A NOTE ON TRACE SCALING ACTIONS AND FUNDAMENTAL GROUPS OF C^*-ALGEBRAS

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ABSTRACT. Using Effros-Handelman-Shen theorem and Elliott's classification theorem of AF algebras, we show that there exists a unital simple AF algebra A with unique trace such that $A \otimes \mathbb{K}$ admits no trace scaling action of the fundamental group of A.

1. Introduction

Let M be a factor of type II_1 with a normalized trace τ. Murray and von Neumann introduced the fundamental group $\mathcal{F}(M)$ of M in [13]. They showed that if M is hyperfinite, then $\mathcal{F}(M) = \mathbb{R}_+^\times$. Since then there has been many works on the computation of the fundamental groups. Voiculescu [23] showed that $\mathcal{F}(L(\mathbb{F}_\infty))$ of the group factor of the free group $\mathbb{F}_\infty$ contains the positive rationals and Radulescu proved that $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+^\times$ in [20]. Connes [3] showed that if G is an ICC group with property (T), then $\mathcal{F}(L(G))$ is a countable group. Popa showed that any countable subgroup of $\mathbb{R}_+^\times$ can be realized as the fundamental group of some factor of type II_1 in [17]. Furthermore Popa and Vaes [18] exhibited a large family $\mathcal{S}$ of subgroups of $\mathbb{R}_+^\times$, containing $\mathbb{R}_+^\times$ itself, all of its countable subgroups, as well as uncountable subgroups with any Hausdorff dimension in $(0,1)$, such that for each $G \in \mathcal{S}$ there exist many free ergodic measure preserving actions of $\mathbb{F}_\infty$ for which the associated II_1 factor M has the fundamental group equal to G. In our previous paper [15] (see also [14]), we introduced the fundamental group $\mathcal{F}(A)$ of a simple unital C^*-algebra A with a normalized trace τ based on the computation of Picard groups by Kodaka [10], [11] and [12]. The fundamental group $\mathcal{F}(A)$ is defined as the set of the numbers $\tau \otimes Tr(p)$ for some projection $p \in M_n(A)$ such that $pM_n(A)p$ is isomorphic to A. We computed the fundamental groups of several C^*-algebras and showed that any countable subgroup of $\mathbb{R}_+^\times$ can be realized as the fundamental group of a separable simple unital C^*-algebra with unique trace [16].

The fundamental group of a II_1-factor M is equal to the set of trace-scaling constants for automorphisms of $M \otimes B(\mathcal{H})$. We have a similar fact, that is, the fundamental group of a C^*-algebra A is equal to the set of trace-scaling constants for automorphisms of $A \otimes \mathbb{K}$ [15] (see also [14]). It is of interest to know whether $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$. In the case where M is a factor of type II_1, the existence of a trace scaling (continuous) $\mathbb{R}_+^\times$-action on $M \otimes B(\mathcal{H})$ is equivalent to the existence of a

2000 *Mathematics Subject Classification.* Primary 46L40, Secondary 06F20.
Key words and phrases. Fundamental group; Trace scaling action; Dimension group: Effros-Handelman-Shen theorem.

type III_1 factor having a core isomorphic to $M \otimes B(\mathcal{H})$ by the continuous decomposition of type III_1 factors. (See [22] and [4].) Hence this question is important in the theory of von Neumann algebras. Radulescu showed that $L(\mathbb{F}_\infty) \otimes B(\mathcal{H})$ admits a trace scaling action of $\mathbb{R}_+^\times$ in [21]. Therefore there exists a type III_1 factor having a core isomorphic to $L(\mathbb{F}_\infty) \otimes B(\mathcal{H})$. Popa and Vaes [19] showed that there exists a II_1 factor M such that $\mathcal{F}(M) = \mathbb{R}_+^\times$ and $M \otimes B(\mathcal{H})$ admits no trace scaling (continuous) action of $\mathbb{R}_+^\times$.

In this paper we consider trace scaling actions on certain AF algebras. If A is a UHF algebra, then $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$. Using Effros-Handelman-Shen theorem and Elliott's classification theorem of AF algebra, we show that there exists a unital simple AF algebra A with unique trace such that $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$. Note that there exist remarkable works of the classification of trace scaling automorphisms in [1], [7] and [8]. But we do not consider the classification of trace scaling actions in this paper.

2. Examples

We recall some definitions in [15]. Let A be a unital simple C^*-algebra with a unique normalized trace τ and Tr the usual unnormalized trace on $M_n(\mathbb{C})$. Put

$$\mathcal{F}(A) := \{\tau \otimes Tr(p) \in \mathbb{R}_+^\times \mid p \text{ is a projection in } M_n(A) \text{ such that } pM_n(A)p \cong A\}.$$

Then $\mathcal{F}(A)$ is a multiplicative subgroup of $\mathbb{R}_+^\times$ by Theorem 3.1. in [15]. For an additive subgroup E of $\mathbb{R}$ containing 1, we define the positive inner multiplier group $IM_+(E)$ of E by

$$IM_+(E) = \{t \in \mathbb{R}_+^\times \mid t \in E, t^{-1} \in E, \text{ and } tE = E\}.$$

Then we have $\mathcal{F}(A) \subset IM_+(\tau_*(K_0(A)))$ by Proposition 3.7 in [15]. This obstruction enables us to compute fundamental groups easily. For $x \in (A \otimes \mathbb{K})_+$, set $\hat{\tau}(x) = \sup\{\tau \otimes Tr(y) : y \in \cup_n M_n(A), y \leq x\}$. Define $\mathcal{M}_\tau^+ = \{x \geq 0 : \hat{\tau}(x) < \infty\}$ and $\mathcal{M}_\tau = \mathrm{span}\mathcal{M}_\tau^+$. Then $\hat{\tau}$ is a densely defined (with the domain $\mathcal{M}_\tau$) lower semi-continuous trace on $A \otimes \mathbb{K}$. Since the normalize trace on A is unique, the lower semi-continuous densely defined trace on $A \otimes \mathbb{K}$ is unique up to constant multiple. We define the set of trace-scaling constants for automorphisms:

$$\mathfrak{S}(A) := \{\lambda \in \mathbb{R}_+^\times \mid \hat{\tau} \circ \alpha = \lambda \hat{\tau} \text{ for some } \alpha \in \mathrm{Aut}(A \otimes K(\mathcal{H}))\ \}.$$

Then $\mathcal{F}(A) = \mathfrak{S}(A)$ by Proposition 3.28 in [15]. Therefore it is of interest to know whether $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$.

It is clear that if the fundamental group of A is singly generated, $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$. See [15] and [16] for such examples. We shall show some examples of AF algebras A such that $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$.

Example 2.1. Consider a UHF algebra $M_{2^\infty 3^\infty}$. Then the fundamental group of $M_{2^\infty 3^\infty}$ is a multiplicative subgroup generated by 2 and 3. Hence $\mathcal{F}(M_{2^\infty 3^\infty})$ is isomorphic to $\mathbb{Z}^2$ as a group. Since $M_{2^\infty 3^\infty} \otimes \mathbb{K}$ is isomorphic to $M_{2^\infty} \otimes \mathbb{K} \otimes M_{3^\infty} \otimes \mathbb{K}$, there exists a trace scaling $\mathbb{Z}^2$-action on $M_{2^\infty 3^\infty} \otimes \mathbb{K}$.

In general, if A is a UHF algebra, then $\mathcal{F}(A)$ is a free abelian group (see [15]) and $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$.

Example 2.2. Let A be a unital simple AF algebra such that $K_0(A) = \mathbb{Z}+\mathbb{Z}\sqrt{3}$, $K_0(A)_+ = (\mathbb{Z}+\mathbb{Z}\sqrt{3})\cap\mathbb{R}_+$ and $[1]_0 = 1$. Then $\mathcal{F}(A) = \{(2+\sqrt{3})^n : n \in \mathbb{Z}\}$ (see [15]). Consider $B = M_{5^\infty} \otimes A$. Then it is easily seen that $\tau_*(K_0(B)) = \mathbb{Z}[\frac{1}{5}] + \mathbb{Z}[\frac{1}{5}]\sqrt{3}$. We shall show that $IM_+(\tau_*(K_0(B)))$ is generated by 5 and $2+\sqrt{3}$. Since $\tau_*(K_0(B))$ is a subring of $\mathbb{R}$, $IM_+(\tau_*(K_0(B)))$ is a group of positive invertible elements. Define a map N of $\mathbb{Z}[\frac{1}{5}] + \mathbb{Z}[\frac{1}{5}]\sqrt{3}$ to $\mathbb{Z}[\frac{1}{5}]$ by $N(a + b\sqrt{3}) = a^2 - 3b^2$ for any $a, b \in \mathbb{Z}[\frac{1}{5}]$. If $a + b\sqrt{3}$ is an invertible element in $\mathbb{Z}[\frac{1}{5}]+\mathbb{Z}[\frac{1}{5}]\sqrt{3}$, then there exists an integer n such that $N(a+b\sqrt{3}) = 5^n$. Therefore elementary computations based on elementary number theory show that $IM_+(\tau_*(K_0(B)))$ is generated by 5 and $2+\sqrt{3}$. (See, for example, [9].) Hence we see that $\mathcal{F}(B) = \{5^n(2+\sqrt{3})^m : n, m \in \mathbb{Z}\}$ and $B \otimes \mathbb{K}$ admits a trace scaling action.

We shall show that there exists a unital simple AF algebra A with unique trace such that $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$. Define

$$E = \{(\frac{j+k\sqrt{3}}{5^{6i}}, \begin{pmatrix} x \\ y \end{pmatrix}) \in \mathbb{R}\times\mathbb{Z}^2 \mid i,j,k,x,y \in \mathbb{Z}, x \equiv j \bmod 9, y \equiv k \bmod 3\}$$

$$E_+ = \{(r, \begin{pmatrix} x \\ y \end{pmatrix}) \in E : r > 0\} \cup \{(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix})\} \text{ and } [u]_0 = (1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Then there exists a simple AF-algebra A with a unique normalized trace τ such that $(K_0(A), K_0(A)_+, [1_A]_0) = (E, E_+, u)$ by Effros-Handelman-Shen theorem [5].

Lemma 2.3. With notation as above the fundamental group of A is equal to the multiplicative group generated by 5 and $2+\sqrt{3}$.

Proof. Since $\tau_*(K_0(A))$ is equal to $\mathbb{Z}[\frac{1}{5}] + \mathbb{Z}[\frac{1}{5}]\sqrt{3}$, $\mathcal{F}(A)$ is a subgroup of $\{5^n(2+\sqrt{3})^m : n, m \in \mathbb{Z}\}$ by an argument in Example 2.2. Define an additive homomorphism $\phi : E \to E$ by

$$\phi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = (5r, \begin{pmatrix} 5 & 9 \\ 6 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).$$

Elementary computations show that ϕ is a well-defined order isomorphism of E with $\phi(u) = (5, \begin{pmatrix} 5 \\ 6 \end{pmatrix})$. There exist a natural number n and a projection p in $M_n(A)$ such that $[p]_0 = (5, \begin{pmatrix} 5 \\ 6 \end{pmatrix})$ and $\tau \otimes Tr(p) = 5$. Since $(K_0(pM_n(A)p), K_0(pM_n(A)p)_+, [p]_0) = (E, E_+, (5, \begin{pmatrix} 5 \\ 6 \end{pmatrix}))$, there exists an isomorphism $f : A \to pM_n(A)p$ with $f_* = \phi$ by Elliott's classification theorem of AF algebra [6]. Therefore $5 \in \mathcal{F}(A)$. Define an additive homomorphism $\psi : E \to E$ by

$$\psi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = ((2+\sqrt{3})r, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).$$

A same argument shows $2+\sqrt{3}\in\mathcal{F}(A)$. Consequently $\mathcal{F}(A)$ is the multiplicative group generated by 5 and $2+\sqrt{3}$. $\square$

We shall consider the order automorphisms of (E, E_+).

Lemma 2.4. Let ϕ be an order automorphism of (E, E_+). Then there exist integers a, b, c, d and a positive invertible element λ in $\mathbb{Z}[\frac{1}{5}]+\mathbb{Z}[\frac{1}{5}]\sqrt{3}$ such that $ad-bc=\pm 1$ and

$$\phi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = (\lambda r, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}).$$

Moreover if $\lambda = 5$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 6 & 2 \end{pmatrix} \mod 9$$

and if $\lambda = 2+\sqrt{3}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix} \mod 9.$$

Proof. We denote by $(\phi_1((r, \begin{pmatrix} x \\ y \end{pmatrix})), \phi_2((r, \begin{pmatrix} x \\ y \end{pmatrix})))$ the element $\phi((r, \begin{pmatrix} x \\ y \end{pmatrix}))$ for any $(r, \begin{pmatrix} x \\ y \end{pmatrix}) \in E$. Consider a subgroup F generated by $(0, \begin{pmatrix} 9 \\ 0 \end{pmatrix})$ and $(0, \begin{pmatrix} 0 \\ 3 \end{pmatrix})$. Then F is an ϕ-invariant subgroup because ϕ is an order isomorphism. Hence there exist integers m_1, m_2, m_3 and m_4 such that $m_1m_3 - m_2m_4 = \pm 1$ and $\phi_2((0, \begin{pmatrix} x \\ y \end{pmatrix})) = \begin{pmatrix} m_1 & 3m_2 \\ \frac{m_3}{3} & m_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ for any $(0, \begin{pmatrix} x \\ y \end{pmatrix}) \in F$. Furthermore we see that there exists a positive invertible element λ in $\mathbb{Z}[\frac{1}{5}]+\mathbb{Z}[\frac{1}{5}]\sqrt{3}$ such that $\phi_1((r, \begin{pmatrix} x \\ y \end{pmatrix})) = \lambda r$. Since $5^{6i}\phi((\frac{9}{5^{6i}}, \begin{pmatrix} 0 \\ 0 \end{pmatrix})) = \phi((9, \begin{pmatrix} 0 \\ 0 \end{pmatrix}))$ for any $i \in \mathbb{Z}$, we see that $\phi((9, \begin{pmatrix} 0 \\ 0 \end{pmatrix})) = (9\lambda, \begin{pmatrix} 0 \\ 0 \end{pmatrix})$. This observation and easy computations show that $\phi((1, \begin{pmatrix} 1 \\ 0 \end{pmatrix})) = (\lambda, \begin{pmatrix} m_1 & 3m_2 \\ \frac{m_3}{3} & m_4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix})$ and $\frac{m_3}{3} \in \mathbb{Z}$. In a similar way, we see that $\phi((\sqrt{3}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})) = (\lambda, \begin{pmatrix} m_1 & 3m_2 \\ \frac{m_3}{3} & m_4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix})$. It is easily seen that ϕ is determined by the values of $\phi((1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}))$, $\phi((\sqrt{3}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}))$, $\phi((0, \begin{pmatrix} 9 \\ 0 \end{pmatrix}))$ and $\phi((0, \begin{pmatrix} 0 \\ 3 \end{pmatrix}))$. Therefore there exist integers a, b, c, d and a positive invertible element λ in $\mathbb{Z}[\frac{1}{5}]+\mathbb{Z}[\frac{1}{5}]\sqrt{3}$ such that $ad-bc=\pm 1$ and

$$\phi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = (\lambda r, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}).$$

Let $\lambda = 5$, then $a \equiv 5 \mod 9$, $b \equiv 0 \mod 9$, $c \equiv 0 \mod 3$ and $d \equiv 5 \mod 3$ by the definition of E. If $ad - bc = 1$, then $d \equiv 5^5 \mod 9$, $-b \equiv 0 \mod 9$,

$-c \equiv 0 \bmod 3$ and $a \equiv 5^5 \bmod 3$ because ϕ is an isomorphism. Therefore elementary computations show

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 6 & 2 \end{pmatrix} \bmod 9.$$

If $ad - bc = -1$, then then $-d \equiv 5^5 \bmod 9$, $b \equiv 0 \bmod 9$, $c \equiv 0 \bmod 3$ and $-a \equiv 5^5 \bmod 3$. There does not exist a integer a such that $a \equiv 5 \bmod 9$ and $-a \equiv 5^5 \bmod 3$. Therefore we reach a conclusion in the case $\lambda = 5$. In the case $\lambda = 2 + \sqrt{3}$, a similar argument proves the lemma. □

Theorem 2.5. There exists a unital simple AF algebra A with unique trace such that $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$.

Proof. Let

$$E = \{(\frac{j + k\sqrt{3}}{5^{6i}}, \begin{pmatrix} x \\ y \end{pmatrix}) \in \mathbb{R}\times\mathbb{Z}^2 \mid i, j, k, x, y \in \mathbb{Z}, x \equiv j \bmod 9, y \equiv k \bmod 3\}$$

$$E_+ = \{(r, \begin{pmatrix} x \\ y \end{pmatrix}) \in E : r > 0\} \cup \{(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix})\} \text{ and } [u]_0 = (1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Then there exists a simple AF algebra A with a unique normalized trace τ such that $(K_0(A), K_0(A)_+, [1_A]_0) = (E, E_+, u)$ by Effros-Handelman-Shen theorem [5]. By Lemma 2.3, $\mathcal{F}(A) = \{5^n(2 + \sqrt{3})^m : n, m \in \mathbb{Z}\}$. Let α be an automorphism of $A \otimes \mathbb{K}$ such that $\hat{\tau} \circ \alpha = 5\hat{\tau}$ and β an automorphism of $A \otimes \mathbb{K}$ such that $\hat{\tau} \circ \alpha = (2 + \sqrt{3})\hat{\tau}$. Then α_* and β_* are order isomorphisms of $(K_0(A), K_0(A)_+)$. Lemma 2.4 and elementary computations show that $\alpha_* \circ \beta_* \neq \beta_* \circ \alpha_*$. Therefore $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$. □

Remark 2.6. Let A be a unital simple C^*-algebra with a unique normalized trace τ. We denote by $\mathrm{Pic}(A)$ the Picard group of A (see [2]). Assume that the normalized trace on A separates equivalence classes of projections. Then we have the following exact sequence [15] (see also [10]).

$$1 \longrightarrow \mathrm{Out}(A) \xrightarrow{\rho_A} \mathrm{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1\cdot$$

If $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$, then $\mathrm{Pic}(A)$ is isomorphic to a semidirect product of $\mathrm{Out}(A)$ with $\mathcal{F}(A)$. Example 2.1 and Example 2.2 are such examples. We do not know whether there exists a simple C^*-algebra A with a unique normalized trace τ such that the normalized trace on A separates equivalence classes of projections and $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$.

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