q^2\widehat{F}_p\mathbf{M} \neq \mathbf{0} \quad \text{and} \quad \widehat{F}_q\widehat{F}_p^2\widehat{F}_q\mathbf{M} \neq \mathbf{0}, \tag{4.5.1}$$

$$\widehat{F}_p\widehat{F}_q^2\widehat{F}_p\mathbf{M} = \widehat{F}_q\widehat{F}_p^2\widehat{F}_q\mathbf{M}, \tag{4.5.2}$$

$$\widehat{\varepsilon}_q(\widehat{F}_p\mathbf{M}) = \widehat{\varepsilon}_q(\widehat{F}_p^2\widehat{F}_q\mathbf{M}) \quad \text{and} \quad \widehat{\varepsilon}_p(\widehat{F}_q\mathbf{M}) = \widehat{\varepsilon}_p(\widehat{F}_q^2\widehat{F}_p\mathbf{M}). \tag{4.5.3}$$

Here we note that $p \neq q$. Indeed, if $p = q$, then it follows from the second equation of (4.4.7) that $\widehat{\Phi}_p(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_q(\mathbf{M}) - 1$, which contradicts the assumption that $\widehat{\Phi}_p(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_p(\mathbf{M}) + 1$.

A key to the proof of (4.5.1)–(4.5.3) is Claim 1 below. For an interval I in $\mathbb{Z}$, we define a dominant integral weight $\lambda_I \in \mathfrak{h}_I$ for $\mathfrak{g}_I^{\vee}$ by:

$$\langle \lambda_I, \alpha_i \rangle = \langle \widehat{\lambda}, \widehat{\alpha}_{\overline{i}} \rangle \quad \text{for } i \in I. \tag{4.5.4}$$

As mentioned in the proof of Proposition 4.4.4 (2), $\mathbf{M}_I \in \mathcal{BZ}_I$ is contained in $\mathcal{BZ}_I(\lambda_I) \subset \mathcal{BZ}_I$; recall from Theorem 2.3.7 that $\mathcal{BZ}_I(\lambda_I)$ is isomorphic, as a crystal for $U_q(\mathfrak{g}_I^{\vee})$, to the crystal basis $\mathcal{B}_I(\lambda_I)$.

Claim 1. *Let $r, t \in \mathbb{Z}$ be such that $\overline{r} = p$, $\overline{t} = q$, and $0 < |r - t| < \ell$. Assume that an interval I in $\mathbb{Z}$ satisfies the following conditions:*

(a1) $I \in \mathrm{Int}(\mathbf{M}; e, r) \cap \mathrm{Int}(\mathbf{M}; s_r, r)$;

(a2) $I \in \mathrm{Int}(\mathbf{M}; e, t) \cap \mathrm{Int}(\mathbf{M}; s_t, t)$;

(a3) $I \in \mathrm{Int}(\widehat{F}_p\mathbf{M}; e, t) \cap \mathrm{Int}(\widehat{F}_p\mathbf{M}; s_t, t)$;

(a4) $I \in \mathrm{Int}(\widehat{F}_q\mathbf{M}; e, r) \cap \mathrm{Int}(\widehat{F}_q\mathbf{M}; s_r, r)$.

(i) *We have $\Phi_r(\mathbf{M}_I) = \widehat{\Phi}_p(\mathbf{M}) > 0$ and $\Phi_t(\mathbf{M}_I) = \widehat{\Phi}_q(\mathbf{M}) > 0$, and hence $F_r\mathbf{M}_I \neq \mathbf{0}$ and $F_t\mathbf{M}_I \neq \mathbf{0}$. Also, we have $\Phi_t(F_r\mathbf{M}_I) = \Phi_t(\mathbf{M}_I) + 1$ and $\Phi_r(F_t\mathbf{M}_I) = \Phi_r(\mathbf{M}_I) + 1$.*

(ii) *We have*

$$F_rF_t^2F_r\mathbf{M}_I \neq \mathbf{0} \quad \text{and} \quad F_tF_r^2F_t\mathbf{M}_I \neq \mathbf{0},$$

$$F_rF_t^2F_r\mathbf{M}_I = F_tF_r^2F_t\mathbf{M}_I,$$

$$\varepsilon_t(F_r\mathbf{M}_I) = \varepsilon_t(F_r^2F_t\mathbf{M}_I) \quad \text{and} \quad \varepsilon_r(F_t\mathbf{M}_I) = \varepsilon_r(F_t^2F_r\mathbf{M}_I).$$

Proof of Claim 1. (i) We write $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O}; \widehat{\lambda})$ and $\Theta(\mathbf{M})$ as: $\mathbf{M} = (M_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}}$ and $\Theta(\mathbf{M}) = (M_\xi)_{\xi \in \Xi_{\mathbb{Z}}}$, respectively. Then, we compute:

$$\begin{aligned} \Phi_r(\mathbf{M}_I) &= M_{\varpi_r^I} - M_{s_r \varpi_r^I} + \langle \lambda_I, \alpha_r \rangle \quad \text{by (2.3.7)} \\ &= M_{\Lambda_r} - M_{s_r \Lambda_r} + \langle \lambda_I, \alpha_r \rangle \quad \text{by condition (a1).} \end{aligned}$$

Since r is congruent to p modulo $\ell + 1$ by assumption, we have $r = \sigma^n(p)$ for some $n \in \mathbb{Z}$. Hence, by Remark 4.3.2,

$$\begin{aligned} M_{\Lambda_r} &= M_{\Lambda_{\sigma^n(p)}} = M_{\sigma^n(\Lambda_p)} = M_{\Lambda_p}, \\ M_{s_r\Lambda_r} &= M_{s_{\sigma^n(p)}\Lambda_{\sigma^n(p)}} = M_{\sigma^n(s_p\Lambda_p)} = M_{s_p\Lambda_p}. \end{aligned}$$

Also, by the definition of λ_I, we have $\langle \lambda_I, \alpha_r \rangle = \langle \widehat{\lambda}, \widehat{\alpha}_p \rangle$. Substituting these into the above, we obtain

$$\Phi_r(\mathbf{M}_I) = M_{\Lambda_p} - M_{s_p\Lambda_p} + \langle \widehat{\lambda}, \widehat{\alpha}_p \rangle = \widehat{\Phi}_p(\mathbf{M}) \quad \text{by (4.4.5).}$$

Since $\widehat{\Phi}_p(\mathbf{M}) > 0$ by the assumption that $\widehat{F}_p\mathbf{M} \neq \mathbf{0}$, we get $\Phi_r(\mathbf{M}_I) = \widehat{\Phi}_p(\mathbf{M}_I) > 0$, as desired. Similarly, we can show that $\Phi_t(\mathbf{M}_I) = \widehat{\Phi}_q(\mathbf{M}) > 0$.

Now, we write $\widehat{F}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O}; \widehat{\lambda})$ and $\Theta(\widehat{F}_p\mathbf{M})$ as: $\widehat{F}_p\mathbf{M} = (M'_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}}$ and $\Theta(\widehat{F}_p\mathbf{M}) = (M'_\xi)_{\xi \in \Xi_{\mathbb{Z}}}$, respectively. Since $L(\varpi_t^I, p) = \emptyset \subset \{r\}$ (recall that $0 < |r - t| < \ell$), we have

$$\begin{aligned} M'_{\Lambda_t} &= M'_{\varpi_t^I} \quad \text{by condition (a3)} \\ &= (\widehat{F}_p\mathbf{M})_{\varpi_t^I} = (F_r\mathbf{M})_{\varpi_t^I} \quad \text{by Remark 4.3.3} \\ &= (F_r\mathbf{M}_I)_{\varpi_t^I} \quad \text{by conditions (a1), (a2), and the definition of } F_rM. \end{aligned}$$

Also, it follows from (3.1.4) that $\{i \in \mathbb{Z} \mid \langle h_i, s_t\varpi_t^I \rangle > 0\} \subset \{t-1, t+1\}$. Since $0 < |r-t| < \ell$, it is easily seen that $r + (\ell+1)n > t + 1$ and $r - (\ell+1)n < t - 1$ for every $n \in \mathbb{Z}_{>0}$. Hence we deduce that $L(s_t\varpi_t^I, p) \subset \{r\}$. Using this fact, we can show in exactly the same way as above that $M'_{s_t\Lambda_t} = (F_r\mathbf{M}_I)_{s_t\varpi_t^I}$. Therefore,

$$\begin{aligned} \Phi_t(F_r\mathbf{M}_I) &= (F_r\mathbf{M}_I)_{\varpi_t^I} - (F_r\mathbf{M}_I)_{s_t\varpi_t^I} + \langle \lambda_I, \alpha_t \rangle \quad \text{by (2.3.7)} \\ &= M'_{\Lambda_t} - M'_{s_t\Lambda_t} + \langle \lambda_I, \alpha_t \rangle \\ &= M'_{\Lambda_q} - M'_{s_q\Lambda_q} + \langle \widehat{\lambda}, \widehat{\alpha}_q \rangle \quad \text{by Remark 4.3.2 and the definition of } \lambda_I \\ &= \widehat{\Phi}_q(\widehat{F}_p\mathbf{M}) \quad \text{by (4.4.5).} \end{aligned}$$

Because $\widehat{\Phi}_q(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_q(\mathbf{M}) + 1$ by our assumption, and $\widehat{\Phi}_q(\mathbf{M}) = \Phi_t(\mathbf{M}_I)$ as shown above, we obtain $\Phi_t(F_r\mathbf{M}_I) = \widehat{\Phi}_q(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_q(\mathbf{M}) + 1 = \Phi_t(\mathbf{M}_I) + 1$, as desired. The equation $\Phi_r(F_t\mathbf{M}_I) = \Phi_r(\mathbf{M}_I) + 1$ can be shown similarly.

(ii) Because $\mathcal{BZ}_I(\lambda_I)$ is isomorphic, as a crystal for $U_q(\mathfrak{g}_I^\vee)$, to the crystal basis $\mathcal{B}_I(\lambda_I)$ by Theorem 2.3.7, this crystal satisfies condition (C5) of Theorem A.1.1. Hence the equations in part (ii) follow immediately from part (i). This proves Claim 1. ∎

First we show (4.5.1); we only prove that $\widehat{F}_p\widehat{F}_q^2\widehat{F}_p\mathbf{M} \neq \mathbf{0}$, since we can prove that $\widehat{F}_q\widehat{F}_p^2\widehat{F}_q\mathbf{M} \neq \mathbf{0}$ similarly. Recall that $\widehat{F}_p\mathbf{M} \neq \mathbf{0}$ by our assumption. Also, since $\widehat{F}_q\mathbf{M} \neq \mathbf{0}$ by our assumption, it follows from Proposition 4.4.4 (2) that $\widehat{\Phi}_q(\mathbf{M}) > 0$. Therefore, we have $\widehat{\Phi}_q(\widehat{F}_p\mathbf{M}) = \widehat{\Phi}_q(\mathbf{M}) + 1 \geq 2$ by our assumption, which implies that $\widehat{F}_q^2\widehat{F}_p\mathbf{M} \neq \mathbf{0}$ by Proposition 4.4.4 (2). We set $\mathbf{M}'' := \widehat{F}_q^2\widehat{F}_p\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O}; \widehat{\lambda})$, and write $\mathbf{M}''$ and $\Theta(\mathbf{M}'')$ as: $\mathbf{M}'' = (M''_\gamma)_{\gamma \in \Gamma_{\mathbb{Z}}}$ and $\Theta(\mathbf{M}'') = (M''_\xi)_{\xi \in \Xi_{\mathbb{Z}}}$, respectively. In order to show that $\widehat{F}_p\widehat{F}_q^2\widehat{F}_p\mathbf{M} = \widehat{F}_p\mathbf{M}'' \neq \mathbf{0}$, it suffices to show that

$$\widehat{\Phi}_p(\mathbf{M}'') = M''_{\Lambda_p} - M''_{s_p\Lambda_p} + \langle \widehat{\lambda},\, \widehat{\alpha}_p \rangle > 0$$

by Proposition 4.4.4 (2) along with equation (4.4.5). We define $r,\, t \in \mathbb{Z}$ by:

$$(r, t) = \begin{cases} (p, q) & \text{if } |p - q| < \ell, \\ (\ell,\, \ell + 1) & \text{if } p = \ell \text{ and } q = 0, \\ (\ell + 1,\, \ell) & \text{if } p = 0 \text{ and } q = \ell. \end{cases} \tag{4.5.5}$$

Let K be an interval in $\mathbb{Z}$ such that $r,\, t \in K$, and take an interval I in $\mathbb{Z}$ satisfying conditions (a1)–(a4) in Claim 1 and the following:

(b1) $I \in \mathrm{Int}(\mathbf{M}''; e, r) \cap \mathrm{Int}(\mathbf{M}''; s_r, r)$;

(b2) $I \in \mathrm{Int}(\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$.

It follows from Remark 4.3.2 and condition (b1) that $M''_{\Lambda_p} = M''_{\Lambda_r} = M''_{\varpi_r^I}$. Also,

$$\begin{aligned} M''_{\varpi_r^I} &= (\widehat{F}_q^2\widehat{F}_p\mathbf{M})_{\varpi_r^I} = (\widehat{f}_q^2\widehat{f}_p\mathbf{M})_{\varpi_r^I} \quad \text{by the definitions of } \widehat{F}_q \text{ and } \widehat{F}_p \\ &= (\widehat{f}_t^2\widehat{f}_r\mathbf{M})_{\varpi_r^I} \quad \text{by (4.3.4).} \end{aligned}$$

Here we note that $L(\varpi_r^I, r) = \{r\}$ and $L(\varpi_r^I, t) = \emptyset$ since $0 < |r - t| < \ell$. Therefore, we deduce from Lemma 4.3.11 (with $p = r$, $q = t$, $\widehat{X} = \widehat{f}_t^2\widehat{f}_r$, $\gamma = \varpi_r^I$, and $L_r = \{r\}$) that $f_t^2f_r\mathbf{M} \neq \mathbf{0}$ and $(\widehat{f}_t^2\widehat{f}_r\mathbf{M})_{\varpi_r^I} = (f_t^2f_r\mathbf{M})_{\varpi_r^I}$. Since $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}(I, K)$ by condition (b2), we see from Lemma 3.3.4 (2) that $(f_t^2f_r\mathbf{M})_I = f_t^2f_r\mathbf{M}_I$, and hence that $(f_t^2f_r\mathbf{M})_{\varpi_r^I} = (f_t^2f_r\mathbf{M}_I)_{\varpi_r^I}$. Also, because $r,\, t \in \mathbb{Z}$ satisfies the conditions that $\overline{r} = p$, $\overline{t} = q$, and $0 < |r - t| < \ell$, and because the interval I satisfies conditions (a1)–(a4) of Claim 1, it follows from Claim 1 (ii) that $F_t^2F_r\mathbf{M}_I \neq \mathbf{0}$, and hence $f_t^2f_r\mathbf{M}_I = F_t^2F_r\mathbf{M}_I$. Putting the above together, we obtain $M''_{\Lambda_p} = (F_t^2F_r\mathbf{M}_I)_{\varpi_r^I}$. Similarly, we can show that $M''_{s_p\Lambda_p} = (F_t^2F_r\mathbf{M}_I)_{s_r\varpi_r^I}$. Consequently, we see that

$$\begin{aligned} \widehat{\Phi}_p(\mathbf{M}'') &= M''_{\Lambda_p} - M''_{s_p\Lambda_p} + \langle \widehat{\lambda},\, \widehat{\alpha}_p \rangle \\ &= (F_t^2F_r\mathbf{M}_I)_{\varpi_r^I} - (F_t^2F_r\mathbf{M}_I)_{s_r\varpi_r^I} + \langle \lambda_I,\, \alpha_r \rangle \\ &= \Phi_r(F_t^2F_r\mathbf{M}_I) \quad \text{by (2.3.7) together with Theorem 2.3.7} \\ &> 0 \quad \text{by Claim 1 (ii).} \end{aligned}$$

Thus we have shown (4.5.1).

Next we show equation (4.5.2). Define $r,\, t \in \mathbb{Z}$ as in (4.5.5). Since $\widehat{F}_p\widehat{F}_q^2\widehat{F}_p\mathbf{M} \neq \mathbf{0}$ and $\widehat{F}_q\widehat{F}_p^2\widehat{F}_q\mathbf{M} \neq \mathbf{0}$ by (4.5.1), it follows from the definitions of $\widehat{F}_p$ and $\widehat{F}_q$ along with (4.3.4) that

$$\widehat{F}_p\widehat{F}_q^2\widehat{F}_p\mathbf{M} = \widehat{f}_p\widehat{f}_q^2\widehat{f}_p\mathbf{M} = \widehat{f}_r\widehat{f}_t^2\widehat{f}_r\mathbf{M},$$
$$\widehat{F}_q\widehat{F}_p^2\widehat{F}_q\mathbf{M} = \widehat{f}_q\widehat{f}_p^2\widehat{f}_q\mathbf{M} = \widehat{f}_t\widehat{f}_r^2\widehat{f}_t\mathbf{M}.$$

Therefore, it suffices to show that

$$(\widehat{f}_r\widehat{f}_t^2\widehat{f}_r\mathbf{M})_\gamma = (\widehat{f}_t\widehat{f}_r^2\widehat{f}_t\mathbf{M})_\gamma \quad \text{for all } \gamma \in \Gamma_\mathbb{Z}.$$

Fix $\gamma \in \Gamma_\mathbb{Z}$, and take a finite subset L_r of $r + (\ell+1)\mathbb{Z}$ such that $L_r \supset L(\gamma, r)$ and such that $L_t := \bigl\{u + (t-r) \mid u \in L_r\bigr\} \supset L(\gamma, t)$. We infer from Lemma 4.3.11 that

$$(\widehat{f}_r\widehat{f}_t^2\widehat{f}_r\mathbf{M})_\gamma = (f_{L_r}f_{L_t}^2f_{L_r}\mathbf{M})_\gamma \quad \text{and} \quad (\widehat{f}_t\widehat{f}_r^2\widehat{f}_t\mathbf{M})_\gamma = (f_{L_t}f_{L_r}^2f_{L_t}\mathbf{M})_\gamma.$$

Let us write L_r and L_t as: $L_r = \bigl\{r_1,\, r_2,\, \dots,\, r_a\bigr\}$ and $L_t = \bigl\{t_1,\, t_2,\, \dots,\, t_a\bigr\}$, respectively, where $t_b = r_b + (t-r)$ for each $1 \leq b \leq a$; note that $0 < |r_b - t_b| < \ell$ for all $1 \leq b \leq a$. Let K be an interval in $\mathbb{Z}$ containing $L_r \cup L_t$, and take an interval I in $\mathbb{Z}$ satisfying the following:

(a1)' $I \in \mathrm{Int}(\mathbf{M}; e, r_b) \cap \mathrm{Int}(\mathbf{M}; s_{r_b}, r_b)$ for all $1 \leq b \leq a$;

(a2)' $I \in \mathrm{Int}(\mathbf{M}; e, t_b) \cap \mathrm{Int}(\mathbf{M}; s_{t_b}, t_b)$ for all $1 \leq b \leq a$;

(a3)' $I \in \mathrm{Int}(\widehat{F}_p\mathbf{M}; e, t_b) \cap \mathrm{Int}(\widehat{F}_p\mathbf{M}; s_{t_b}, t_b)$ for all $1 \leq b \leq a$;

(a4)' $I \in \mathrm{Int}(\widehat{F}_q\mathbf{M}; e, r_b) \cap \mathrm{Int}(\widehat{F}_q\mathbf{M}; s_{r_b}, r_b)$ for all $1 \leq b \leq a$;

(c1) $\gamma \in \Gamma_I$;

(c2) $I \in \mathrm{Int}(\mathbf{M}; v, k)$ for all $v \in W_K$ and $k \in K$.

Then, since $\mathbf{M} \in \mathcal{BZ}_\mathbb{Z}(I, K)$ by condition (c2), we see from Lemma 3.3.4 (3) that

$$(f_{L_r}f_{L_t}^2f_{L_r}\mathbf{M})_I = f_{L_r}f_{L_t}^2f_{L_r}\mathbf{M}_I \quad \text{and} \quad (f_{L_t}f_{L_r}^2f_{L_t}\mathbf{M})_I = f_{L_t}f_{L_r}^2f_{L_t}\mathbf{M}_I,$$

and hence, by condition (c1), that

$$(f_{L_r}f_{L_t}^2f_{L_r}\mathbf{M})_\gamma = (f_{L_r}f_{L_t}^2f_{L_r}\mathbf{M}_I)_\gamma \quad \text{and} \quad (f_{L_t}f_{L_r}^2f_{L_t}\mathbf{M})_\gamma = (f_{L_t}f_{L_r}^2f_{L_t}\mathbf{M}_I)_\gamma.$$

Thus, in order to show that $(\widehat{f}_r\widehat{f}_t^2\widehat{f}_r\mathbf{M})_\gamma = (\widehat{f}_t\widehat{f}_r^2\widehat{f}_t\mathbf{M})_\gamma$, it suffices to show that

$$f_{L_r}f_{L_t}^2f_{L_r}\mathbf{M}_I = f_{L_t}f_{L_r}^2f_{L_t}\mathbf{M}_I. \tag{4.5.6}$$

We now define

$$X_b := (F_{r_b}F_{t_b}^2F_{r_b})\cdots(F_{r_2}F_{t_2}^2F_{r_2})(F_{r_1}F_{t_1}^2F_{r_1}),$$
$$Y_b := (F_{t_b}F_{r_b}^2F_{t_b})\cdots(F_{t_2}F_{r_2}^2F_{t_2})(F_{t_1}F_{r_1}^2F_{t_1}),$$

for $0 \leq b \leq a$; X_0 and Y_0 are understood to be the identity map on $\mathcal{BZ}_I(\lambda_I)$. We will show by induction on b that $X_b\mathbf{M}_I \neq \mathbf{0}$, $Y_b\mathbf{M}_I \neq \mathbf{0}$, and $X_b\mathbf{M}_I = Y_b\mathbf{M}_I$ for all $0 \leq b \leq a$. If $b = 0$, then there is nothing to prove. Assume, therefore, that $b > 0$. Note that $\mathbf{M}_I \in \mathcal{BZ}_I(\lambda_I)$ (see the comment preceding Claim 1). Hence, $X_{b-1}\mathbf{M}_I \in \mathcal{BZ}_I(\lambda_I)$ since $X_{b-1}\mathbf{M}_I \neq \mathbf{0}$ by the induction hypothesis. Because $\mathcal{BZ}_I(\lambda_I) \cong \mathcal{B}_I(\lambda_I)$ as crystals for $U_q(\mathfrak{g}_I^\vee)$ by Theorem 2.3.7, we have

$$\Phi_{r_b}(X_{b-1}\mathbf{M}_I) = \max\bigl\{N \geq 0 \mid F_{r_b}^N X_{b-1}\mathbf{M}_I \neq \mathbf{0}\bigr\}.$$

Here, observe that $F_{r_b}X_{b-1} = X_{b-1}F_{r_b}$ by the definition of X_{b-1} since for $1 \leq c \leq b-1$,

$$|r_b - r_c| \geq \ell + 1, \quad \text{and} \quad |r_b - t_c| \geq \underbrace{|r_b - r_c|}_{\geq \ell+1} - \underbrace{|r_c - t_c|}_{<\ell} > (\ell+1) - \ell = 1. \tag{4.5.7}$$

As a result, we have

$$\max\bigl\{N \geq 0 \mid F_{r_b}^N X_{b-1}\mathbf{M}_I \neq \mathbf{0}\bigr\} = \max\bigl\{N \geq 0 \mid F_{r_b}^N \mathbf{M}_I \neq \mathbf{0}\bigr\} = \Phi_{r_b}(\mathbf{M}_I),$$

and hence $\Phi_{r_b}(X_{b-1}\mathbf{M}_I) = \Phi_{r_b}(\mathbf{M}_I)$. Recall that for each $1 \leq b \leq a$, the integers r_b and t_b are such that $\overline{r_b} = p$, $\overline{t_b} = q$, and $0 < |r_b - t_b| < \ell$, and that the interval I satisfies conditions (a1)'–(a4)', which are just conditions (a1)–(a4) of Claim 1, with r and t replaced by r_b and t_b, respectively. Consequently, it follows from Claim 1 (i) that $\Phi_{r_b}(\mathbf{M}_I) = \widehat{\Phi}_p(\mathbf{M}) > 0$, and hence $\Phi_{r_b}(X_{b-1}\mathbf{M}_I) = \Phi_{r_b}(\mathbf{M}_I) = \widehat{\Phi}_p(\mathbf{M}) > 0$. Similarly, we can show that $\Phi_{t_b}(X_{b-1}\mathbf{M}_I) = \Phi_{t_b}(\mathbf{M}_I) = \widehat{\Phi}_q(\mathbf{M}) > 0$. Moreover, since $F_{t_b}X_{b-1} = X_{b-1}F_{t_b}$ and $F_{r_b}X_{b-1} = X_{b-1}F_{r_b}$, we have

$$\begin{aligned}\Phi_{r_b}(F_{t_b}X_{b-1}\mathbf{M}_I) &= \max\bigl\{N \geq 0 \mid F_{r_b}^N F_{t_b} X_{b-1}\mathbf{M}_I \neq \mathbf{0}\bigr\} \\ &= \max\bigl\{N \geq 0 \mid F_{r_b}^N F_{t_b}\mathbf{M}_I \neq \mathbf{0}\bigr\} \\ &= \Phi_{r_b}(F_{t_b}\mathbf{M}_I).\end{aligned}$$

Also, it follows from Claim 1 (i) that $\Phi_{r_b}(F_{t_b}\mathbf{M}_I) = \Phi_{r_b}(\mathbf{M}_I) + 1$; note that $\Phi_{r_b}(\mathbf{M}_I) = \Phi_{r_b}(X_{b-1}\mathbf{M}_I)$ as shown above. Combining these, we get $\Phi_{r_b}(F_{t_b}X_{b-1}\mathbf{M}_I) = \Phi_{r_b}(X_{b-1}\mathbf{M}_I) + 1$. Similarly, we have $\Phi_{t_b}(F_{r_b}X_{b-1}\mathbf{M}_I) = \Phi_{t_b}(X_{b-1}\mathbf{M}_I) + 1$. Here we remark that the crystal $\mathcal{BZ}_I(\lambda_I) \cong \mathcal{B}_I(\lambda_I)$ satisfies condition (C5) of Theorem A.1.1. Therefore, we obtain

$$X_b\mathbf{M}_I = F_{r_b}F_{t_b}^2F_{r_b}X_{b-1}\mathbf{M}_I \neq \mathbf{0} \quad \text{and} \quad F_{t_b}F_{r_b}^2F_{t_b}X_{b-1}\mathbf{M}_I \neq \mathbf{0},$$

and

$$\mathbf{0} \neq X_b\mathbf{M}_I = F_{r_b}F_{t_b}^2F_{r_b}X_{b-1}\mathbf{M}_I = F_{t_b}F_{r_b}^2F_{t_b}X_{b-1}\mathbf{M}_I.$$

Also, since $X_{b-1}\mathbf{M}_I = Y_{b-1}\mathbf{M}_I$ by the induction hypothesis, we obtain

$$Y_b\mathbf{M}_I = F_{t_b}F_{r_b}^2F_{t_b}Y_{b-1}\mathbf{M}_I = F_{t_b}F_{r_b}^2F_{t_b}X_{b-1}\mathbf{M}_I \neq \mathbf{0},$$

and

$$X_b\mathbf{M}_I = F_{t_b}F_{r_b}^2F_{t_b}X_{b-1}\mathbf{M}_I = F_{t_b}F_{r_b}^2F_{t_b}Y_{b-1}\mathbf{M}_I = Y_b\mathbf{M}_I.$$

Thus, we have shown that $X_b\mathbf{M}_I \neq \mathbf{0}$, $Y_b\mathbf{M}_I \neq \mathbf{0}$, and $X_b\mathbf{M}_I = Y_b\mathbf{M}_I$ for all $0 \leq b \leq a$, as desired.

Since $X_a \mathbf{M}_I \neq \mathbf{0}$, we have

$$\begin{aligned} X_a \mathbf{M}_I &= (F_{r_a} F_{t_a}^2 F_{r_a}) \cdots (F_{r_2} F_{t_2}^2 F_{r_2})(F_{r_1} F_{t_1}^2 F_{r_1}) \mathbf{M}_I \\ &= (f_{r_a} f_{t_a}^2 f_{r_a}) \cdots (f_{r_2} f_{t_2}^2 f_{r_2})(f_{r_1} f_{t_1}^2 f_{r_1}) \mathbf{M}_I \\ &= f_{L_r} f_{L_t}^2 f_{L_r} \mathbf{M}_I \quad \text{by Theorem 2.3.4;} \end{aligned}$$

on the crystal $\mathcal{BZ}_I \cong \mathcal{B}_I(\infty)$, we have $f_{r_b} f_{r_c} = f_{r_c} f_{r_b}$ and $f_{t_b} f_{t_c} = f_{t_c} f_{t_b}$ for all $1 \leq b,\, c \leq a$, and $f_{r_b} f_{t_c} = f_{t_c} f_{r_b}$ for all $1 \leq b,\, c \leq a$ with $b \neq c$ (see (4.5.7)). Similarly, we can show that $Y_a \mathbf{M}_I = f_{L_t} f_{L_r}^2 f_{L_t} \mathbf{M}_I$. Since $X_a \mathbf{M}_I = Y_a \mathbf{M}_I$ as shown above, we obtain (4.5.6), and hence (4.5.2).

Finally, we show (4.5.3); we give a proof only for the first equation, since the proof of the second one is similar. Define $r,\, t \in \mathbb{Z}$ as in (4.5.5); note that $\widehat{a}_{pq} = a_{rt}$ and $\widehat{a}_{qp} = a_{tr}$ by the definitions. Let K be an interval in $\mathbb{Z}$ such that $r,\, t \in K$, and take an interval I in $\mathbb{Z}$ satisfying conditions (a1)–(a4) in Claim 1, conditions (b1), (b2) in the proof of (4.5.1) with $\mathbf{M}'' = \widehat{F}_q^2 \widehat{F}_p \mathbf{M}$ and r replaced by $\widehat{F}_p^2 \widehat{F}_q \mathbf{M}$ and t, respectively, and the following:

(d) $I \in \mathrm{Int}(\mathbf{M}; e, t-1) \cap \mathrm{Int}(\mathbf{M}; e, t) \cap \mathrm{Int}(\mathbf{M}; e, t+1)$.

Then, we see from the proof of Claim 1 (i) that $\widehat{\Phi}_q(\widehat{F}_p \mathbf{M}) = \Phi_t(F_r \mathbf{M}_I)$. Therefore,

$$\begin{aligned} \widehat{\varepsilon}_q(\widehat{F}_p \mathbf{M}) &= \widehat{\Phi}_q(\widehat{F}_p \mathbf{M}) - \langle \mathrm{Wt}(\widehat{F}_p \mathbf{M}),\, \widehat{\alpha}_q \rangle \\ &= \Phi_t(F_r \mathbf{M}_I) - \langle \mathrm{Wt}(\mathbf{M}) - \widehat{h}_p,\, \widehat{\alpha}_q \rangle \\ &= \Phi_t(F_r \mathbf{M}_I) - \langle \widehat{\lambda} + \mathrm{wt}(\mathbf{M}) - \widehat{h}_p,\, \widehat{\alpha}_q \rangle. \end{aligned} \tag{4.5.8}$$

Let us compute the value $\langle \mathrm{wt}(\mathbf{M}),\, \widehat{\alpha}_q \rangle$. We deduce from the definition (4.3.2) of $\mathrm{wt}(\mathbf{M})$ along with Remark 4.3.2 that $\langle \mathrm{wt}(\mathbf{M}),\, \widehat{\alpha}_q \rangle = -M_{\Lambda_{q-1}} + 2M_{\Lambda_q} - M_{\Lambda_{q+1}}$. Also,

$$\begin{aligned} -M_{\Lambda_{q-1}} + 2M_{\Lambda_q} - M_{\Lambda_{q+1}} &= -M_{\Lambda_{t-1}} + 2M_{\Lambda_t} - M_{\Lambda_{t+1}} \quad \text{by Remark 4.3.2} \\ &= -M_{\varpi_{t-1}^I} + 2M_{\varpi_t^I} - M_{\varpi_{t+1}^I} = \langle \mathrm{wt}(\mathbf{M}_I),\, \alpha_t \rangle \quad \text{by condition (d).} \end{aligned}$$

Hence we obtain $\langle \mathrm{wt}(\mathbf{M}),\, \widehat{\alpha}_q \rangle = \langle \mathrm{wt}(\mathbf{M}_I),\, \alpha_t \rangle$. In addition, note that $\langle \widehat{\lambda},\, \widehat{\alpha}_q \rangle = \langle \lambda_I,\, \alpha_t \rangle$ by the definition (4.5.4) of $\lambda_I \in \mathfrak{h}_I$, and that $\langle \widehat{h}_p,\, \widehat{\alpha}_q \rangle = \widehat{a}_{pq} = a_{rt} = \langle h_r,\, \alpha_t \rangle$. Substituting these equations into (4.5.8), we see that

$$\begin{aligned} \widehat{\varepsilon}_q(\widehat{F}_p \mathbf{M}) &= \Phi_t(F_r \mathbf{M}_I) - \langle \lambda_I + \mathrm{wt}(\mathbf{M}_I) - h_r,\, \alpha_t \rangle \\ &= \Phi_t(F_r \mathbf{M}_I) - \langle \mathrm{Wt}(\mathbf{M}_I) - h_r,\, \alpha_t \rangle \\ &= \Phi_t(F_r \mathbf{M}_I) - \langle \mathrm{Wt}(F_r \mathbf{M}_I),\, \alpha_t \rangle = \varepsilon_t(F_r \mathbf{M}_I). \end{aligned}$$

Now, the same argument as in the proof of (4.5.1) yields $\widehat{\Phi}_q(\widehat{F}_p^2 \widehat{F}_q \mathbf{M}) = \Phi_t(F_r^2 F_t \mathbf{M}_I)$. Using this, we can show in exactly the same way as above that $\widehat{\varepsilon}_q(\widehat{F}_p^2 \widehat{F}_q \mathbf{M}) = \varepsilon_t(F_r^2 F_t \mathbf{M}_I)$. Since we know from Claim 1 (ii) that $\varepsilon_t(F_r \mathbf{M}_I) = \varepsilon_t(F_r^2 F_t \mathbf{M}_I)$, we conclude that $\widehat{\varepsilon}_q(\widehat{F}_p \mathbf{M}) = \widehat{\varepsilon}_q(\widehat{F}_p^2 \widehat{F}_q \mathbf{M})$, as desired. Thus we have shown (4.5.3). This completes the proof of the theorem. □

Proof of Theorem 4.4.1. Recall from [Kas, §8.1] that the crystal basis $\widehat{\mathcal{B}}(\infty)$ can be regarded as the "direct limit" of $\widehat{\mathcal{B}}(\widehat{\lambda})$'s as $\widehat{\lambda} \in \widehat{\mathfrak{h}}$ tends to infinity, i.e., as $\langle \widehat{\lambda}, \widehat{\alpha}_i \rangle \to +\infty$ for all $i \in \widehat{I}$. Also, by using (4.4.1), we can verify that the direct limit of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O}; \widehat{\lambda})$'s (as $\widehat{\lambda} \in \widehat{\mathfrak{h}}$ tends to infinity) is nothing but $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$. Consequently, the crystal $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ is isomorphic to the crystal basis $\widehat{\mathcal{B}}(\infty)$. This proves Theorem 4.4.1. □

A Appendix.

A.1 Characterization of some crystal bases in the simply-laced case. In this appendix, let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix indexed by a finite set I such that $a_{ij} \in \{0, -1\}$ for all $i, j \in I$ with $i \neq j$. Let $\mathfrak{g}$ be the (simply-laced) Kac-Moody algebra over $\mathbb{C}$ associated to this generalized Cartan matrix A, with Cartan subalgebra $\mathfrak{h}$, and simple coroots h_i, $i \in I$. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $\mathfrak{g}$. For a dominant integral weight $\lambda \in \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ for $\mathfrak{g}$, let $\mathcal{B}(\lambda)$ denote the crystal basis of the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ.

Let $\mathcal{B}$ be a crystal for $U_q(\mathfrak{g})$, equipped with the maps wt, e_p, f_p $(p \in I)$, and ε_p, φ_p $(p \in I)$. We assume that $\mathcal{B}$ is semiregular in the sense of [HK, p.86]; namely, for $x \in \mathcal{B}$ and $p \in I$,

$$\varepsilon_p(x) = \max\{N \geq 0 \mid e_p^N x \neq \mathbf{0}\} \in \mathbb{Z}_{\geq 0},$$
$$\varphi_p(x) = \max\{N \geq 0 \mid f_p^N x \neq \mathbf{0}\} \in \mathbb{Z}_{\geq 0},$$

where $\mathbf{0}$ is an additional element, which is not contained in $\mathcal{B}$. Let X denote the crystal graph of the crystal $\mathcal{B}$. We further assume that the crystal graph X is connected. The following theorem is a restatement of results in [S].

Theorem A.1.1. *Keep the setting above. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight for $\mathfrak{g}$. The crystal $\mathcal{B}$ is isomorphic, as a crystal for $U_q(\mathfrak{g})$, to the crystal basis $\mathcal{B}(\lambda)$ if and only if $\mathcal{B}$ satisfies the following conditions* (C1)–(C6):

(C1) *If $x \in \mathcal{B}$ and $p, q \in I$ are such that $p \neq q$ and $e_p x \neq \mathbf{0}$, then $\varepsilon_q(x) \leq \varepsilon_q(e_p x)$ and $\varphi_q(e_p x) \leq \varphi_q(x)$.*

(C2) *Let $x \in \mathcal{B}$, and $p, q \in I$. Assume that $e_p x \neq \mathbf{0}$ and $e_q x \neq \mathbf{0}$, and that $\varepsilon_q(e_p x) = \varepsilon_q(x)$. Then, $e_p e_q x \neq \mathbf{0}$, $e_q e_p x \neq \mathbf{0}$, and $e_p e_q x = e_q e_p x$.*

(C3) *Let $x \in \mathcal{B}$, and $p, q \in I$. Assume that $e_p x \neq \mathbf{0}$ and $e_q x \neq \mathbf{0}$, and that $\varepsilon_q(e_p x) = \varepsilon_q(x) + 1$ and $\varepsilon_p(e_q x) = \varepsilon_p(x) + 1$. Then, $e_p e_q^2 e_p x \neq \mathbf{0}$, $e_q e_p^2 e_q x \neq \mathbf{0}$, and $e_p e_q^2 e_p x = e_q e_p^2 e_q x$. Moreover, $\varphi_q(e_p x) = \varphi_q(e_p^2 e_q x)$ and $\varphi_p(e_q x) = \varphi_p(e_q^2 e_p x)$.*

(C4) *Let $x \in \mathcal{B}$, and $p, q \in I$. Assume that $f_p x \neq \mathbf{0}$ and $f_q x \neq \mathbf{0}$, and that $\varepsilon_q(f_p x) = \varepsilon_q(x)$. Then, $f_p f_q x \neq \mathbf{0}$, $f_q f_p x \neq \mathbf{0}$, and $f_p f_q x = f_q f_p x$.*

(C5) *Let $x \in \mathcal{B}$, and $p, q \in I$. Assume that $f_p x \neq \mathbf{0}$ and $f_q x \neq \mathbf{0}$, and that $\varphi_q(f_p x) = \varphi_q(x) + 1$ and $\varphi_p(f_q x) = \varphi_p(x) + 1$. Then, $f_p f_q^2 f_p x \neq \mathbf{0}$, $f_q f_p^2 f_q x \neq \mathbf{0}$, and $f_p f_q^2 f_p x = f_q f_p^2 f_q x$. Moreover, $\varepsilon_q(f_p x) = \varepsilon_q(f_p^2 f_q x)$ and $\varepsilon_p(f_q x) = \varepsilon_p(f_q^2 f_p x)$.*

(C6) *There exists an element* $x_0 \in \mathcal{B}$ *of weight* λ *such that* $e_p x_0 = \mathbf{0}$ *and* $\varphi_p(x_0) = \langle h_p,\, \lambda \rangle$ *for all* $p \in I$.

(Sketch of) Proof. First we prove the "if" part. Recall that the crystal graph X of the crystal $\mathcal{B}$ is an I-colored directed graph. We will show that X is A-regular in the sense of [S, Definition 1.1]. It is obvious that X satisfies condition (P1) on page 4809 of [S] since $\mathcal{B}$ is assumed to be semiregular. Also, it follows immediately from the axioms of a crystal that X satisfies condition (P2) on page 4809 of [S]. Now we note that for $x \in \mathcal{B}$ and $p \in I$, $\varepsilon(x,p)$ (resp., $\delta(x,p)$) in the notation of [S] agrees with $\varphi_p(x)$ (resp., $-\varepsilon_p(x)$) in our notation. Hence, for $x \in \mathcal{B}$ and $p,\, q \in I$ with $e_p x \neq \mathbf{0}$, $\Delta_p \delta(x,q)$ (resp., $\Delta_p \varepsilon(x,q)$) in the notation of [S] agrees with $-\varepsilon_q(e_p x) + \varepsilon_q(x)$ (resp., $\varphi_q(e_p x) - \varphi_q(x)$) in our notation. Hence, in our notation, we can rewrite condition (P3) on page 4809 of [S] as: $\bigl\{-\varepsilon_q(e_p x) + \varepsilon_q(x)\bigr\} + \bigl\{\varphi_q(e_p x) - \varphi_q(x)\bigr\} = a_{pq}$ for $x \in \mathcal{B}$ and $p,\, q \in I$ such that $p \neq q$ and $e_p x \neq \mathbf{0}$. From the axioms of a crystal, we have

$$\begin{aligned}
\varphi_q(e_p x) - \varepsilon_q(e_p x) &= \langle h_q,\, \mathrm{wt}(e_p x) \rangle = \langle h_q,\, \alpha_p \rangle + \langle h_q,\, \mathrm{wt}\, x \rangle \\
&= a_{qp} + \langle h_q,\, \mathrm{wt}\, x \rangle, \\
\varphi_q(x) - \varepsilon_q(x) &= \langle h_q,\, \mathrm{wt}\, x \rangle.
\end{aligned}$$

Thus, condition (P3) on page 4809 of [S] holds for the crystal graph X. Similarly, in our notation, we can rewrite condition (P4) on page 4809 of [S] as: $-\varepsilon_q(e_p x) + \varepsilon_q(x) \leq 0$ and $\varphi_q(e_p x) - \varphi_q(x) \leq 0$ for $x \in \mathcal{B}$ and $p,\, q \in I$ such that $p \neq q$ and $e_p x \neq \mathbf{0}$, which is equivalent to condition (C1). In addition, note that for $x \in \mathcal{B}$ and $p,\, q \in I$ with $f_p x \neq \mathbf{0}$, $\nabla_p \delta(x,q)$ (resp., $\nabla_p \varepsilon(x,q)$) in the notation of [S] agrees with $-\varepsilon_q(x) + \varepsilon_q(f_p x)$ (resp., $\varphi_q(x) - \varphi_q(f_p x)$) in our notation. In is easy to check that conditions (P5) and (P6) on page 4809 of [S] are equivalent to conditions (C2) and (C3), respectively. Similarly, it is easily seen that conditions (P5') and (P6') on page 4809 of [S] are equivalent to conditions (C4) and (C5), respectively. Thus, we have shown that the crystal graph X is A-regular.

We know from [S, §3] that the crystal graph of the crystal basis $\mathcal{B}(\lambda)$ is A-regular. Also, it is obvious that the highest weight element u_λ of $\mathcal{B}(\lambda)$ satisfies the condition that $e_p u_\lambda = \mathbf{0}$ and $\varphi_p(u_\lambda) = \langle h_p,\, \lambda \rangle$ for all $p \in I$ (cf. condition (C6)). Therefore, we conclude from [S, Proposition 1.4] that the crystal graph X of the crystal $\mathcal{B}$ is isomorphic, as an I-colored directed graph, to the crystal graph of the crystal basis $\mathcal{B}(\lambda)$; note that $x_0 \in \mathcal{B}$ corresponds to $u_\lambda \in \mathcal{B}(\lambda)$ under this isomorphism. Since the crystal graphs of $\mathcal{B}$ and $\mathcal{B}(\lambda)$ are both connected, and since $x_0 \in \mathcal{B}$ and $u_\lambda \in \mathcal{B}(\lambda)$ are both of weight λ, it follows that the crystal $\mathcal{B}$ is isomorphic to the crystal basis $\mathcal{B}(\lambda)$. This proves the "if" part.

The "only if" part is now clear from the argument above. Thus we have proved the theorem. □

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