ORTHOGONALITY RELATIONS FOR MULTIVARIATE KRAWTCHOUK POLYNOMIALS

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ABSTRACT. The orthogonality relations of multivariate Krawtchouk polynomials are discussed. For two variables case, an necessary and sufficient condition of orthogonality are given by Rahman and Grünbaum in [1]. In this note, a simple proof of an necessary and sufficient condition of orthogonality are given for the general case.

Key Words: Multivariate orthogonal polynomial, Hypergeometric function

1. INTRODUCTION

Set

$$X(n,N) = \{\boldsymbol{x} = (x_0, x_1, \cdots, x_{n-1}) \in \mathbb{N}_0^n \mid |\boldsymbol{x}| = N\},$$

where $\mathbb{N}_0$ is the set of nonnegative integers and $|\boldsymbol{x}| = x_0 + x_1 + \cdots + x_{n-1}$. Put the multinomial coefficient $\binom{N}{\boldsymbol{x}} = \binom{N}{x_0,\cdots,x_{n-1}}$ for $\boldsymbol{x} \in X(n,N)$. Let $A = (a_{ij})_{1\le i,j\le n-1}$ be a complex matrix. Fix an $\boldsymbol{x} \in X(n,N)$. We define the functions $\phi_A(\boldsymbol{x};\boldsymbol{m})$ of $\boldsymbol{m} \in X(n,N)$ by the following generating function;

$$\Phi_N(A;\boldsymbol{x}) = \prod_{i=0}^{n-1}(\sum_{i=0}^{n-1} a_{ji}t_j)^{x_i} = \sum_{\boldsymbol{m}\in X(n,N)} \binom{N}{\boldsymbol{m}} \phi_A(\boldsymbol{x};\boldsymbol{m})\boldsymbol{t}^{\boldsymbol{m}},$$

where $\boldsymbol{t}^{\boldsymbol{m}} = t_0^{m_0}t_1^{m_1}\cdots t_{n-1}^{m_{n-1}}$ and $a_{0j} = a_{i0} = 1$ for $0 \le i,j \le n-1$. We know an hypergeometric expression of $\phi_A(\boldsymbol{x};\boldsymbol{m})$;

$$\phi_A(\boldsymbol{x};\boldsymbol{m}) = \sum_{\substack{\sum_{i,j} c_{ij}\le N \\ (c_{ij})\in M_{n-1}(\mathbb{N}_0)}} \frac{\prod_{i=1}^{n-1}(-x_i)_{\sum_{j=1}^{n-1} c_{ij}} \prod_{j=1}^{n-1}(-m_j)_{\sum_{i=1}^{n-1} c_{ij}}}{(-N)_{\sum_{i,j} c_{ij}}} \frac{\prod(1-a_{ij})^{c_{ij}}}{\prod a_{ij}!},$$

where $M_{n-1}(\mathbb{N}_0)$ is the set of square matrices of degree $n-1$ with non negative integer elements. This type of hypergeometric functions are originally defined by Aomoto and Gelfand for general parameters. We are interest in the aspects of discrete orthogonal polynomials of these functions with weights

$$b_n(\boldsymbol{x};N;\boldsymbol{\eta}_{(i)}) = \binom{N}{\boldsymbol{x}} \prod_{j=0}^{n-1} \eta_{ji}^{x_j}$$

for $\boldsymbol{\eta}_{(i)} = (\eta_{0i},\cdots,\eta_{n-1i})$ $(i=1,2)$. In a special case, when $n=2$, they are well known and called the Krawtchouk polynomials. Moreover they appear as the zonal spherical functions of Gelfand pairs of wreath products or character algebras [2, 3]. In the paper

[1], Grünbaum and Rahman discuss and determine the necessary and sufficient conditions of the orthogonality of $\phi_A(\boldsymbol{x};\boldsymbol{m})$'s for $n=3$. They make full use of the techniques of the theory special functions like integral expressions, transformation formulas, and so on. In this paper, we give an simple proof of the Grünbaum and Rahman's condition for general cases.

For A we define an $n\times n$ matrix $A_0=(a_{ij})_{0\le i,j\le n-1}$ by putting $a_{0j}=a_{i0}=1$ ($0\le \forall i,j\le n-1$). In the below, we see that orthogonality relation comes from a orthogonal property of the matrix A_0. Our main state is

Theorem 1.1. *The orthogonality relation*

$$\sum_{\boldsymbol{x}\in X(n,N)} b_n(\boldsymbol{x};N;\boldsymbol{\eta}_{(i)})\phi_A(\boldsymbol{x};\boldsymbol{m})\overline{\phi_A(\boldsymbol{x};\boldsymbol{m'})}=\delta_{\boldsymbol{m},\boldsymbol{m'}}\frac{\boldsymbol{\eta}_{(2)}^{\boldsymbol{m}}}{\binom{N}{\boldsymbol{m}}}\ (\forall \boldsymbol{m},\boldsymbol{m'}\in X(n,N))$$

holds if and only if

$$A_0{}^*D_1A_0=D_2, \tag{1}$$

where $D_i=\mathrm{diag}(\eta_{0i},\eta_{1i},\cdots,\eta_{n-1i})$ *is a diagonal matrix* $(i=1,2)$ *and* $A_0^*=\overline{{}^tA_0}$.

Remark 1.2. [1] *Under the setting above, by putting* $n=3$ *and* $A=\begin{bmatrix}1-u_1 & 1-u_2\\ 1-v_1 & 1-v_2\end{bmatrix}$, *one can recover the formula (1.10) of Grünbaum and Rahman's paper* [1].

Remark 1.3. [2, 3] *In the group theoretical aspect,* A_0 *is suitable for a table of the zonal spherical functions of a Gelfand pair* (G,H) *and* $D^{(i)}$ *consists of cardinalities of each double cosets* ($i=1$) *and dimensions of each irreducible components in* 1_H^G. *Under these setting,* $\phi_A(\boldsymbol{x};\boldsymbol{m})$*'s are the zonal spherical functions of a Gelfand pair* $(G\wr S_N, H\wr S_N)$.

2. Proof of Theorem 1.1

First we assume the condition (1). Let $\mathbf{e}_i=(\delta_{0i},\delta_{1i},\cdots,\delta_{ni})$ be an i-th unit vector $(0\le i\le n-1)$. Put $\mathbf{s}=(s_0,s_1,\cdots,s_{n-1})$ and $\mathbf{t}=(t_0,t_1,\cdots,t_{n-1})$. We compute

$$\prod_{i=0}^{n-1}(s_i\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i}=\prod_{i=0}^{n-1}\left(\sum_{j=0}^{n-1}a_{ij}t_j\right)^{x_i}\prod_{i=0}^{n-1}s_i^{x_i}=\Phi_N(A,x)\prod_{i=0}^{n-1}s_i^{x_i}. \tag{2}$$

By multiplying (2) by $\binom{N}{\boldsymbol{x}}$ and summing over $X(n,N)$, we have

$$\sum_{\boldsymbol{x}\in X(n,N)}\binom{N}{\boldsymbol{x}}\prod_{i=0}^{n-1}(s_i\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i}=(\sum_{i=0}^{n-1}s_i\mathbf{e}_iA_0\ {}^t\mathbf{t})^N=(\mathbf{s}A_0\ {}^t\mathbf{t})^N. \tag{3}$$

We change the variables, say

$$\mathbf{s}=\mathbf{u}A_0^*D_1,$$

where $\mathbf{u} = (u_0, u_1, \cdots, u_{n-1})$. This change of variables is same as $s_i = \eta_{i1}\mathbf{e}_i\overline{A_0}\ {}^t\mathbf{u}$. We substitute $\mathbf{s}$ for (3). Then the right hand side of (3) turns into

$$(\mathbf{s}A\ {}^t\mathbf{t})^N = (\mathbf{u}D_2\ {}^t\mathbf{t})^N = \sum_{\boldsymbol{m}\in X(n,N)} \binom{N}{\boldsymbol{m}} \prod_{i=0}^{n} (\eta_{i2}u_i t_i)^{m_i}. \tag{4}$$

Under this substitution, we consider the the left hand side of (3) and have

$$\begin{aligned}
\sum_{\boldsymbol{x}\in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} (s_i\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i} &= \sum_{\boldsymbol{x}\in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} s_i^{x_i} \prod_{i=0}^{n-1} (\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i} \\
&= \sum_{\boldsymbol{x}\in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} (\eta_{i1}\mathbf{e}_i\overline{A_0}\ {}^t\mathbf{u})^{x_i} \prod_{i=0}^{n-1} (\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i} \\
&= \sum_{\boldsymbol{x}\in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} \eta_{i1}{}^{x_i} \prod_{i=0}^{n-1} (\mathbf{e}_i\overline{A_0}\ {}^t\mathbf{u})^{x_i} \prod_{i=0}^{n-1} (\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i}.
\end{aligned}$$

We expand the last two products of the formula above in terms of $\mathbf{u}^{\boldsymbol{m}}\mathbf{t}^{\boldsymbol{m}'}$'s;

$$\prod_{i=0}^{n-1} (\mathbf{e}_i\overline{A_0}\ {}^t\mathbf{u})^{x_i} \prod_{i=0}^{n-1} (\mathbf{e}_iA_0\ {}^t\mathbf{t})^{x_i} = \sum_{\boldsymbol{m},\boldsymbol{m}'\in X(n,N)} \binom{N}{\boldsymbol{m}} \binom{N}{\boldsymbol{m}'} \phi_A(\boldsymbol{x};\boldsymbol{m})\overline{\phi_A(\boldsymbol{x};\boldsymbol{m}')}\mathbf{u}^{\boldsymbol{m}}\mathbf{t}^{\boldsymbol{m}'}.$$

Now the left side of (3) turns to

$$\sum_{\boldsymbol{m},\boldsymbol{m}'\in X(n,N)} \binom{N}{\boldsymbol{m}} \binom{N}{\boldsymbol{m}'} \left(\sum_{\boldsymbol{x}\in X(n,N)} \prod_{i=0}^{n-1} \eta_{i1}^{x_i} \binom{N}{\boldsymbol{x}} \phi_A(\boldsymbol{x};\boldsymbol{m})\overline{\phi_A(\boldsymbol{x};\boldsymbol{m}')} \right) \mathbf{u}^{\boldsymbol{m}}\mathbf{t}^{\boldsymbol{m}'}. \tag{5}$$

We can conclude, comparing coefficients of $\mathbf{u}^k\mathbf{t}^{k'}$ of (4) and (5), that

$$\sum_{\boldsymbol{x}\in X(n,N)} \prod_{i=0}^{n-1} \eta_{i1}^{x_i} \binom{N}{\boldsymbol{x}} \phi_A(\boldsymbol{x};\boldsymbol{m})\overline{\phi_A(\boldsymbol{x};\boldsymbol{m}')} = \frac{\prod_{i=0}^{n-1} \eta_{i2}^{m_i}}{\binom{N}{\boldsymbol{m}}} \delta_{\boldsymbol{m}\boldsymbol{m}'}. \tag{6}$$

Conversely we assume the formula (6). When we put $N = 1$, then (6) is reduced to

$$\sum_{i=1} \eta_{i1}\phi_A(\boldsymbol{e}_i;\boldsymbol{e}_j)\overline{\phi_A(\boldsymbol{e}_i;\boldsymbol{e}_{j'})} = \eta_{i2}\delta_{jj'}. \tag{7}$$

From definition of $\phi_{\boldsymbol{x}}(\boldsymbol{m})$'s, we have

$$\phi_A(\boldsymbol{e}_i;\boldsymbol{e}_j) = a_{ij}.$$

Therefore (7) is same as

$$A_0{}^*D_1A_0 = D_2.$$

References

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