The Cauchy problem for a class of two-dimensional nonlocal nonlinear wave equations governing anti-plane shear motions in elastic materials

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Abstract. This paper is concerned with the analysis of the Cauchy problem of a general class of two-dimensional nonlinear nonlocal wave equations governing anti-plane shear motions in nonlocal elasticity. The nonlocal nature of the problem is reflected by a convolution integral in the space variables. The Fourier transform of the convolution kernel is nonnegative and satisfies a certain growth condition at infinity. For initial data in L^2 Sobolev spaces, conditions for global existence or finite time blow-up of the solutions of the Cauchy problem are established.

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1. Introduction

In the present paper we consider the initial value problem

$$w_{tt}=\left(\beta*\frac{\partial F}{\partial w_x}\right)_x+\left(\beta*\frac{\partial F}{\partial w_y}\right)_y, \quad (x,y)\in\mathbb{R}^2, \quad t>0, \tag{1.1}$$

$$w(x,y,0)=\varphi(x,y), \quad w_t(x,y,0)=\psi(x,y), \tag{1.2}$$

where (1.1) models anti-plane shear motions in nonlinear nonlocal elasticity. In (1.1)-(1.2), $w=w(x,y,t)$, F is a nonlinear function of $|\nabla w|^2\equiv(w_x^2+w_y^2)$ with $F(0)=0$, the subscripts denote partial derivatives and

$$(\beta*u)(x,y)=\int_{\mathbb{R}^2}\beta(x-x',y-y')u(x',y')dx'dy'$$

denotes convolution of β and u. The kernel $\beta(x,y)$ is assumed to be an integrable function whose Fourier transform, $\widehat{\beta}(\xi_1,\xi_2)$, satisfies

$$0\leq\widehat{\beta}(\xi)\leq C(1+|\xi|^2)^{-r/2}, \quad \text{for all } \xi=(\xi_1,\xi_2), \tag{1.3}$$

where C is a positive constant and $r\geq 2$. The aim of this paper is to establish the well-posedness of the initial value problem (1.1)-(1.2), as well as the global existence and blow-up of solutions for a wide class of the kernel functions $\beta(x,y)$. The number r in (1.3) is closely related to the smoothness of β and, consequently, as r gets larger the regularizing effect of the nonlocal behavior increases. This situation is clearly observed through a comparison of Theorem 3.7 and Theorem 3.8.

Although the model requires the nonlinearity to be of the form $F(w_x^2+w_y^2)$, it is possible to extend our results to $F(w_x,w_y)$-type nonlinearities. Similarly, in the model the kernel β is a function of the modulus $|(x,y)|$, but we do not require this restriction on β in our work. Basically, the approach presented here extends the techniques used for the one-dimensional nonlinear nonlocal Boussinesq-type wave equations in the previous studies [1, 2, 3] to the two-dimensional wave equation given by (1.1). It is worthwhile observing that when β is taken as the Dirac measure in (1.1), one recovers the quasilinear wave equation for anti-plane shear motions of the conventional theory of elasticity.

The plan of the paper is as follows: In Section 2 we give a brief formulation of the anti-plane shearing problem of nonlocal elasticity. In Section 3 we present a local existence theory for solutions of the Cauchy problem (1.1)-(1.2) for given initial data in suitable Sobolev spaces. In Section 4 we prove global existence of solutions of (1.1)-(1.2) assuming some positivity condition on the nonlinear function F together with enough smoothness on the initial data. Finally, in Section 5 we discuss finite time blow-up of solutions.

Throughout the paper, $\widehat{u}(\xi)=\mathcal{F}(u)(\xi)=\int_{\mathbb{R}^2}e^{-iz\cdot\xi}u(z)dz$ and $\mathcal{F}^{-1}(\widehat{u})(z)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{iz\cdot\xi}\widehat{u}(\xi)d\xi$ denote the Fourier transform and inverse Fourier transform, respectively, where $z=(x,y)$, $\xi=(\xi_1,\xi_2)$, $dz=dxdy$, and $d\xi=d\xi_1d\xi_2$. Furthermore, $H^s(\mathbb{R}^2)$ denotes the L^2 Sobolev space on $\mathbb{R}^2$. For the H^s norm we use the Fourier transform representation $\|u\|_s^2=\int_{\mathbb{R}^2}(1+|\xi|^2)^s|\widehat{u}(\xi)|^2\mathrm{d}\xi$. Also, $\|u\|_\infty$ and $\|u\|$

indicate the L^∞ and L^2 norms, respectively, and $\langle u, v\rangle$ refers to the inner product of u and v in $L^2(\mathbb{R}^2)$.

2. The Model

In this section we discuss how equation (1.1) can be derived to describe the propagation of a finite amplitude transverse wave in a nonlocally elastic medium. Before stating our derivation, we need to introduce the concept of nonlocal elasticity. One of the major drawbacks in the conventional theory of elasticity is that it does not include any intrinsic length scale and consequently does not take into account the long range forces that become increasingly important at small scales. As a result, the conventional theory of elasticity is incapable of predicting, for instance, (i) the dispersive nature of harmonic waves in crystal lattices and (ii) the boundedness of the stress field near the tip of a crack. In order to overcome such deficiencies various generalizations of the conventional theory of elasticity have been proposed. One such generalization is the theory of nonlocal elasticity which has been developed by Kröner [4], Eringen and Edelen [5], Kunin [6], Rogula [7], Eringen [8, 9] over the last several decades (For more recent studies on the subject of generalized theories of elasticity, see, for instance, [10, 11, 12, 13, 14, 15] and references therein). What distinguishes the theory of nonlocal elasticity from the conventional theory of elasticity is that the stress at a point depends on the strain field at every point in the body. Although there has been a considerable amount of research done on small scale effects within the context of the theory of nonlocal elasticity, they are mostly restricted to linear models. Recently, in [1, 2, 3] various Cauchy problems based on a one-dimensional nonlinear model of nonlocal elasticity have been studied. Here, we show how the approach in those studies is extended to the dynamic anti-plane shearing problem of nonlinear nonlocal elasticity.

Consider a homogeneous nonlocally elastic medium. Identify a material point $\mathbf{X}$ of the medium by its rectangular Cartesian coordinates in a reference configuration: $\mathbf{X} = (X_1, X_2, X_3)$. We assume that the reference configuration is unstressed. Let $\mathbf{x}(\mathbf{X}, t) = (x_1(\mathbf{X}, t), x_2(\mathbf{X}, t), x_3(\mathbf{X}, t))$ denote the position of the same point at time t. Then the displacement and the deformation gradient are given by $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$ and $\mathbf{A}(\mathbf{X}, t) = \text{Grad } \mathbf{x}(\mathbf{X}, t)$, respectively. We suppose that a (local) strain energy density function $F(\mathbf{A})$ per unit volume of the undeformed reference configuration exists, i.e., the material is (locally) hyperelastic and that it sustains a nontrivial anti-plane shear motion. In the conventional theory of elasticity, a constitutive equation of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{A}) \equiv \partial F(\mathbf{A})/\partial \mathbf{A}$ holds for a hyperelastic material (see equation (4.3.7) of [16]), where $\boldsymbol{\sigma}$ is the nominal stress tensor (note that some authors use its transpose referred to as the first Piola-Kirchhoff stress tensor). In the theory of nonlocal elasticity the (nonlocal) stress tensor $\mathbf{S}$ is related to the (local) stress tensor $\boldsymbol{\sigma}$ through the constitutive relation $\mathbf{S} = \mathbf{S}(\mathbf{X}, t) \equiv \int \beta(|\mathbf{X} - \mathbf{Y}|)\boldsymbol{\sigma}(\mathbf{A}(\mathbf{Y}, t))d\mathbf{Y}$ where $\beta(|\mathbf{X} - \mathbf{Y}|)$ is a kernel function that weights the contribution of the local stresses to the nonlocal stresses. In the absence of body forces, the (Lagrangean) equation of motion (see equation (3.4.4) of [16]) is given by $\rho_0 \ddot{\mathbf{x}} = \text{Div}\, \mathbf{S}$ where ρ_0 is the mass density of the medium and a superposed dot indicates the material time derivative. The only difference between the equations of the conventional theory of elasticity and those of the nonlocal model presented here is due to the constitutive equations.

Now we consider an anti-plane shear motion of the form

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1, X_2, t) \tag{2.1}$$

for a nonlocally elastic material, where the out-of-plane displacement w is the only non-zero component of displacement, i.e. $u_1 = u_2 \equiv 0$ and $u_3 \equiv w(X_1, X_2, t)$. We henceforth replace the arguments X_1 and X_2 of the displacement w with x and y, respectively, and denote partial differentiations with subscript letters. A lengthy computation shows that for isotropic materials the strain energy density function F is a function of $w_x^2 + w_y^2$ alone: $F = F(w_x^2 + w_y^2)$. Furthermore, the equation of motion reduces to the scalar partial differential equation $\rho_0 w_{tt} = (\beta * \sigma_{13})_x + (\beta * \sigma_{23})_y$ where σ_{13} and σ_{23} are the (local) shear stresses arising due to the anti-plane shear motion and they are given by $\sigma_{13} = \partial F/\partial w_x$ and $\sigma_{23} = \partial F/\partial w_y$. The computations are identical to those in the conventional formulation of nonlinear elasticity, provided we replace the nonlocal stress tensor with the local stress of conventional theory of elasticity [17]. The nonlocal behavior is represented by the convolution integral. Thus, without loss of generality, if we make a suitable non-dimensionalization of the equation of motion (see [2] for the non-dimensionalization in one-dimensional case) and use the same symbols to avoid a proliferation of notation, or simply take the mass density to be 1, we get (1.1). Equation (1.1) is consistent with that of the conventional formulation of nonlinear elasticity. In other words, when β is taken as the Dirac measure to eliminate the nonlocal effect, (1.1) reduces to the quasilinear wave equation governing anti-plane shear motions in the conventional theory of nonlinear elasticity (see for instance equation (7.10) of [17] or equation (2.2) of [18]). A list of the most commonly used one-dimensional kernel functions that satisfy the one-dimensional version of the condition given in (1.3) is presented in [2]. We now present three examples of two-dimensional kernel functions used in the literature.

(i) The Gaussian kernel [19]: $\beta(x, y) = (2\pi)^{-1} e^{-(x^2+y^2)/2}$. We have $\widehat{\beta}(\xi_1, \xi_2) = e^{-(\xi_1^2+\xi_2^2)/2}$. This is a highly regularizing kernel as can be observed by the fact that we can take any r in (1.3).

(ii) The modified Bessel function kernel [19]: $\beta(x, y) = (2\pi)^{-1} K_0(\sqrt{x^2 + y^2})$ where K_0 is the modified Bessel function of the second kind of order zero. Since $\widehat{\beta}(\xi_1, \xi_2) = (1 + \xi_1^2 + \xi_2^2)^{-1}$, for this special case we have $r = 2$ in (1.3). Note that β is Green's function for the operator $(1 - \Delta)$ where Δ denotes the two-dimensional Laplacian. In this case (1.1) becomes

$$w_{tt} - \Delta w_{tt} = \left(\frac{\partial F}{\partial w_x}\right)_x + \left(\frac{\partial F}{\partial w_y}\right)_y .$$

Letting $F(s) = \frac{1}{2}s + G(s)$ we obtain the more familiar form

$$w_{tt} - \Delta w - \Delta w_{tt} = \left(\frac{\partial G}{\partial w_x}\right)_x + \left(\frac{\partial G}{\partial w_y}\right)_y .$$

(iii) The bi-Helmholtz type kernel [20]:

$$\beta(x,y) = \frac{1}{2\pi(c_1^2 - c_2^2)}[K_0(\sqrt{x^2+y^2}/c_1) - K_0(\sqrt{x^2+y^2}/c_2)]$$

where c_1 and c_2 are real and positive constants. Since $\widehat{\beta}(\xi_1,\xi_2) = [1+\gamma_1(\xi_1^2+\xi_2^2)+\gamma_2(\xi_1^2+\xi_2^2)^2]^{-1}$ where $\gamma_1 = c_1^2 + c_2^2$ and $\gamma_2 = c_1^2 c_2^2$ we have $r = 4$. As above, β is Green's function for the operator $(1 - \gamma_1\Delta + \gamma_2\Delta^2)$. Then (1.1) becomes

$$w_{tt} - \Delta w - \gamma_1 \Delta w_{tt} + \gamma_2 \Delta^2 w_{tt} = \left(\frac{\partial G}{\partial w_x}\right)_x + \left(\frac{\partial G}{\partial w_y}\right)_y .$$

In the remainder of this paper we discuss the question of well-posedness of the Cauchy problem (1.1)-(1.2).

3. Local Existence and Uniqueness of Solutions

In the present section, we prove existence and uniqueness of solutions over a small time interval. Local well-posedness is established by converting the initial value problem (1.1)-(1.2) into a system of Banach space-valued ordinary differential equations. Thus (1.1)-(1.2) is equivalent to the system

$$w_t = v, \quad w(0) = \varphi, \tag{3.1}$$

$$v_t = Kw, \quad v(0) = \psi \tag{3.2}$$

where the operator K is defined as

$$Kw = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y . \tag{3.3}$$

The Banach space X^s will be defined as

$$X^s = \{w \in H^s(\mathbb{R}^2);\ w_x, w_y \in L^\infty(\mathbb{R}^2)\},$$

endowed with the norm

$$\|w\|_{s,\infty} = \|w\|_s + \|w_x\|_\infty + \|w_y\|_\infty . \tag{3.4}$$

The following two lemmas [21] are useful in the proof.

Lemma 3.1 *(Sobolev Embedding Theorem) If* $s > \dfrac{n}{2} + k$ *, then* $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. In particular when* $n = 2$, $\|\ |\nabla u|\ \|_\infty \le C\|u\|_s$ *for* $s > 2$.

Lemma 3.2 *Let* $s \ge 0$ *and let* $u_1, u_2 \in H^s \cap L^\infty$*. Then* $u_1 u_2 \in H^s$ *and*

$$\|u_1 u_2\|_s \le C(\|u_1\|_s \|u_2\|_\infty + \|u_1\|_\infty \|u_2\|_s).$$

The two lemmas below [22, 23] will be used to control the nonlinear terms.

Lemma 3.3 *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$ with $f(0) = 0$. Then for any $u \in H^s \cap L^\infty$, we have $f(u) \in H^s \cap L^\infty$. Moreover there is some constant $A(M)$ depending on M such that for all $u \in H^s \cap L^\infty$ with $\|u\|_\infty \leq M$*

$$\|f(u)\|_s \leq A(M)\|u\|_s \ .$$

Lemma 3.4 *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $u_1, u_2 \in H^s \cap L^\infty$ with $\|u_1\|_\infty \leq M$, $\|u_2\|_\infty \leq M$ and $\|u_1\|_s \leq M$, $\|u_2\|_s \leq M$ we have*

$$\|f(u_1) - f(u_2)\|_s \leq B(M)\|u_1 - u_2\|_s \ .$$

In our case the nonlinearities are of the form

$$\frac{\partial F}{\partial w_x}(|\nabla w|^2) = 2w_x F'(|\nabla w|^2), \qquad \frac{\partial F}{\partial w_y}(|\nabla w|^2) = 2w_y F'(|\nabla w|^2)$$

where F' denotes the derivative of F. It follows from repeated applications of Lemma 3.2 that for the above terms Lemmas 3.3 and 3.4 take the following forms:

Lemma 3.5 *Let $s \geq 1$, $F \in C^{[s]+1}(\mathbb{R})$. Then for any $w \in X^s$, we have*

$$\frac{\partial F}{\partial w_x}(|\nabla w|^2) \in H^{s-1}, \qquad \frac{\partial F}{\partial w_y}(|\nabla w|^2) \in H^{s-1}.$$

Moreover there is some constant $A(M)$ depending on M such that for all $w \in X^s$ with $\|\ |\nabla w|\ \|_\infty \leq M$

$$\|\frac{\partial F}{\partial w_x}(|\nabla w|^2)\|_{s-1} \leq A(M)\|w\|_s,$$
$$\|\frac{\partial F}{\partial w_y}(|\nabla w|^2)\|_{s-1} \leq A(M)\|w\|_s \ .$$

Lemma 3.6 *Let $s \geq 1$, $F \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $w_1, w_2 \in X^s$ with $\|w_1\|_{s,\infty} \leq M$, $\|w_2\|_{s,\infty} \leq M$ we have*

$$\|\frac{\partial F}{\partial w_x}(|\nabla w_1|^2) - \frac{\partial F}{\partial w_x}(|\nabla w_2|^2)\|_{s-1} \leq B(M)\|w_1 - w_2\|_s,$$
$$\|\frac{\partial F}{\partial w_y}(|\nabla w_1|^2) - \frac{\partial F}{\partial w_y}(|\nabla w_2|^2)\|_{s-1} \leq B(M)\|w_1 - w_2\|_s \ .$$

When $s > 2$ we have the following local well posedness result.

Theorem 3.7 *Suppose $s > 2$, $r \geq 2$ and $\varphi, \psi \in H^s(\mathbb{R}^2)$. Then there is some $T > 0$ such that the Cauchy problem (1.1)-(1.2) is well posed with solution $w(x, y, t)$ in $C^2([0, T], H^s(\mathbb{R}^2))$.*

Proof. Let $w \in H^s(\mathbb{R}^2)$. For $s > 2$, by the Sobolev Embedding Theorem we have $|\nabla w| \in L^\infty(\mathbb{R}^2)$. Thus $X^s = H^s(\mathbb{R}^2)$ and the norm $\|w\|_{s,\infty}$ can be replaced by the equivalent H^s norm $\|w\|_s$. Since (1.1)-(1.2) is equivalent to (3.1)-(3.2), we will use the standard well posedness result for systems of ordinary differential equations [24].

Obviously, all we need is to show that the operator K of (3.3) is locally Lipschitz on X^s. We first show that K maps X^s into X^s. We estimate the convolution as

$$\|\beta * u\|_s = \|(1+\xi^2)^{s/2}\widehat{\beta}(\xi)\widehat{u}(\xi)\| \le C\|(1+\xi^2)^{(s-r)/2}\widehat{u}(\xi)\| = C\|u\|_{s-r},$$

where we have used inequality (1.3). By Lemma 3.5 for $\|\,|\nabla w|\,\|_\infty \le M$

$$\begin{aligned}\|\left(\beta * \frac{\partial F}{\partial w_x}\right)_x\|_s &\le \|\beta * \frac{\partial F}{\partial w_x}\|_{s+1} \le C\|\frac{\partial F}{\partial w_x}\|_{s+1-r} \\ &\le CA(M)\|w\|_{s+2-r} \le CA(M)\|w\|_s\end{aligned}$$

where we have used $r \ge 2$. The same holds for the term $\left(\beta * \frac{\partial F}{\partial w_y}\right)_y$ and

$$\|Kw\|_s \le CA(M)\|w\|_{s+2-r} \le CA(M)\|w\|_s. \tag{3.5}$$

Similarly, for $w_1, w_2 \in X^s$ with $\|w_1\|_s \le M$ and $\|w_2\|_s \le M$, by Lemma 3.6

$$\begin{aligned}\|\left(\beta * \frac{\partial F}{\partial w_x}(|\nabla w_1|^2)\right)_x - \left(\beta * \frac{\partial F}{\partial w_x}(|\nabla w_2|^2)\right)_x\|_s &\le CB(M)\|w_1 - w_2\|_{s+2-r} \\ &\le CB(M)\|w_1 - w_2\|_s.\end{aligned}$$

As above, the same holds for the term $\left(\beta * \frac{\partial F}{\partial w_y}\right)_y$. So, K is locally Lipschitz on X^s and thus the local well posedness of the Cauchy problem is established. ■

When $r > 3$ in (1.3), the extra regularizing effect of β allows us to improve the result in Theorem 3.7 to the case of $s \ge 1$.

Theorem 3.8 *Suppose* $s \ge 1$, $r > 3$, *and* $\varphi, \psi \in X^s$. *Then there is some* $T > 0$ *such that the Cauchy problem (1.1)-(1.2) is well posed with solution* $w(x, y, t)$ *in* $C^2([0, T], X^s)$.

Proof. Similar to the proof of Theorem 3.7 it suffices to show that the map K given in (3.3) is locally Lipschitz on X^s. Recall that $\|Kw\|_{s,\infty} = \|Kw\|_s + \|(Kw)_x\|_\infty + \|(Kw)_y\|_\infty$. The term $\|Kw\|_s$ can be estimated by $\|w\|_s$ as above. For $\epsilon = r - 3 > 0$ we have

$$\begin{aligned}\|(Kw)_x\|_\infty &\le C\|(Kw)_x\|_{1+\epsilon} \le C\|Kw\|_{2+\epsilon} \le C\|Kw\|_{s+1+\epsilon} \\ &\le CA(M)\|w\|_{s+3+\epsilon-r} = CA(M)\|w\|_s.\end{aligned} \tag{3.6}$$

where we have used (3.5) and the Sobolev Embedding Theorem. The same holds for $(Kw)_y$ and a similar estimate as in the proof of Theorem 3.7 shows that K is locally Lipschitz on X^s. ■

The solution of (1.1)-(1.2) can be extended to a maximal interval $[0, T_{\max})$ where finite $T_{\max}$ is characterized by the blow up condition

$$\limsup_{t \to T_{\max}^-} (\|w(t)\|_{s,\infty} + \|w_t(t)\|_{s,\infty}) = \infty.$$

Obviously $T_{\max} = \infty$, i.e. there is a global solution if and only if for any $T < \infty$

$$\limsup_{t \to T^-} (\|w(t)\|_{s,\infty} + \|w_t(t)\|_{s,\infty}) < \infty.$$

The lemma below characterizes the type of blow-up; namely blow-up occurs in the L^∞-norm of $|\nabla w|$.

Lemma 3.9 *Suppose that the conditions of Theorem 3.7 or Theorem 3.8 hold. Then there is a global solution of the Cauchy problem (1.1)-(1.2) if and only if for any* $T > 0$

$$\limsup_{t\to T^-}(\|w_x(t)\|_\infty + \|w_y(t)\|_\infty) < \infty.$$

Proof. Since

$$\|w_x(t)\|_\infty + \|w_y(t)\|_\infty \leq \|w(t)\|_{s,\infty},$$

it suffices to prove that if the solution exists for $t \in [0, T)$ and $\|w_x(t)\|_\infty + \|w_y(t)\|_\infty \leq M$ for all $0 \leq t < T$ then both $\|w(t)\|_{s,\infty}$ and $\|w_t(t)\|_{s,\infty}$ stay bounded. Integrating equation (1.1) twice and calculating the resulting double integral as an iterated integral, we obtain

$$w(t) \;= \varphi + t\psi + \int_0^t (t-\tau)(Kw)(\tau)d\tau, \tag{3.7}$$

$$w_t(t) = \psi + \int_0^t (Kw)(\tau)d\tau. \tag{3.8}$$

But, by (3.5), $\|(Kw)(\tau)\|_s \leq CA(M)\|w(\tau)\|_{s+2-r} \leq CA(M)\|w(\tau)\|_s$ where the constant $A(M)$ depends only on M. Hence

$$\|w(t)\|_s + \|w_t(t)\|_s \leq \|\varphi\|_s + (1+T)\|\psi\|_s + (1+T)CA(M)\int_0^t \|w(\tau)\|_s d\tau,$$

and Gronwall's Lemma gives

$$\|w(t)\|_s + \|w_t(t)\|_s \leq (\|\varphi\|_s + (1+T)\|\psi\|_s)e^{(1+T)CA(M)T} \tag{3.9}$$

for all $t \in [0, T)$. We now estimate $\|w_{tx}(t)\|_\infty$. The estimate for $\|w_{ty}(t)\|_\infty$ follows similarly. In the case of Theorem 3.7 (where $s > 2$), by the Sobolev Embedding Theorem,

$$\|w_{tx}(t)\|_\infty \leq C\|w_t(t)\|_s$$

so that (3.9) applies. In the case of Theorem 3.8 (where $r > 3$), from (3.8) and (3.6)

$$\|w_{tx}(t)\|_\infty \leq \|\psi_x\|_\infty + T\|(Kw)_x(t)\|_\infty \leq \|\psi_x\|_\infty + CA(M)T\|w(t)\|_s$$

and again (3.9) applies. ■

4. Conservation of Energy and Global Existence

In the present section we will prove that locally well defined solutions can be extended to the entire time.

In the study of global existence of solutions the conservation of energy plays a key role. First, time invariance of the energy functional will be shown. To this end, we define the linear operator R as $R^p u = \mathcal{F}^{-1}\left((\widehat{\beta}(\xi))^{-\frac{p}{2}}\widehat{u}(\xi)\right)$ where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\widehat{\beta}(\xi)$ is defined in (1.3). Then $R^{-2}u = \beta * u$. Multiplying by R^2, (1.1) can be rewritten as

$$R^2 w_{tt} = \left(\frac{\partial F}{\partial w_x}\right)_x + \left(\frac{\partial F}{\partial w_y}\right)_y. \tag{4.1}$$

Here we have used the fact that convolution commutes with derivatives in the distribution sense, i.e. $(\beta * u)_x = \beta * u_x$.

Lemma 4.1 *Suppose that the conditions of Theorem 3.7 or Theorem 3.8 hold and the solution of the Cauchy problem (1.1)-(1.2) exists in $C^2([0,T), X^s)$. If $R\psi \in L^2(\mathbb{R}^2)$, then $Rw_t(t) \in L^2(\mathbb{R}^2)$ for all $t \in [0,T)$. Moreover, if $R\varphi \in L^2(\mathbb{R}^2)$, then $Rw(t) \in L^2(\mathbb{R}^2)$ for all $t \in [0,T)$.*

Proof. Formally, from (3.8) we have

$$Rw_t(t) = R\psi + \int_0^t (RKw)(\tau)d\tau \ . \tag{4.2}$$

Note that

$$RKw = \left(\alpha * \frac{\partial F}{\partial w_x}\right)_x + \left(\alpha * \frac{\partial F}{\partial w_y}\right)_y ,$$

where $\widehat{\alpha}(\xi) = (\widehat{\beta}(\xi))^{1/2}$. Then similar to the derivation of (3.5) (replacing β by α and hence r by $r/2$) we get

$$\|(RKw)(\tau)\|_{s+\frac{r}{2}-2} \le C\|w(\tau)\|_s.$$

Since either ($s > 2$ and $r \ge 2$) or ($s \ge 1$ and $r > 3$), in both cases we have $s + \frac{r}{2} - 2 > 0$. Thus the right-hand side of (4.2) belongs to $L^2(\mathbb{R}^2)$ and the conclusion follows. The second statement follows similarly from (3.7). ■

Lemma 4.2 *Suppose that the solution of the Cauchy problem (1.1)-(1.2) exists on some interval $[0,T)$. If $R\psi \in L^2(\mathbb{R}^2)$ and the function $F(|\nabla\varphi|^2)$ belongs to $L^1(\mathbb{R}^2)$, then for any $t \in [0,T)$ the energy*

$$E(t) = \frac{1}{2}\|Rw_t(t)\|^2 + \int_{\mathbb{R}^2} F(|\nabla w(t)|^2)dxdy \tag{4.3}$$

is constant in $[0,T)$.

Proof. By Lemma 4.1, $Rw_t(t) \in L^2(\mathbb{R}^2)$. Multiplying (4.1) by w_t, integrating in x and y, and using Parseval's identity we get

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2}(\widehat{\beta}(\xi))^{-1}|\widehat{w}_t(\xi,t)|^2 d\xi + \int_{\mathbb{R}^2} F'(|\nabla w(t)|^2)\frac{\partial}{\partial t}(|\nabla w(t)|^2)dxdy \\ &= \frac{d}{dt}\int_{\mathbb{R}^2} \left(\frac{1}{2}(Rw_t(t))^2 + F(|\nabla w(t)|^2)\right) dxdy \end{aligned}$$

which implies the conservation of energy. ■

The main result of this section is the following theorem.

Theorem 4.3 *Let $s \ge 1$ and $r > 4$. Let $\varphi, \psi \in X^s$, $R\psi \in L^2(\mathbb{R}^2)$ and $F(|\nabla\varphi|^2) \in L^1(\mathbb{R}^2)$. If there is some $k > 0$ so that $F(u) \ge -ku$ for all $u \ge 0$, then the Cauchy problem (1.1)-(1.2) has a global solution in $C^2([0,\infty), X^s)$.*

Proof. By Theorem 3.8 the Cauchy problem is locally well-posed. Assume $w \in C^2([0,T), X^s)$ for some $T > 0$. Since $F(u) \ge -ku$, for all $t \in [0,T)$ we have

$$\begin{aligned} \|Rw_t(t)\|^2 &= 2E(0) - 2\int_{\mathbb{R}^2} F(|\nabla w(t)|^2)dxdy, \\ &\le 2E(0) + 2k\int_{\mathbb{R}^2} |\nabla w(t)|^2 dxdy, \end{aligned} \tag{4.4}$$

where $E(0)$ is the initial energy. On the other hand we have

$$\begin{aligned}\|Rw_t(t)\|^2 &= \int_{\mathbb{R}^2} (\widehat{\beta}(\xi))^{-1}|\widehat{w}_t(\xi,t)|^2 d\xi,\\ &\geq C^{-1}\int_{\mathbb{R}^2}(1+|\xi|^2)^{\frac{r}{2}}|\widehat{w}_t(\xi,t)|^2 d\xi,\\ &= C^{-1}\|w_t(t)\|^2_{\frac{r}{2}},\end{aligned} \tag{4.5}$$

where (1.3) is used. Combining (4.4) and (4.5)

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2_{\frac{r}{2}} &\leq 2\|w(t)\|_{\frac{r}{2}}\|w_t(t)\|_{\frac{r}{2}}\\ &\leq \|w(t)\|^2_{\frac{r}{2}} + \|w_t(t)\|^2_{\frac{r}{2}}\\ &\leq \|w(t)\|^2_{\frac{r}{2}} + C\|Rw_t(t)\|^2\\ &\leq \|w(t)\|^2_{\frac{r}{2}} + 2C(E(0)+k\|w_x(t)\|^2 + k\|w_y(t)\|^2)\\ &\leq 2CE(0) + (1+4Ck)\|w(t)\|^2_{\frac{r}{2}},\end{aligned}$$

where $\|w_x(t)\| \leq \|w(t)\|_1 \leq \|w(t)\|_{\frac{r}{2}}$ and $\|w_y(t)\| \leq \|w(t)\|_1 \leq \|w(t)\|_{\frac{r}{2}}$ are used. Gronwall's lemma implies that $\|w\,(t)\|_{\frac{r}{2}}$ stays bounded in $[0,T)$. As $r>4$ we have $\frac{r}{2}-1>1$ and the Sobolev Embedding Theorem implies

$$\|w_x(t)\|_\infty \leq \|w_x(t)\|_{\frac{r}{2}-1} \leq \|w(t)\|_{\frac{r}{2}}.$$

We conclude that $\|w_x(t)\|_\infty$ and similarly $\|w_y(t)\|_\infty$ also stay bounded in $[0,T)$. By Lemma 3.9, this implies a global solution. ■

5. Blow up

In this section a blow result for (1.1)- (1.2) will be presented. The following lemma [25] will be used to prove blow up of solutions in finite time.

Lemma 5.1 *Suppose that* $\mathcal{H}(t)$, $t \geq 0$, *is a positive, twice differentiable function satisfying* $\mathcal{H}''\mathcal{H} - (1+\nu)(\mathcal{H}')^2 \geq 0$ *where* $\nu > 0$. *If* $\mathcal{H}(0) > 0$ *and* $\mathcal{H}'(0) > 0$, *then* $\mathcal{H}(t) \to \infty$ *as* $t \to t_1$ *for some* $t_1 \leq \mathcal{H}(0)/\nu\mathcal{H}'(0)$.

Theorem 5.2 *Suppose that the solution,* w, *of the Cauchy problem (1.1)-(1.2) exists,* $R\varphi$, $R\psi \in L^2(\mathbb{R}^2)$ *and* $F(|\nabla\varphi|^2) \in L^1(\mathbb{R}^2)$. *If there exists a positive number* ν *such that*

$$uF'(u) \leq (1+2\nu)F(u) \quad \textit{for all } u \geq 0,$$

and

$$E(0) = \frac{1}{2}\|R\psi\|^2 + \int_{\mathbb{R}^2} F(|\nabla\varphi|^2)dxdy < 0,$$

then the solution, w, *of the Cauchy problem (1.1)-(1.2) blows up in finite time.*

Proof. We assume that the global solution to (1.1)-(1.2) exists. Then, by Lemma 4.1, $Rw(t),\ Rw_t(t) \in L^2(\mathbb{R}^2)$ for all $t > 0$. Let $\mathcal{H}(t) = \|Rw(t)\|^2 + b(t+t_0)^2$ where b and t_0 are positive constants to be determined later. Then we have

$$\begin{aligned}\mathcal{H}' &= 2\langle Rw_t, Rw\rangle + 2b(t+t_0)\\ \mathcal{H}'' &= 2\|Rw_t\|^2 + 2\langle Rw_{tt}, Rw\rangle + 2b.\end{aligned}$$

Note that $\mathcal{H}'(0) = 2\langle R\varphi, R\psi\rangle + 2bt_0 > 0$ for sufficiently large t_0. Using the inequality $uF'(u) \le (1+2\nu)F(u)$ together with (4.4) and (4.5) we have

$$\begin{aligned}\langle Rw_{tt}, Rw\rangle &= \langle R^2 w_{tt}, w\rangle\\ &= -2\int_{\mathbb{R}^2} |\nabla w|^2 F'(|\nabla w|^2)dxdy\\ &\ge -2(1+2\nu)\int_{\mathbb{R}^2} F(|\nabla w|^2)dxdy\\ &= (1+2\nu)\left(\|Rw_t\|^2 - 2E(0)\right),\end{aligned}$$

so that

$$\mathcal{H}'' \ge 4(1+\nu)\|Rw_t\|^2 - 4(1+2\nu)E(0) + 2b.$$

On the other hand, using $2ab \le a^2 + b^2$ and Cauchy-Schwarz inequalities we have

$$\begin{aligned}(\mathcal{H}')^2 &= 4\left[\langle Rw, Rw_t\rangle + b(t+t_0)\right]^2\\ &\le 4\left(\|Rw\|^2\ \|Rw_t\|^2 + 2b(t+t_0)\|Rw\|\ \|Rw_t\| + b^2(t+t_0)^2\right)\\ &\le 4\left(\|Rw\|^2\ \|Rw_t\|^2 + b\|Rw\|^2 + b\|Rw_t\|^2(t+t_0)^2 + b^2(t+t_0)^2\right).\end{aligned}$$

Thus

$$\begin{aligned}&\mathcal{H}''\mathcal{H} - (1+\nu)(\mathcal{H}')^2\\ &\qquad \ge \left(4(1+\nu)\|Rw_t\|^2 - 4(1+2\nu)E(0) + 2b\right)\left(\|Rw\|^2 + b(t+t_0)^2\right)\\ &\qquad - 4(1+\nu)\left(\|Rw\|^2\ \|Rw_t\|^2 + b\|Rw\|^2 + b\|Rw_t\|^2(t+t_0)^2 + b^2(t+t_0)^2\right)\\ &\qquad = -2(1+2\nu)(b+2E(0))\mathcal{H}.\end{aligned}$$

Now if we choose $b \le -2E(0)$, this gives

$$\mathcal{H}''(t)\mathcal{H}(t) - (1+\nu)\left(\mathcal{H}'(t)\right)^2 \ge 0.$$

According to the Blow-up Lemma 5.1, this implies that $\mathcal{H}(t)$, and thus $\|Rw(t)\|^2$ blows up in finite time. ■

Finally, we conclude with a short discussion on the condition $E(0) < 0$. If $F(u) \ge 0$ for all $u \ge 0$, then by Theorem 4.3, there is a global solution. If $F(u)$ is negative on some interval I, we can choose φ with support in I so that $\int_{\mathbb{R}^2} F(|\nabla\varphi|^2)dxdy < 0$. Hence, when $\psi = 0$ or $R\psi$ is sufficiently small we get $E(0) < 0$. This also shows that blow up may occur even for small initial data.

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References

[1] Duruk N, Erkip A and Erbay H A 2009 A higher-order Boussinesq equation in locally nonlinear theory of one-dimensional nonlocal elasticity *IMA Journal of Applied Mathematics* **74** 97-106

[2] Duruk N, Erbay H A and Erkip A 2010 Global existence and blow-up for a class of nonlocal nonlinear Cauchy problems arising in elasticity *Nonlinearity* **23** 107-118

[3] Duruk N, Erbay H A and Erkip A 2010 Blow-up and global existence for a general class of nonlocal nonlinear coupled wave equations *Journal of Differential Equations* (in press) (DOI information: 10.1016/j.jde.2010.09.002)

[4] Kröner E 1967 Elasticity theory of materials with long range cohesive forces *International Journal of Solids and Structures* **3** 731-742

[5] Eringen A C and Edelen D G B 1972 On nonlocal elasticity *International Journal of Engineering Science* **10** 233-248

[6] Kunin I A 1982 *Elastic Media with Microstructure* vol. I and II (Berlin:Springer)

[7] Rogula D 1982 *Nonlocal Theory of Material Media* (Berlin:Springer)

[8] Eringen A C 1992 Vistas of nonlocal continuum physics *International Journal of Engineering Science* **30** 1551-1565

[9] Eringen A C 2002 *Nonlocal Continuum Field Theories* (New York: Springer)

[10] Silling S A 2000 Reformulation of elasticity theory for discontinuities and long-range forces *Journal of the Mechanics and Physics of Solids* **48** 175-209

[11] Polizzotto C 2001 Nonlocal elasticity and related variational principles *International Journal of Solids and Structures* **38** 7359-7380

[12] Chen Y P, Lee J D and Eskandarian A 2003 Examining the physical foundation of continuum theories from the viewpoint of phonon dispersion relation *International Journal of Engineering Science* **41** 61-83

[13] Arndt M and Griebel M 2005 Derivation of higher order gradient continuum models from atomistic models for crystalline solids *Multiscale Modeling and Simulation* **4** 531-562

[14] Blanc X, Le Bris C and Lions P L 2007 Atomistic to continuum limits for computational materials science *ESAIM-Mathematical Modelling and Numerical Analysis* **41** 391-426

[15] Huang Z X 2006 Formulations of nonlocal continuum mechanics based on a new definition of stress tensor *Acta Mechanica* **187** 11-27

[16] Ogden R W 1997 *Non-Linear Elastic Deformations* (New York:Dover)

[17] Horgan C O 1995 Anti-plane shear deformations in linear and nonlinear solid mechanics *SIAM Review* **37** 53-81

[18] Lott D A, Antman S S and Szymczak W G 2001 The quasilinear wave equation for antiplane shearing of nonlinearly elastic bodies *Journal of Computational Physics* **171** 201-226

[19] Eringen A C 1983 On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves *Journal of Applied Physics* **54** 4703-4710

[20] Lazar M, Maugin G A and Aifantis E C 2006 On a theory of nonlocal elasticity of bi-Helmholtz type and some applications *International Journal of Solids and Structures* **43** 1404-1421

[21] Taylor M E 1996 *Partial Differential Equations I: Basic Theory* (New York: Springer)

[22] Constantin A and Molinet L 2002 The initial value problem for a generalized Boussinesq equation *Differential and Integral Equations* **15** 1061-1072

[23] Wang S and Chen G 2006 Cauchy problem of the generalized double dispersion equation *Nonlinear Analysis-Theory Methods and Applications* **64** 159-173

[24] Ladas G E and Lakshmikantham V 1972 *Differential Equations in Abstract Spaces* (New York: Academic Press)

[25] Kalantarov V K and Ladyzhenskaya O A 1978 The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types *Journal of Soviet Mathematics* **10** 53-70