# COMPLETENESS OF THE BETHE ANSATZ
# ON WEYL ALCOVES

E. EMSIZ

ABSTRACT. We prove the completeness of the Bethe ansatz eigenfunctions of the Laplacian on a Weyl alcove with repulsive boundary condition at the walls. For the root system of type **A** this amounts to the result of Dorlas of the completeness of the Bethe ansatz eigenfunctions of the quantum Bose gas on the circle with repulsive delta-function interaction.

## 1. INTRODUCTION

In a celebrated paper Lieb and Liniger [16] introduced the quantum Bose gas on the circle with repulsive delta-function interaction and using the Bethe ansatz method showed that it is exactly solvable. C. N Yang and C. P. Yang [21] proved that the corresponding Bethe ansatz equations are controlled by a strictly convex function. Fundamental progress was made by Korepin [14], who proved Gaudin's [7, 8] compact determinantal formula for the $L^2$-norms of the Bethe ansatz eigenfunctions and Dorlas [4], who proved the completeness and orthogonality of these functions using a lattice version of the quantum inverse scattering methods [15].

A crucial insight by Gaudin [7, 8] was that the quantum Bose gas on the line with a delta-function potential has a natural generalization in the context of the root system of semi-simple complex Lie algebras. Gutkin and Sutherland [10, 9] pushed these generalization further by introducing affine root system version. The quantum Hamiltonian now has a potential expressible as a weighted sum of delta-functions at the associated affine root hyperplanes. In [5, 6] the affine Weyl group invariant quantum eigenvalue problem for these systems in the repulsive regime was studied using representation theory of the trigonometric Cherednik algebra and Dunkl operators. From this perspective the quantum Bose gas of quantum particles with repulsive delta-potential on the circle essentially corresponds to the affine root system of type **A**. A large part of [5, 6] deals with extending many of the above properties of the quantum Bose gas to the root system versions ([6] deals with quantum spin-particle systems).

In this paper we continue with the study of the general root system versions. The main result is the completeness of the Bethe ansatz eigenfunctions in the sense that they span a dense subspace of the Hilbert space $L^2(A)$ of quadratically integrable functions with respect to the standard Euclidean measure on a Weyl alcove $A$. The question of orthogonality and norm formulae for the root system versions turns out to be much harder to crack and for general root systems we only have conjectural statements.

The paper is organized as follows. In the following sections we introduce the quantum integrable systems and summarize the main results of [5, 6] on the quantum spectral eigenvalue problem. The completeness will be deduced from a continuity argument of Dorlas [4] at zero coupling, which we recall in Section 3. An important part is played by making sense (in Section 5) of the formal quantum Hamiltonians as positive self-adjoint operators on $L^2(A)$. Section 4 is preparatory and deals with the non-interacting case and the necessary Sobolev space theory to define the forms to make sense of the aforementioned formal Hamiltonians as positive self-adjoint operators on $L^2(A)$.

## 2. COMPLETENESS

We will freely use concepts and facts on root systems, see e.g. [12] for a detailed exposition. Let $V$ be an Euclidean space of dimension $n$ and $R$ a finite, irreducible crystallographic root system spanning $V$. We denote $\langle \cdot, \cdot \rangle$ for the inner product on $V$ and $\| \cdot \|$ for the corresponding norm. The co-root of a root $\alpha \in R$ is $2\alpha/\|\alpha\|^2$. We fix a positive system $R^+$ and denote the corresponding basis of simple roots by $I = \{\alpha_1, \ldots, \alpha_n\}$. We let $\alpha_0 = -\varphi$, where $\varphi$ denotes the highest root of $R^+$. The corresponding Weyl alcove

$$A = \{v \in V | 0 < \langle \alpha, v \rangle < 1, \forall \alpha \in R^+\} \tag{1}$$

is bounded by the walls

$$V_0 = \{v \in V \mid \langle \alpha_0, v \rangle + 1 = 0\}, \tag{2}$$

$$V_i = \{v \in V \mid \langle \alpha_i, v \rangle = 0\}, \quad i = 1, \ldots, n \tag{3}$$

The space of repulsive coupling constants $\mathcal{K}$ is the set of all $k_\alpha \in (0, \infty)$, $\alpha \in R$ such that $k_\alpha = k_\beta$ whenever $\|\alpha\| = \|\beta\|$. We identify $\mathcal{K}$ naturally with $(0, \infty)$ (if $R$ has one root lenght) or $(0, \infty)^2$ (respectively two root lenghts).

For any $k \in \mathcal{K}$ we will consider the following spectral problem for the Laplacian in the alcove $A$ with repulsive boundary condition at the walls, i.e.

$$-\Delta f = E f \quad \text{on } A, \tag{4}$$

$$(\partial_{\alpha_i^\vee} f)(v) = k_{\alpha_i} f(v), \quad v \in V_i \cap \overline{A}, \quad i = 0, 1, \ldots, n, \tag{5}$$

(note that $\alpha_i^\vee$ points inward to $A$) where $\Delta$ refers to the Laplacian and $\partial_v$ to the partial derivative in the direction of $v$.

Let $s_i : V \to V$, $i = 0, 1, \ldots, n$ denote the orthogonal reflection in the wall $V_i$:

$$s_0(v) = v - (\langle \alpha_0, v \rangle + 1)\alpha_0^\vee,$$

$$s_i(v) = v - \langle \alpha_i, v \rangle \alpha_i^\vee, i = 1, 2 \ldots, n$$

The reflections $s_1, \ldots, s_n$ generate the Weyl group $W$ associated with $R$ while $s_0, s_1, \ldots, s_n$ generate the affine Weyl group $W^a$. A second important presentation of $W^a$ is given by $W^a = W \ltimes Q^\vee$, with $Q^\vee$ the co-root lattice generated by $\alpha^\vee$, $\alpha \in R$, acting by translations on $V$.

It was shown in [5, Theorem 2.6],[6] that the Bethe ansatz function

$$\phi_\lambda^k(v) = \frac{1}{\#W} \sum_{w \in W} c_k(w\lambda) e^{i\langle w\lambda, v \rangle}, \quad v \in \overline{A}, \tag{6}$$

solves the boundary value problem (4), (5) with eigenvalue $E = \|\lambda\|^2$ if

$$c_k(\lambda) = \prod_{\alpha \in R^+} \frac{\langle \lambda, \alpha^\vee \rangle - i k_\alpha}{\langle \lambda, \alpha^\vee \rangle} \tag{7}$$

and the spectral parameter $\lambda \in V$ satisfies the Bethe ansatz equations (BAE)

$$e^{i \langle \lambda, \alpha_j^\vee \rangle} = \prod_{\beta \in R^+} \left( \frac{\langle \lambda, \beta^\vee \rangle + i k_\beta}{\langle \lambda, \beta^\vee \rangle - i k_\beta} \right)^{\langle \alpha_j^\vee, \beta \rangle}, \quad j = 1, 2, \dots, n. \tag{8}$$

The cone of dominant weights is denoted by

$$\mathcal{P}^+ = \{ \lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}, \quad \forall \alpha \in R^+ \}.$$

while $\mathcal{P}^{++} = \rho + \mathcal{P}^+$ is the cone of strictly dominant weights ($\rho$ is the half sum of positive roots). The function $S_k : \mathcal{P}^{++} \times V \to \mathbb{R}$ defined by

$$S_k(\mu, v) = \frac{1}{2} \|v\|^2 - 2\pi \langle v, \mu \rangle + \frac{1}{2} \sum_{\alpha \in R} \|\alpha\|^2 \int_0^{\langle v, \alpha^\vee \rangle} \arctan\left( \frac{t}{k_\alpha} \right) dt \tag{9}$$

is strictly convex and assumes a global minimum at $\widehat{\mu}_k \in V$. Moreover it is known [5, Propositions 2.9 and 2.10] that $\widehat{\mu}_k$ solves the BAE (8) and lies in the fundamental Weyl chamber

$$V_+ = \{ v \in V \mid \langle v, \alpha^\vee \rangle > 0 \quad \forall \alpha \in R^+ \}. \tag{10}$$

The main results of this paper states that the Bethe anstaz eigenfunctions are complete in the Hilbert space $L^2(A) = L^2(A, dv)$ of quadratically integrable functions on $A$ with respect to the Euclidean measure $dv$:

**Theorem 1.** *The Bethe ansatz eigenfunctions* $\phi_{i\widehat{\mu}_k}^k$, $\mu \in \mathcal{P}^{++}$ *span a dense subspace of* $L^2(A)$.

One sees easily that $\phi_{\widehat{\mu}_k}^k$ and $\phi_{\widehat{\eta}_k}^k$ are orthogonal if $\|\widehat{\mu}_k\| \neq \|\widehat{\eta}_k\|$ (see e.g. (16)). Full orthogonality is to the knowledge of the author only conjectural for general $R$, as well as the following $L^2$-norm formula conjecture:

$$\frac{1}{|A|} \int_A \left| \phi_{\widehat{\mu}_k}^k(v) \right|^2 dv = \frac{|c_k(\widehat{\mu}_k)|^2 \det B_{\widehat{\mu}_k}^k}{\#W}, \quad \forall \mu \in \mathcal{P}^{++}, \tag{11}$$

Here $B_v^k : V \times V \to \mathbb{R}$ is the Hessian of $S_k(\mu, \cdot)$ at $v \in V$ (see also [5, (9.3)]) and $|A| = \int_A dv$.

An interesting consequence of (11) would be $\lim_{k \to 0} \det B_{\rho_k}^k = \#W$, an identity very similar to the limit formula [11, (3.5.14)] for the Heckman-Opdam hypergeometric functions associated to $R$.

As alluded to in the introduction, for $R$ of type **A** the norm formulae and orthogonality conjectures were proved by Korepin and Dorlas, respectively. A new proof of the orthogonality in this case was recently obtained by van Diejen [3] using an integrable lattice dicretization of the quantum Bose-gas on the circle. The norm formulae (11) were also very recently checked [1] for all $R$ with rank less or equal to 3, thus including the important test case of $R$ of type $G_2$.

The upgrade from type **A** to other classical root systems amounts to adding particular reflection terms to the physical model, see e.g. [7, 8, 2] (and [5, 6] for more references). For the exceptional root systems there is not such a convincing interpretation as a physical model (however, see [2] for the exceptional root system $R$ of type $G_2$).

## 3. Continuity

Following Dorlas [4] we will deduce the completeness from a continuity argument at zero coupling:

**Theorem 2** ([4]). *Let $\{H_t\}_{t\in[0,\infty)}$ be a family of positive self-adjoint operators acting on a Hilbert space $\mathfrak{H}$. Assume that $t \mapsto H_t$ is monotonically nondecreasing and right-continuous in the strong-resolvent sense and that $H_0$ has compact resolvent. Suppose that there exists a set of linearly independent eigenfunctions $\{\phi_n^t\}_{n\in N}$ for $H_t$, parameterized by a discrete set $N$, which depends continuously on $t \in [0,\infty)$, and which is complete at $t = 0$. Then the linear subspace spanned by the $\phi_n^t$ $(n \in N)$ is dense in $\mathfrak{H}$ for all $t \in [0,\infty)$.*

In [5, 6] the eigenvalue value problem (4), (5) was interpreted as the $W^a$-invariant spectral problem of the following formal quantum Hamiltonian

$$\mathcal{H}_k = -\Delta + \sum_{\alpha \in R, m \in \mathbb{Z}} k_\alpha \delta(\langle \alpha, \cdot \rangle + m) \tag{12}$$

and was studied with the aid of trigonometric Cherednik algebra and Dunkl operators. Here $W^a$ acts on the space of $\mathbb{C}$-valued functions $f$ as follows,

$$(wf)(v) = f(w^{-1}v), \quad (w \in W^a, v \in V). \tag{13}$$

We will show that for the $W^a$-invariant theory the formal $\mathcal{H}_k$ can be interpreted as self-adjoint operators on the fundamental domain $A$ for the action of $W^a$ on $V$, and moreover they will be the operators $H_k$ in Theorem 2 (see Section 5 for the exact statement). The functions $\phi_n^t$, $n \in N$ will be $\phi_{\hat{\mu}_k}^k$, $\mu \in \mathcal{P}^{++}$. As for the case $k \equiv 0$ (free boundary value problem), note that the plane wave

$$\phi_\lambda^0(v) = \frac{1}{\#W} \sum_{w \in W} e^{iw\lambda(v)}, \quad v \in \overline{A}, \tag{14}$$

solves (4), (5) iff $\lambda = \widehat{\mu}_0 = 2\pi(\mu - \rho)$, $\mu \in \mathcal{P}^{++}$. The continuity of the eigenfunctions, one of the conditions of Theorem 2 is guaranteed by:

**Proposition 3.** *Let $\mu \in \mathcal{P}^{++}$. Then $k \mapsto \phi_{\hat{\mu}_k}^k$ defines a continuous map from $\mathcal{K} \cup \{0\}$ to $L^2(A)$.*

Since $(v, k) \mapsto S_k(\mu, v)$ is differentiable in $k \in \mathcal{K}$, $v \in V$ the implicit function theorem shows that $k \mapsto \hat{\mu}_k$ is smooth from $\mathcal{K}$ to $V_+$, and therefore $k \mapsto \phi_{\hat{\mu}_k}^k$ is continuous from $\mathcal{K}$ to $L^2(A)$. The problematic point is continuity in the boundary point $k \equiv 0$. We will actually not use any differentiability, Proposition 3 will be proved by a purely topological argument, based on the following lemma (which we formulate in greater generality than strictly necessary since the proof is not more difficult).

**Lemma 4.** *Let $X$ and $Y$ be locally compact Hausdorff spaces with $Y$ first countable. Let $f$ be a continuous real-valued function on $X \times Y$ such that $f(\cdot, y)$ has a unique global minimum at $\gamma_y$ for $y \in Y$. Define the map $\gamma : Y \to X$ by $y \mapsto \gamma_y$. Assume also there is a compact $D \subset X$ such that $\gamma(Y) \subset D$. Then $\gamma$ is continuous.*

*Proof.* Assume $\gamma$ is not continuous in a point $\eta \in Y$. We will show that this leads to a contradiction. There is a neighborhood $N$ of $\gamma_\eta$ with compact closure and a sequence $y_1, y_2, \ldots$ in $Y$ with $y_j \to \eta$ and $\gamma_{y_j} \in X \backslash N$ for all $j$, and where $X \backslash N$

is the complement of $N$ in $X$. Defining $m_0 = \min_{x \in D \cap (X \setminus N)} (f(x, \eta) - f(\gamma_\eta, \eta))$, we have $m_0 > 0$ by definition of $\gamma_\eta$ and compactness of $D$. Choose an $\varepsilon$ such that $0 < \varepsilon < m_0/4$. Choose a neighborhood $U$ of $\eta$ with compact closure such that $|f(x, \eta) - f(x, y)| \le \varepsilon$ for $y \in U$ and $x \in D$. Hence for $y \in U$ we have $f(\gamma_y, \eta) - f(\gamma_y, y) \le \varepsilon$. Furthermore by the definition of $\gamma_y$ we have $f(\gamma_y, y) \le f(\gamma_\eta, y)$, and continuity of $f$ in $y$ implies that there is a neighborhood $U'$ of $\eta$ such that $f(\gamma_\eta, y) \le f(\gamma_\eta, \eta) + \varepsilon$ for $y \in U'$, thus $f(\gamma_y, y) \le f(\gamma_\eta, \eta) + \varepsilon$ for $y \in U'$. Therefore $f(\gamma_y, \eta) - f(\gamma_\eta, \eta) \le 2\varepsilon < m_0/2$ for $y \in U \cap U'$, which leads to a contradiction for $y = y_j$ and $j$ large by the definition of $m_0$. $\qquad\square$

*Proof of Proposition 3.* Let $X = V$, $Y = \mathcal{K} \cup \{0\}$ and $f(v, k) = S_k(\mu, v)$, where $S_0(\mu, v) = \frac{1}{2}\|v\|^2 - 2\pi\langle v, \mu\rangle + \pi \sum_{\alpha \in R^+} |\langle v, \alpha\rangle|$. Then $S_0(\mu, v)$ is continuous (but not differentiable), strictly convex in $v$ and has a unique global minimum that is obtained at $\widehat{\mu}_0 = \mu - \rho$. Lemma 4 with [5, Propositions 2.10] yields that $k \mapsto \widehat{\mu}_k$ is continuous on $\mathcal{K} \cup \{0\}$, and in particular $\lim_{k \to 0} \widehat{\mu}_k = \widehat{\mu}_0 = 2\pi(\mu - \rho)$. By Section 9 of [5] $\widehat{\mu}_k \in V$ is determined by the equation $\widehat{\mu}_k + \sigma_{\widehat{\mu}_k}^k = 2\pi\mu$ where $\sigma_\lambda^k = 2 \sum_{\alpha \in R^+} \arctan\left(\langle \lambda, \alpha^\vee\rangle / k_\alpha\right) \alpha$. Taking the inproduct of this with a $v \in V_+$ gives $\arctan(\langle \widehat{\mu}_k, \alpha^\vee\rangle / k_\alpha) \to \pi/2$ as $k \to 0$, and whence $\langle \widehat{\mu}_k, \alpha^\vee\rangle / k_\alpha \to +\infty$ as $k \to 0$. In particular $c_k(\widehat{\mu}_k)$ is continuos in $k \in \mathcal{K} \cup \{0\}$, which together with (6) yields the proposition. $\qquad\square$

In the following section we will make sense of $\mathcal{H}_0 = -\Delta$ as a self-adjoint positive operator $H_0$. The form to make sense of (12) as a self-adjoint operator on $L^2(A)$ will be defined in the final section. It will be a perturbation of the form corresponding to $H_0$. For this we will show in the following section that the form domain of the form corresponding to $H_0$ is the Sobolev space of once weak differentiable functions on $A$ while the form itself is equal to the Sobolev inner product.

## 4. The Laplacian on the Weyl alcove

The domain of any operator $H$ will be denoted by $D(H)$ and its form and form domain by $q_H$ and $Q(H)$, respectively.

One checks easily that a function $f \in D_0 = \{f_{|\overline{A}} | f \in C^\infty(V)^{W^a}\}$ satisfies

$$(\partial_{\alpha_i^\vee} f)(v) = 0, \quad v \in V_i \cap \overline{A}, \ i = 0, \dots, n \tag{15}$$

By Stokes' theorem we have for suitable $f, g$,

$$\int_A ((\Delta f)g - f\Delta g) = \sum_{j=0}^n \frac{1}{\|\alpha_j^\vee\|} \int_{V_j \cap \overline{A}} (f\overline{\partial_{\alpha_j^\vee} g} - (\partial_{\alpha_j^\vee} f)\overline{g}). \tag{16}$$

Whence $-\Delta$ with domain $D_0$ is symmetric on $L^2(A)$, and moreover also positive by another application of Stokes' theorem. Furthermore:

**Proposition 5.** $(-\Delta, D_0)$ *is essentially self-adjoint and $D_0$ is an operator core for the unique positive self-adjoint extension $(H_0, D(H_0))$. Moreover the $\phi_\mu^0 \in D_0 \subset D(H_0)$, $\mu \in 2\pi\mathcal{P}^+$ form a complete set of orthogonal functions and $H_0\phi_\mu^0 = \|\mu\|^2\phi_\mu^0$. Furthermore $H_0$ has compact resolvent.*

*Proof.* The first statement follows from the second one and [18, Theorem X.39]. We have $\phi_\mu^0 \in D_0$ because (14) extends directly to a $W^a = W \ltimes Q^\vee$-invariant function in $C(V)^{W^a}$. Orthogonality of the $\phi_\mu^0$ is shown by an explicit calculation and $H_0\phi_\mu^0 = -\Delta\phi_\mu^0 = \|\mu\|^2\phi_\mu^0$. Density of the $\phi_\mu^0$ follows from an elementary

application of the Stone-Weierstrass theorem. The last statement follows from [19, Theorem XIII.64]. $\qquad\square$

For a $f \in L^2(A)$ and $\eta \in V$ one says that $\partial_\eta f \in L^2(A)$ weakly provided that there is a $g \in L^2(A)$ such that $\partial_\eta f = g$ as distributions on $A$. It is a standard result in Sobolev space theory that $W^{1,2} = \{f \in L^2(A) \mid \partial_\eta f \in L^2(A), \ \forall \eta \in V\}$ is a Hilbert space with inner product

$$(f,g)_{W^{1,2}} = \int_A f(v)\overline{g(v)}dv + \sum_{j=1}^n \int_A (\partial_{\eta_j} f)(v)\overline{(\partial_{\eta_j} g)(v)}dv, \qquad (17)$$

where $\{\eta_1, \ldots, \eta_n\}$ denotes an orthonormal basis for $V$ ((17) is in particular independent of the choice of orthonormal basis). Stokes' theorem with (15) yields that $h_0(f,g) = (f,g)_{W^{1,2}}$ for $f,g \in D_0 \subset Q_0$. Making use of the geometry of the affine hyperplane arrangement associated with the root system $R$ we can prove for the symmetric domain $A$:

**Proposition 6.** *The form domain $Q_0$ of $h_0$ is equal to the Sobolev space $W^{1,2}$ of once weak differentiable functions on $A$ and $h_0 = (\cdot, \cdot)_{W^{1,2}}$.*

*Proof.* Since $D_0$ is an operator core for $(H_0, D(H_0))$ (Proposition 5) and $h_0 = (\cdot, \cdot)_{W^{1,2}}$ on $D_0$ it suffices to show that $D_0$ is a dense subspace of $W^{1,2}$ in the Sobolev topology for the first statement. The second statement then follows because $D_0$ is a form core for $(h_0, Q_0)$. Since $C^\infty(\overline{A}) = \{f_{|\overline{A}} \mid f \in C^\infty(V)\}$ is dense in $W^{1,2}$ (see e.g. [20, Theorem I.3.6]), it suffices to show that every $\phi \in C^\infty(\overline{A})$ can be approximated in $W^{1,2}$ by functions from $D_0$. Fix therefore a $\phi \in C^\infty(\overline{A})$. Denote the unique $W^a$-invariant extension of $\phi$ to $V$ by $u$, thus $u \in C(V)^{W^a}$ (see (13)).

We will use a regularization procedure. For a $r > 0$ and $v \in V$ let $\bar{B}(v,r) = \{v' \in V \mid \|v' - v\| \leq r\}$. Choose a $j \in C_c^\infty(V)^W$ with $j \geq 0$ and support in $\bar{B}(0,1)$, and $\int_V j(v)dv = 1$. For $\varepsilon > 0$ define $j_\varepsilon \in C_c^\infty(V)^W$ by $j_\varepsilon(v) = j(v/\varepsilon)/\varepsilon^n$. Consider the regularization $u_\varepsilon \in C^\infty(V)$ of $u$, $u_\varepsilon(v) = \int_V j_\varepsilon(v - v')u(v')dv' = \int_V j_\varepsilon(v')u(v - v')dv'$. Since $u \in L^1_{loc}(V, dv)^{W^a}$ we have $u_\varepsilon \in C^\infty(V)^{W^a}$ and in particular $u_{\varepsilon|\overline{A}} \in D_0$. We shall show that $u_{\varepsilon|\overline{A}} \to \phi$ in the Sobolev norm as $\varepsilon \searrow 0$.

For any $\eta \in V$ consider the function in $u^\eta \in L^1_{loc}(V, dv)$ defined uniquely by $u^\eta = \partial_\eta u$ on $V_{reg}$. We claim that $\partial_\eta u_\varepsilon = (u^\eta)_\varepsilon$ for all $\varepsilon > 0$. For $t > 0$ consider the difference-quotient function $g_{t,\eta}(v) = (u(v + t\eta) - u(v))/t$ for $v \in V$. Then $g_{t,\eta}$ is continuous on $V$ and $Q^\vee$-periodic. It is easily seen that for every $v \in V$ and $0 < t < \text{diam}(A)$ the line segment between $v$ and $v + t\eta$ lies in at most $\#W/2$ alcoves. Since $u_{|\overline{C}} \in C^\infty(\overline{C})$ for all alcoves $C = wA$ ($w \in W^a$), multiple applications of the mean value theorem (but at most $\#W/2$ times) together with the $W^a$-invariance and continuity of $u$ yields the bound $\#W M_\eta/2$ for $|g_{t,\eta}|$ on $V$ for $t \in (0, \text{diam}(A))$ and where $M_\eta$ denotes the supremum of $\partial_\eta u$ on $A$. Whence $|j_\varepsilon g_{t,\eta}(v - \cdot)|$ is bounded by $(\#W M_\eta)/2 \sup_V j/(\varepsilon^n)$ (and has support in $\bar{B}(0,\varepsilon)$) for all $v \in V$. Since $\lim_{t \to 0} g_{t,\eta} = u^\eta$ pointwise on $V_{reg}$, Lebesgue's dominated convergence theorem (integrating over $\bar{B}(0,\varepsilon)$) yields then indeed $\partial_\eta u_\varepsilon = (u^\eta)_\varepsilon$ for all $\varepsilon > 0$.

Together with $u_\varepsilon \to u$ uniformly on compact subsets of $A$, which follows from $u_{|A} \in C(A)$ and an elementary uniform continuity argument, it follows that $\partial_\eta u_\varepsilon \to \partial_\eta u$ point-wise on $A$. Note that if a locally integrable function is bounded by a $M \geq 0$, then so does its regularization. Applying this observation to $u^\eta$ and

using $\partial_\eta u_\varepsilon = (u^\eta)_\varepsilon$ shows that $|\partial_\eta(u_\varepsilon - u)|$ is bounded by $2M_\eta$ uniformly on $A$. A simple application of Lebesgue's dominated convergence theorem now gives $\|\partial_\eta u_\varepsilon - \partial_\eta u\|_{L^2(A)} \to 0$ as $\varepsilon \to 0$. Furthermore $\|u_\varepsilon - u\|_{L^2(A)} \to 0$ as $\varepsilon \to 0$ since $u_\varepsilon \to u$ uniformly on $\overline{A}$ and whence $u_{\varepsilon|A} \to u_{|A} = \phi$ in the Sobolev norm, concluding the proof. $\qquad\square$

In the following final section we will finish the proof of Theorem 1 on the completeness of the Bethe ansatz eigenfunctions on the Weyl alcove $A$.

## 5. Proof of the completeness

We denote the restriction of the Euclidean measure on $V$ to $\partial A$ also by $dv$. The following non-trivial result is a special case of [20, Theorem I.8.7],

**Theorem 7** (Trace operator). *There exist a continuous linear operator*

$$\mathcal{B} : W^{1,2} \to L^2(\partial A, dv) \tag{18}$$

*such that $\mathcal{B}\phi = \phi_{|\partial A}$ for $\phi \in W^{1,2} \cap C^1(\overline{A})$, where $C^1(\overline{A}) = \{f_{|\overline{A}} | f \in C^1(V)\}$.*

We now are able to define the form to make sense of (12) as a positive self-adjoint operator.

**Definition 8.** *We denote by $\delta_k$ the following form with domain $W^{1,2}$,*

$$\delta_k(f,g) = \sum_{j=0}^{n} \frac{k_{a_j}}{\|a_j^\vee\|} \int_{\overline{A} \cap V_j} \mathcal{B}f(v)\overline{\mathcal{B}g(v)} \, dv$$

*and by $h_k$ the form $h_0 + \delta_k$ with domain $W^{1,2}$ ($= Q_0$ by Proposition 6).*

By continuity of $\mathcal{B}$ there is a $c > 0$ such that

$$\delta_1(f,f) \le c\left((f,f)_{W^{1,2}} + (f,f)_{L^2(A)}\right), \quad \forall f \in W^{1,2} \tag{19}$$

For root system of type **A** our $\delta_k$ differs from Dorlas' (see [4, (2.7)]) in the sense that his $\delta_k$ has no integration over an affine wall, which is explained by the fact that we work in the center-of-mass coordinates.

We write $k \le k'$ if $k_\alpha \le k'_\alpha$ for all $\alpha \in R$. Following Dorlas [4] we are now in position to make sense of the formal Hamiltonian (12) as a positive self-adjoint operator.

**Proposition 9.** *(i) There is a unique positive self-adjoint operator $H_k$ on $L^2(A)$ with form $(h_k, W^{1,2})$. Furthermore $D_0$ is a form core for $h_k$.*
*(ii) The $H_k$ are non-decreasing in $k$, i.e. $H_k \le H_{k'}$ if $k \le k'$.*
*(iii) $k \mapsto H_k$ is right-continuous in $k \in \mathcal{K} \cup \{0\}$ in the strong-resolvent sense.*
*(iv) Let $\mu \in \mathcal{P}^{++}$. Then $\phi^k_{\hat{\mu}_k} \in D(H_k)$ and $H_k\phi^k_{\hat{\mu}_k} = \|\hat{\mu}_k\|^2\phi^k_{\hat{\mu}_k}$*

*Proof.* The uniqueness of $H_k$ follows because a self-adjoint operator is uniquely defined by its form by the first representation theorem [13, Theorem VI.2.1]. For the existence we use induction on $k_{\max} = \max\{k_\alpha | \alpha \in R\}$, starting from $k \equiv 0$ using Proposition 5. Assume therefore that $H_{k^0}$ is defined for some $k^0 \ge 0$ with the properties mentioned in (i). Let $\beta$ be the form $\delta_{k-k^0}$. Thus $h_k = h_{k^0} + \beta$ is a perturbation of $h_{k^0}$ by $\beta$. Then it follows immediately from the non-decreasing of the forms $h_k$ and (19) that $\beta(f,f) \le c(k-k^0)_{\max}(h_{k^0}(f,f) + (f,f))$ for all $f \in W^{1,2}$. The KLMN theorem [18, Theorem X.17], [13, Theorem VI.3.9] now yield the existence of the operators $H_k$ for those $k \in \mathcal{K}$ satisfying $k^0 \le k < k^0 + 1/c$. Whence

by induction they are defined for all $k \in \mathcal{K}$, yielding (i). Statement (ii) follows from the non-decreasing of the $h_k$. Statement (iii) follows from [13, Theorem VI.2.1(iii)] and $h_k(\phi^k_{\hat{\mu}_k}, \psi) = (-\Delta\phi^k_{\hat{\mu}_k}, \psi)_{L^2(A)}$ for all $\psi \in D_0$, which itself follows from the definition of $h_k$, symmetry of $h_k$ and (16). As for (iv), let $k_1 \geq k_2 \geq \cdots \geq k^0$ be a sequence in $\mathcal{K} \cup \{0\}$ s.t. $k_j \to k^0$. Then $h_{k_1} \geq h_{k_2} \geq \cdots \geq 0$ as forms on $W^{1,2}$. Now $h_{k^0}(\phi, \phi) = \inf_{k>k^0} h_k(\phi, \phi)$, $\forall \phi \in D_0$, yielding $h_{k^0} = \lim_{j\to\infty} h_{k_j} = \inf_{j\in\mathbb{N}} h_{k_j}$ by the first part of [17, Theorem S16] and (iv) follows by the second part of [17, Theorem S16]. $\qquad\square$

*Proof of Theorem 1.* Denote the space of $\mathbb{R}$-valued polynomials on $V$ over $\mathbb{R}$ by $P(V)$. To any $p \in P(V)$ one can associate in a natural way a constant coefficient differential operator $\partial_p$. Since $\partial_p\phi^k_\lambda = p(\lambda)\phi^k_\lambda$ for $p \in P(V)^W$ (see (13)) and $P(V)^W$ separates the orbits $V/W$ it follows that the $\phi^k_{\hat{\mu}_k}$, $\mu \in \mathcal{P}^{++}$ must be linearly independent. The theorem now follows from Theorem 2 applied to $(H_{tk})_{t\in[0,\infty)}$, $(\phi^{tk}_{\hat{\mu}_{tk}})_{\mu\in\mathcal{P}^{++}}$ and using Propositions 3, 5 and 9. $\qquad\square$

## References

[1] M. D. Bustamante, J. F. van Diejen and A. C. de la Maza, *Norm formulae for the Bethe Ansatz on root systems of small rank*, Journal of Physics A: Mathematical and Theoretical **41** (2008), no. 2, 025202–(13pp).

[2] N. Crampé and C. A. S. Young, *Bethe equations for a $\mathfrak{g}_2$ model*, Journal of Physics. A. Mathematical and Theoretical **39** (2006), no. 7, L135–L143.

[3] J. F. van Diejen, *Diagonalization of an integrable discretization of the repulsive delta Bose gas on the circle*, Comm. Math. Phys. **267** (2006), 451–476.

[4] T. C. Dorlas, *Orthogonality and completeness of the Bethe ansatz eigenstates of the nonlinear Schroedinger model*, Comm. Math. Phys. **154** (1993), no. 2, 347–376.

[5] E. Emsiz, E. M. Opdam and J. V. Stokman, *Periodic integrable systems with delta-potentials*, Comm. Math. Phys. **264** (2006), 191–225.

[6] E. Emsiz, E. M. Opdam and J. V. Stokman, *Trigonometric Cherednik algebra at critical level and quantum many-body problems*, Sel. math., New ser. **14** (2009), 571–605.

[7] M. Gaudin, *Boundary energy of a bose gas in one dimension*, Phys. Rev. A **4** (1971), 386–394.

[8] M. Gaudin, *La fonction d'onde de Bethe*, Masson, Paris, 1983.

[9] E. Gutkin, *Integrable systems with delta-potential*, Duke Math. J. **49** (1982), no. 1, 1–21.

[10] E. Gutkin and B. Sutherland, *Completely integrable systems and groups generated by reflections*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 12, 6057–6059.

[11] G. J. Heckman and H. Schlichtkrull, *Harmonic analysis and special functions on symmetric spaces*, Perspectives in Mathematics, vol. 16, Academic Press Inc., San Diego, CA, 1994.

[12] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[13] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.

[14] V. E. Korepin, *Calculation of norms of Bethe wave functions*, Comm. Math. Phys. **86** (1982), no. 3, 391–418.

[15] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge, 1993.

[16] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas, I. The general solution and the ground state*, Phys. Rev. **130** (1963), 1605–1624.

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1980, Revised and Enlarged Edition.

[18] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, 1975.

[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, 1978.

[20] J. Wloka, *Partielle Differentialgleichungen*, B. G. Teubner, Stuttgart, 1982.

[21] C. N. Yang and C. P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. **10** (1969), 1115–1122.

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile

*E-mail address*: eemsiz@inst-mat.utalca.cl