QUANTUM INVARIANTS OF RANDOM 3-MANIFOLDS

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ABSTRACT. We consider the SO(3) Witten-Reshetikhin-Turaev quantum invariants of random 3-manifolds. When the level r is prime, we show that the asymptotic distribution of the absolute value of these invariants is given by the standard Rayleigh distribution and independent of the choice of level. Hence the probability that the quantum invariant certifies the Heegaard genus of a random 3-manifold of a fixed Heegaard genus g is positive but very small, less than 10^{-7} except when $g \leq 3$. We also examine random surface bundles over the circle and find the same distribution for quantum invariants there.

CONTENTS

1. INTRODUCTION

An important class of 3-manifold invariants arises from the topological quantum field theories (TQFTs) introduced by Witten [Wit] and then formalized by Atiyah [Ati1, Ati2]. At the coarsest level, a TQFT associates to each closed 3-manifold M a complex number $Z(M)$. Here, we study the distribution of $Z(M)$ for a random 3-manifold in the sense of Dunfield and W. Thurston [DT2]. Such manifolds are generated by gluing together two handlebodies of large genus by an extremely complicated element in the mapping class group (see Section 2 for details). For the SO(3) quantum invariants at a prime level, we show that the distribution of $|Z(M)|$ is in fact independent of the particular invariant:

1.1. **Theorem.** *Let Z be the* SO(3) *TQFT associated to a prime level $r \geq 5$. Then for a random 3-manifold M, the invariant $|Z(M)|$ is distributed by the standard Rayleigh distribution. In particular, the probability that $|Z(M)|$ is at least some $x \in [0, \infty)$ is:*

$$P\{|Z(M)| \geq x\} = e^{-x^2}$$

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We conjecture that $Z(M)$ itself is distributed by the standard complex Gaussian, though we were not able to prove this. It is known that for each prime level the values of $Z(M)$ form a dense set in $\mathbb{C}$ [Won1].

While Theorem 1.1 has intrinsic interest, our motivation was a classical topological question: when does the rank of $\pi_1(M)$ determine the *Heegaard genus* of M, that is, the minimal genus of a Heegaard splitting of M? This was first posed by Waldhausen [Hak, Wal], and in 1984 Boileau and Zieschang gave the first breakthrough examples where the rank and genus differ [BZ]. However, when M is hyperbolic, there are still no known examples where the rank and genus differ (see [Sou] for a survey). Garoufalidis [Gar] and Turaev [Tur] showed that $|Z(M)|$ gives a lower bound on Heegaard genus (see Theorem 3.6 below), and Wong used this to give an alternate proof that the examples of Boileau and Zieschang have rank smaller than genus [Won2]. This is interesting in part because the technique of [BZ] cannot be applied to any hyperbolic 3-manifold, whereas quantum invariants have no such apparent restrictions. This led us to try to find hyperbolic 3-manifolds where the rank and genus differ and the genus could be certified by $|Z(M)|$.

Unfortunately, in that search we found that quantum invariants typically gave very poor bounds on the Heegaard genus. This can be explained by our results here. In particular, Theorem 4.1 gives the precise distribution of $|Z(M)|$ for a random 3-manifold of Heegaard genus g, and we use this to show:

1.2. **Theorem.** *Fix a prime level $r \geq 5$. Let M come from a random Heegaard splitting of genus g. Then the* SO(3) *quantum invariant of level r gives a sharp lower bound on the genus of M with probability $\left(1-\mu^2\right)^{d-1}$, where $\mu = \frac{2}{\sqrt{r}} \sin\frac{\pi}{r}$, and d is the dimension of the Hilbert space of this TQFT for a surface of genus g.*

Here d depends on both g and r with $d \sim \mu^{2-2g}$ (see Lemma 3.11); since $\mu < 1$, the dimension d increases rapidly in both r and g. Table 1.3 gives the probabilities when r and g are small. As you can see, the probability that $Z(M)$

	$r=5$	$r=7$	$r=11$	$r=13$	$\lvert H_1(M;\mathbb{F}_2)\rvert$
$g=2$	0.274	0.228	0.206	0.202	0.067
$g=3$	0.011	1.6×10^{-5}	1.2×10^{-17}	4.0×10^{-28}	0.007
$g=4$	1.3×10^{-7}	7.5×10^{-42}	1.8×10^{-542}	1.2×10^{-1442}	4.3×10^{-4}
$g=5$	3.6×10^{-25}	1.2×10^{-373}	4.2×10^{-18437}	6.7×10^{-80495}	1.3×10^{-5}

TABLE 1.3. Probabilities that the SO(3) quantum invariants of various levels r give sharp bounds on genus. For comparison, the far right column gives the corresponding probability that the rank of $H_1(M;\mathbb{F}_2)$ certifies the genus, namely $\prod_{k=1}^{g}(2^k+1)^{-1}$ as determined by [DT2, §8.6].

gives a sharp genus bound decreases as either g or r increases, and the bound is very rarely sharp except when $g = 2$ or (marginally) when $(g, r) = (3, 5)$. Unfortunately, the case $g = 2$ isn't interesting for the question of rank versus genus since by Perelman's Geometrization Theorem, any closed 3-manifold where $\pi_1(M)$ is cyclic must have genus one. Moreover, it seems more likely that hyperbolic examples with rank differing from genus exist when $g > 2$. However, Table 1.3 makes it clear that it would be very hard to find such examples where the genus is certified by $Z(M)$ via a brute force search. Of course, it is possible to systematically generate examples where $Z(M)$ does certify the genus by always doing certain powers of Dehn twists, but we had no luck finding examples among such manifolds.

One natural class of hyperbolic 3-manifolds is surface bundles over the circle, and one place we looked for possible examples was the Hall-Schleimer census of small genus 2 bundles [HS]. For the natural notion of random surface bundle, we show here that the asymptotic distribution of $|Z(M)|$ is the same as that for all random 3-manifolds, even though general random 3-manifolds are bundles with probability 0. This can be taken as evidence for the naturality and robustness of this notion of random 3-manifold. We also show that for a generic bundle, no $\mathrm{SO}(3)$ quantum invariant certifies its Heegaard genus. For details, see Section 5.

1.4. **Outline.** In Section 2, we review the notion of a random 3-manifold and make precise how we discuss (asymptotically) their various properties. Then in Section 3, we introduce our preferred formalism for $Z(M)$, in particular focusing on certain (projective) unitary representations of mapping class groups. Section 4 is the core of the paper and contains the proofs of the main results. The two main tools we use are the work of Freedman, Larsen, and Wang [LW, FLW] on the density of these unitary representations, and the Ito-Kawada theorem about equidistribution of random walks in compact groups. Finally, Section 5 discusses what happens for surface bundles.

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2. Random Heegaard splittings

We now review the notion of *random 3-manifold* studied in [DT2]. Fix a positive integer g. Let H be a genus-g handlebody, and denote ∂H by Σ. Let Γ be the mapping class group of Σ. For any mapping class $f \in \Gamma$, we may associate to it a closed 3-manifold M_f obtained by gluing together two copies of H via the homeomorphism f.

To define a random Heegaard splitting of genus g, fix generators T for Γ. A random element w of Γ of complexity ℓ is defined to be the result of a random walk in the generators T of length ℓ, that is, a random word of length ℓ in T. For such a w, we call M_w the *manifold of a random Heegaard splitting* of genus g and complexity ℓ.

We are interested in the properties of such random M_w as the complexity $\ell \to \infty$. A priori, this might depend on the choice of generators T for Γ, though it turns out that many properties behave nicely and are independent of such choices. Consider a function

$$F\colon \text{ (closed 3-manifolds)} \to S,$$

where S is some set; a typical example is $F(M) = \dim H_1(M; \mathbb{F}_2)$. Now let μ_ℓ be the finitely supported atomic measure on S defined by taking $\mu_\ell(\{s\})$ to be the probability that $F(M) = s$, where M is a random Heegaard splitting of genus g and complexity ℓ. If there is a measure ν_g on S so that, for every choice of generating set of Γ, the measures μ_ℓ converge weakly to ν_g as $\ell \to \infty$, then we say that *F is distributed by ν_g for a random 3-manifold of genus g*. When ν_g is atomic, we will say things like "the probability that $F(M) = s$ is $\nu_g(s)$", though properly speaking this is a limiting statement as $\ell \to \infty$.

If the ν_g themselves have a weak limit ν as $g \to \infty$, then we say that *F is distributed by ν for a random 3-manifold*. Of course, not all F have such limiting distributions ν, but for instance $F(M) = \dim H_1(M; \mathbb{F}_p)$ does [DT2, §8.7]. Another example is that, for fixed n, the number of epimorphisms from $\pi_1(M)$ to the alternating group A_n is Poisson distributed with mean 1 [DT2, Thm 7.1]. A geometric example is Maher's result that M is hyperbolic with probability 1 [Mah].

3. WRT invariants via unitary representations

In this section, we review the definition of the Witten-Reshetikhin-Turaev TQFT for SO(3) in terms of a certain unitary representation of an extension of the mapping class group. Though the WRT TQFT was originally defined using the representation theory of $U_q sl(2)$ in [Wit] and [RT], an equivalent definition using skein theory was developed in [BHMV] and [Lic1], among others. We follow the latter approach as summarized in [Lic2], and will also refer to [MR] when necessary.

Fix an odd integer $r \geq 5$ as the level, and let $A = ie^{2\pi i/4r}$. For a closed surface Σ of genus g, the associated finite-dimensional Hilbert space $(V, \langle \cdot, \cdot \rangle)$ has the following form. Let $\mathscr{L} = \{0, 2, 4, \ldots r-3\}$ be the set of even labels. The basis elements of V correspond to the admissible $\mathscr{L}$-labelings of trivalent spines of a handlebody H with $\partial H = \Sigma$. The dimension of V is given by the Verlinde formula [BHMV] as

$$d_g = \frac{1}{2^g} \left(\frac{r}{2}\right)^{g-1} \sum_{j=1}^{r-1} \left(\sin \frac{\pi j}{r}\right)^{2-2g} \tag{3.1}$$

Moreover, for our choice of A, there is an inner product $\langle \cdot, \cdot \rangle$ induced from the splitting $\#_g S^1 \times S^2 = H \cup (-H)$, and these basis elements are orthogonal with respect to $\langle \cdot, \cdot \rangle$. See [Lic2] for details.

Underlying this inner product, and indeed the entire WRT TQFT theory, is a skein element ω which has the key property that if two links related by Kirby handleslides are colored by ω, then the two resulting skeins have the same

Kauffman bracket evaluation. More precisely, we have chosen ω so that the Kauffman bracket of a 0-framed unlink in S^3 colored by ω is the number μ^{-1}, where $\mu^2 = -\left(A^2 - A^{-2}\right)^2/r$ and so $\mu = \frac{2}{\sqrt{r}} \sin \frac{\pi}{r}$. We denote the Kauffman bracket of a (-1)-framed unlink colored by ω by κ^3. Formulas for κ^3 appear in e.g. [BHMV], but here all that is needed is that κ^3 is a $4r^{\text{th}}$ root of unity.

We now describe the (projective) action of the mapping class group Γ of Σ on V. Let $\mathscr{D}_{ns}$ denote the set of Dehn twists along non-separating curves on Σ. These generate Γ, and to start, we define an action of the free group $\mathrm{Free}(\mathscr{D}_{ns})$ on V. To a word w in $\mathrm{Free}(\mathscr{D}_{ns})$, we associate a link $L(w)$ whose components have framing ± 1 and that lives in $\Sigma \times I$, as in the proof of the Dehn-Lickorish theorem. Color the components of $L(w)$ by the skein element ω to obtain $s(w)$. Consider the transformation on V which adds to a collar neighborhood of H the skein $s(w)$. Because ω behaves well under handleslides, this gives a unitary transformation on V. In other words, we have a homomorphism $\rho\colon \mathrm{Free}(\mathscr{D}_{ns}) \to \mathrm{U}(V)$. This construction is the "skein-theoretic version of the original geometric action on Dehn twists" from [MR]. While the representation of $\mathrm{Free}(\mathscr{D}_{ns})$ on V does not descend to Γ itself, it does give a *projective* representation of Γ, i.e. there is a homomorphism $\overline{\rho} : \Gamma \to \mathrm{PU}(V)$ with the property that

$$\begin{array}{ccc} \mathrm{Free}(\mathscr{D}_{ns}) & \xrightarrow{\rho} & \mathrm{U}(V_g) \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{\overline{\rho}} & \mathrm{PU}(V_g) \end{array} \tag{3.2}$$

commutes.

The WRT 3-manifold invariant may be described easily using this language of unitary representations. For any word w in $\mathrm{Free}(\mathscr{D}_{ns})$, let M_w be the closed 3-manifold obtained from the corresponding element of Γ. The *vacuum vector* $v_\emptyset \in V$ is the element corresponding to the empty link. Up to a phase factor involving κ^3, the action of $\rho(w)$ on $v_\emptyset$ determines $Z(M_w)$. The phase factor is in some sense a correction factor for the framing, and depends on the exponent sum $e(w)$ of the word w and on the signature $\sigma_b(w)$ of the linking matrix of $L(w)$. Precisely, the WRT invariant is given by

$$Z(M_w) = \kappa^{-3(\sigma_b(w)+e(w))} \cdot \mu \cdot \langle \rho(w) v_\emptyset, v_\emptyset \rangle \tag{3.3}$$

For example, since the vacuum vector has $|v_\emptyset|^2 = \mu^{-g}$, we have $Z(S^3) = \mu$ and $Z(S^1 \times S^2) = 1$; in general $Z(\#_g S^1 \times S^2) = \mu^{1-g}$. Note that $Z(M)$ here differs from $I_p(M)$ appearing in [MR] by a factor of μ.

It will be convenient to rewrite $Z(M_w)$ using matrix notation. Take the normalized vacuum vector $\frac{v_\emptyset}{|v_\emptyset|}$ as the first basis element in an orthonormal basis for V, and denote the $(1,1)$-entry of $V(w)$ by $\rho(w)_{(1,1)}$. Then,

$$Z(M_w) = \kappa^{-3(\sigma_b(w)+e(w))} \cdot \mu^{1-g} \cdot \rho(w)_{(1,1)}, \tag{3.4}$$

and since κ is a root of unity and μ positive, we have

$$|Z(M_w)| = \mu^{1-g} |\rho(w)_{(1,1)}|. \tag{3.5}$$

In particular, $|Z(M_w)| \leq \mu^{1-g}$, leading to the key observation of Garoufalidis [Gar] and Turaev [Tur]:

3.6. **Theorem.** *Suppose M is a closed orientable 3-manifold with Heegaard genus $g(M)$. Then $|Z(M)| \leq \mu^{1-g(M)}$.*

3.7. **The image of ρ.** We now turn to the properties of ρ that will be needed later. The first proposition asserts that the image of ρ is contained in $\mathrm{SU}(V)$, or in some exceptional cases, a slightly larger group.

3.8. **Proposition.** *Let $r \geq 5$ be odd. The image of the homomorphism*

$$\det \circ \rho \colon \operatorname{Free}(\mathscr{D}_{ns}) \to \mathbb{C}^\times$$

is 1 for all $g \geq 2$, except when both $g = 2$ and $r \equiv 0 \pmod 5$. In the latter case, or when $g = 1$, the image is contained in the subgroup generated by A^4.

Proof. When $g \geq 3$, we need to show that $\det \circ \rho \colon \operatorname{Free}(\mathscr{D}_{ns}) \to \mathbb{C}^\times$ is the trivial homomorphism. Following [MR], there is a central extension $\widetilde{\Gamma}_1$ of Γ to which ρ descends, and in this proof we consider $\widetilde{\Gamma}_1$ as the domain of ρ. As discussed in the proof of Theorem 3.10 of [MR], the abelianization $\widetilde{\Gamma}_1^{ab} = H_1(\widetilde{\Gamma}_1; \mathbb{Z})$ is 0 when $g \geq 3$. Thus the homomorphism $\det \circ \rho \colon \widetilde{\Gamma}_1 \to \mathbb{C}^\times$ must be trivial as claimed, since it factors through the trivial group.

So now we focus on $g = 1, 2$, where $\widetilde{\Gamma}_1^{ab} = H_1(\widetilde{\Gamma}_1; \mathbb{Z}) = \mathbb{Z}$. By Theorem 3.8 and Remark 3.9(ii) of [MR], all the Dehn twists in $\mathscr{D}_{ns}$ become conjugate in $\widetilde{\Gamma}_1$. Thus the abelianization $H_1(\widetilde{\Gamma}_1; \mathbb{Z})$ is generated by any single non-trivial element t_α of $\mathscr{D}_{ns}$. We proceed by calculating the determinant of $\rho(t_\alpha)$ explicitly, and find $\rho(t_\alpha) = A^e$ where

$$e = \begin{cases} \frac{1}{6}(r-3)(r-1)(r+1) & \text{when } g = 1, \\ \frac{r}{5}\left(\frac{r-3}{2}\right)\left(\frac{r-1}{2}\right)\left(\frac{r+1}{2}\right)\left(\frac{r+3}{2}\right) & \text{when both } g = 2 \text{ and } r \equiv 0 \bmod 5, \\ 0 & \text{in all other cases.} \end{cases}$$

In the first two cases, it is easy to see that $4|e$, and so the image of $\det \circ \rho$ is contained in $\langle A^4 \rangle \leq \mathbb{C}^\times$.

First, consider the case $g = 2$. We view the genus 2 handlebody H as a neighborhood of the theta graph, with edges labeled by $a, b, c \in \{0, 2, 4, \ldots, r-3\}$. The basis elements of V correspond to those labelings that are admissible, namely those satisfying

$$a + b + c \leq 2(r-2), \qquad b + c \geq a, \qquad a + c \geq b, \qquad a + b \geq c.$$

Consider a simple closed curve $\alpha \subset \Sigma$ which bounds a disc in H which is dual to the edge labeled by a. Then each basis element (a, b, c) of V is an eigenvector for $\rho(t_\alpha)$, with eigenvalue A^{a^2+2a}. Now $\det \rho(t_\alpha)$ is the product of these eigenvalues, and hence is A^e for $e = \sum_{(a,b,c) \text{ admissible}} a^2 + 2a$.

For a fixed label a, it can be shown that the number of admissible triples containing a is exactly $(a+1)(\frac{r-1-a}{2})$. Thus we can apply standard formulas to show that

$$\begin{aligned} e &= \sum_{\substack{(a,b,c)\\ \text{admissible}}} a^2+2a = \sum_{\text{even } a=0}^{r-3} (a^2+2a)(a+1)\left(\frac{r-1-a}{2}\right) \\ &= \sum_{k=0}^{\frac{r-3}{2}} -8k^4+(4r-16)k^3+(6r-10)k^2+(2r-21)k \\ &= \frac{r}{5}\left(\frac{r-3}{2}\right)\left(\frac{r-1}{2}\right)\left(\frac{r+1}{2}\right)\left(\frac{r+3}{2}\right). \end{aligned}$$

To conclude, we claim $4r$ divides e and hence $\det\rho(t_\alpha) = A^e = 1$ since A is a $4r^{\text{th}}$ root of unity. As $5 \nmid r$, then 5 must divide one of the consecutive integers $\frac{r-3}{2}, \frac{r-1}{2}, \frac{r+1}{2}, \frac{r+3}{2}$ and hence $r|e$. It is also clear that 4 divides a^2+2a since a is even, and hence $4|e$. Thus $4r|e$ as needed.

When $g=1$, we view the solid torus H as a neighborhood of a circle, with the Dehn twist curve α being a meridian linking the circle once. Thus the eigenvalues of $\rho(t_\alpha)$ are A^{a^2+2a} all with multiplicity one. Then $e = \sum_{\text{even } a=0}^{r-2} a^2+2a = \frac{1}{6}(r-3)(r-1)(r+1)$, as claimed. $\square$

The next result is a simple corollary of the results of Freedman, Larsen, and Wang [FLW, LW] that the image of $\overline{\rho}$ is dense in $\mathrm{PU}(V)$ when r is an odd prime.

3.9. **Proposition.** *Let $G \leq \mathrm{U}(V)$ be the closure of the image of ρ. For $g \geq 2$ and $r \geq 5$ prime, the group G contains $\mathrm{SU}(V)$. Moreover $G = \mathrm{SU}(V)$ except when both $g=2$ and $r=5$.*

Proof. First suppose that $(g,r) \neq (2,5)$, so that $G \subset \mathrm{SU}(V)$ by Proposition 3.8. Since ρ is a homomorphism, G is a compact Lie subgroup of $\mathrm{SU}(V)$ and hence $\dim G \leq \dim \mathrm{SU}(V)$. On the other hand, the results of Freedman, Larsen, and Wang (Theorem 3 of [LW]) show that the map $\overline{\rho}$ has dense image. Thus the commutativity of (3.2) implies that G maps onto $\mathrm{PU}(V)$, and so $\dim G \geq \dim \mathrm{PU}(V)$. As $\dim \mathrm{SU}(V) = \dim \mathrm{PU}(V)$ and $\mathrm{SU}(V)$ is connected, we must have $G = \mathrm{SU}(V)$ as claimed.

If $(g,r) = (2,5)$, then we saw in the proof of Proposition 3.8 that the image of $\det\circ\rho$ is contained in the finite subgroup of $\mathbb{C}^*$ generated by A^4. Let G_0 be the connected component of the identity, which must be contained in $\ker(\det\circ\rho)$ since the image of $\det\circ\rho$ is finite. Now as before, we must have $\dim G \geq \dim \mathrm{PU}(V)$. Since $G_0 \leq \mathrm{SU}(V)$ and $\dim G_0 = \dim G$ this forces $G_0 = \mathrm{SU}(V)$ as needed. $\square$

3.10. *Remark.* Note that Proposition 3.9 remains true if we restrict the domain of ρ to any subgroup of $\mathrm{Free}(\mathscr{D}_{ns})$ whose image generates Γ; this is because the proof only uses that the image of $\overline{\rho}$ is dense $\mathrm{PU}(V)$.

We will also need to understand the dimension of V.

3.11. **Lemma.** *Fix an odd integer level $r \geq 5$. Let d_g be the dimension of the WRT representation space V. Then*

$$\lim_{g\to\infty} \frac{d_g}{\mu^{2-2g}} = 1$$

As $\mu < 1$, this implies $d_g \to \infty$ as $g \to \infty$.

Proof. Using (3.1) and the formula for μ we have:

$$\frac{d_g}{\mu^{2-2g}} = \frac{1}{2}\sum_{j=1}^{r-1}\left(\frac{\sin(\pi j/r)}{\sin(\pi/r)}\right)^{2-2g} = \frac{1}{2}\sum_{j=1}^{r-1} a_j^{2g-2} \quad \text{where} \quad a_j = \frac{\sin(\pi/r)}{\sin(\pi j/r)}$$

Now when $j = 1$ or $r-1$, then $\sin(\pi j/r) = \sin(\pi/r)$ and so $a_j = 1$. However, for any other j, we have $\sin(\pi j/r) > \sin(\pi/r)$ and so $0 \leq a_j < 1$. Thus

$$\frac{d_g}{\mu^{2-2g}} = \frac{1}{2}\left(2 + \sum_{j=2}^{r-2} a_j^{2g-2}\right) \tag{3.12}$$

where all a_j left are < 1 and hence $a_j^{2g-2} \to 0$ as $g \to \infty$. As r is fixed, we see that $\lim_{g\to\infty} d_g/\mu^{2-2g} = 1$ as claimed. □

4. Distribution of $|Z(M)|$

In this section, we calculate the distribution of $|Z(M)|$ for random 3-manifolds. To state our results, we first define some distributions. The *standard complex Gaussian* is the probability distribution on $\mathbb{C}$ where the real and imaginary components are independently distributed Gaussians of mean 0 and variance 1/2. The *standard Rayleigh* is the distribution R on $[0, \infty)$ describing the probability distribution of the absolute value of a standard complex Gaussian; explicitly the density function is $2xe^{-x^2}$ and $P\{R \geq x\} = e^{-x^2}$ which has mean $\frac{\sqrt{\pi}}{2}$ and variance $\frac{4-\pi}{4}$. Recall our main result is:

1.1. **Theorem.** *Let Z be the* SO(3) *TQFT associated to a prime level $r \geq 5$. Then for a random 3-manifold M, the invariant $|Z(M)|$ is distributed by the standard Rayleigh.*

Along the way we give a complete picture for each fixed Heegaard genus, using the following distribution. Let X_d be the random variable on $\mathbb{C}$ which gives the distribution of the (1,1)-entry of a unitary matrix of size d with respect to Haar measure; it is well-known, see e.g. [HP, pg. 140] that the density in polar coordinates is $\frac{d-1}{\pi}(1-r^2)^{d-2}\,dA$ for $0 \leq r \leq 1$. Focusing on the absolute value $|X_d|$, we easily calculate

$$P\{|X_d| \geq x\} = \left(1 - x^2\right)^{d-1} \quad \text{for } 0 \leq x \leq 1.$$

We first show

4.1. **Theorem.** *Fix a genus g and prime level $r \geq 5$. Let d_g be the dimension of the Hilbert space V associated to the surface of genus g. Then the invariant $|Z(M)|$ is distributed by $\mu^{1-g}|X_{d_g}|$ for a random Heegaard splitting of genus g. In particular*

$$P\{|Z(M)| \geq x\} = \left(1 - \mu^{2g-2}x^2\right)^{d_g-1} \quad \textit{for } 0 \leq x \leq \mu^{1-g}.$$

Proof. Consider a set T consisting of words $t_1, \dots t_n \in \mathrm{Free}(\mathscr{D}_{ns})$ which together generate Γ. Let w be a random word in T of length ℓ. Then by (3.5), the manifold M_w has $|Z(M_w)| = \mu^{1-g}|\rho(w)_{(1,1)}|$. We need to show that the distribution of $|Z(M_w)|$ limits on $\mu^{1-g}|X_{d_g}|$ as the length $\ell \to \infty$.

Let G be the closure of image of the subgroup $\langle T \rangle$ under ρ. By Proposition 3.9 and Remark 3.10, we know that G contains $\mathrm{SU}(V)$. Now for the random word w, note that $\rho(w)$ is the product of the $\rho(t_i)$'s. Thus by the Ito-Kawada Theorem [KI], the matrix $\rho(w)$ becomes equidistributed on G with respect to Haar measure as $\ell \to \infty$. Thus if X denotes the distribution of the $(1,1)$-entry of a random matrix in G, we have that $|Z(M)|$ is distributed by $\mu^{1-g}|X|$.

Thus it remains to show X and X_d have the same distribution where $d = d_g = \dim V$. If $G = \mathrm{U}(d)$ this is obvious, so since $\mathrm{SU}(V) \leq G$ we must have $\dim G = \dim \mathrm{SU}(V) = \dim \mathrm{U}(V) - 1$. Consider the homomorphism $\phi : S^1 \times G \to \mathrm{U}(d)$ given by $\phi(\zeta, M) = \zeta M$. Since $S^1 \times G \to \mathrm{U}(d)$ is a covering map, the Haar measure (thought of as a differential form) on $\mathrm{U}(d)$ pulls back to (a multiple of) the Haar measure on $S^1 \times G$. Now the pullback of the random variable $|X_d|$ on $\mathrm{U}(d)$ to $S^1 \times G$ is equal to $|X|$, and as the Haar measure pulls back we have that $|X_d|$ and $|X|$ have the same distribution. To see that the distributions themselves are the same, for any $\zeta \in S^1$ consider left multiplication on G by the matrix $A_\zeta \in G$ which is diagonal with entries $(\zeta, \zeta^{-1}, 1, 1, \dots, 1)$. Notice that the absolute value of the (1,1)-entry remains unchanged, but the phase shifts by ζ. Since Haar measure is left-invariant, this shows X to be rotationally symmetric, and hence the same as X_d. □

We now consider the limit as the genus g gets large.

Proof of Theorem 1.1. We need to determine the limit of the random variables $\mu^{1-g}|X_{d_g}|$ as $g \to \infty$. It is well known that $\sqrt{d_g}X_{d_g}$ converges in distribution to the standard complex Gaussian (see e.g. [Nov]), and by Lemma 3.11 we know $\sqrt{d_g} \approx \mu^{1-g}$. Combining these can give that $\mu^{1-g}|X_{d_g}|$ converges to the standard Rayleigh distribution R, but instead we show this directly starting from Theorem 4.1. If we fix $x \geq 0$, then since $\mu < 1$ we have

$$\begin{aligned} \lim_{g\to\infty} P\left\{|Z(M)| \geq x\right\} &= \lim_{g\to\infty} \left(1 - \mu^{2g-2}x^2\right)^{d_g - 1} \\ &= \lim_{g\to\infty} \left(1 - \frac{x^2}{\mu^{2-2g}}\right)^{\mu^{2-2g}} \cdot \lim_{g\to\infty} \left(1 - \frac{x^2}{\mu^{2-2g}}\right)^{\frac{d_g - 1}{\mu^{2-2g}}} \\ &= e^{-x^2} \cdot 1^1 = e^{-x^2} = P\left\{R \geq x\right\} \end{aligned}$$

as needed. □

As $|Z(M)|$ is distributed by the standard Rayleigh distribution, it is very natural to postulate

4.2. **Conjecture.** *Let $r \geq 5$ be prime. For a random 3-manifold M, the* $\mathrm{SO}(3)$ *invariant $Z(M)$ is distributed by a standard complex Gaussian.*

Looking at the proof of Theorem 1.1, in fact we showed that $\mu^{1-g}\rho(w)_{(1,1)}$ is essentially distributed by a standard complex Gaussian when g is large. From (3.4), we see this quantity differs from $Z(M)$ by $\kappa^{-3(\sigma_b(w)+e(w))}$, where κ is a certain $4r^{\text{th}}$ root of unity. Since r is fixed, this tells us the image of $Z(M)$ in $\mathbb{C}/(z \mapsto \kappa^3 z)$ is distributed by the push-forward of the standard complex Gaussian. The exponent sum $e(w)$ is a homomorphism $e\colon \operatorname{Free}(\mathscr{D}_{ns}) \to \mathbb{Z}$ and is thus easy to deal with, but $\sigma_b(w)$ is not a homomorphism. Indeed, the signature σ_b is not Markovian in the sense that $\sigma_b(w \cdot t)$ does not depend just on $\sigma_b(w)$ and t. While it seems almost certain that $\sigma_b(w) + e(w)$ must be uncorrelated with $\rho(w)_{(1,1)}$, and hence $Z(M)$ is the standard complex Gaussian, we were unable to show this. If Conjecture 4.2 is true, it would give an alternate proof that the values of $Z(M)$ are dense in $\mathbb{C}$ [Won1].

Finally, we compute the probability that $Z(M)$ gives a sharp genus bound for each genus g and level r.

Proof of Theorem 1.2. Fix a genus g. By [Mah], a random Heegaard splitting of genus g actually has Heegaard genus g with probability 1. Thus by Theorem 3.6, the quantum invariant will give a sharp lower bound on Heegaard genus whenever $|Z(M)| > \mu^{2-g}$. By Theorem 1.1, this happens with probability $\left(1-\mu^2\right)^{d_g-1}$, as claimed. □

5. Random surface bundles

A simple class of 3-manifolds are those which are surface bundles fibering over the circle. Such manifolds are built from some $f \in \Gamma$ by considering the mapping torus

$$MT_f = \Sigma \times [0,1] \Big/ (p,1) \sim (f(p),0)$$

For reasons of homology, a random 3-manifold in the sense of Section 2 fibers over the circle with probability 0 [DT2, Cor. 8.5], and even among those manifolds with $b_1 > 0$, surface bundles appear to be rare [DT1]. Despite this, we show that for a natural model of random bundles the distribution of $|Z(M)|$ is the same as that of random manifolds more generally. This can be taken as evidence for the naturality and robustness of this notion of random 3-manifolds.

Here, a random bundle with fiber a surface Σ of genus g is defined just as one expects from Section 2: after fixing generators T of the mapping class group Γ of Σ, one considers MT_f for $f \in \Gamma$ a random word in T of length ℓ, as ℓ tends to infinity. If we then send $g \to \infty$, we find:

5.1. **Theorem.** *Consider the* $\mathrm{SO}(3)$ *invariant for a prime level* $r \geq 5$*. Then* $|Z(M)|$ *is distributed by the standard Rayleigh for* M *a random surface bundle.*

Proof. Fix a genus $g > 2$ for the fiber Σ. In the notation of Section 3, if $w \in \operatorname{Free}(\mathscr{D}_{ns})$ then we have (see e.g. [Tur, §IV.7.2]):

$$|Z(MT_w)| = \left|\operatorname{tr}\rho(w)\right|$$

Thus as in the proof of Theorem 4.1, the Ito-Kawada Theorem tells us that $|Z(MT_w)|$ is distributed the same as $|\operatorname{tr} A|$ for $A \in \mathrm{SU}(V)$ chosen with respect to Haar measure. Again as in Theorem 4.1, this is the same as the distribution of $|\operatorname{tr} A|$ for $A \in \mathrm{U}(V)$. Now as $d \to \infty$, the distribution of $\operatorname{tr} A$ for $A \in \mathrm{U}(d)$ converges to the standard complex Gaussian ([DS], see also [PR, Nov]). Thus since $\dim V \to \infty$ as $g \to \infty$ by Lemma 3.11, we have that $|Z(M)|$ is distributed by the standard Rayleigh for a random surface bundle. □

A genus-g surface bundle M has a natural Heegaard splitting of genus $2g+1$ obtained by tubing together two of the fibers. Combining [Mah] with Theorem 4.2 of [Sou] shows that this is the minimal genus splitting with probability 1, and explicit examples where this is the case are easy to construct by taking the monodromy from the Torelli group. However, we claim that the $\mathrm{SO}(3)$ quantum invariants can never certify that M has Heegaard genus $2g+1$. As in the proof of Theorem 5.1, we have $|Z(M)| = |\operatorname{tr} \rho(w)|$, and since $\rho(w)$ is unitary, it follows that $|\operatorname{tr} \rho(w)| \leq \operatorname{tr}(\mathrm{Id}) = d_g$. From (3.12), we see that $d_g \leq \mu^{2-2g} < \mu^{1-2g}$ for any g when $r \geq 5$. Thus $|Z(M)| < \mu^{1-2g}$, and so $Z(M)$ fails to give a sharp bound on Heegaard genus. To summarize, this shows:

5.2. **Proposition.** *There exist closed hyperbolic 3-manifolds of arbitrary large Heegaard genus such that no* $\mathrm{SO}(3)$ *quantum invariant gives a sharp genus bound.*

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