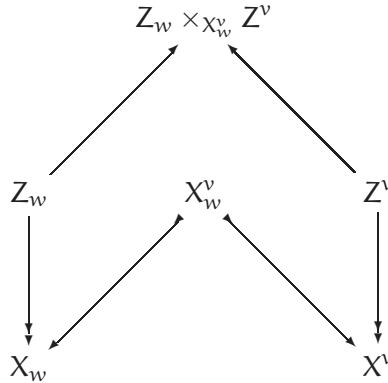# STANDARD MONOMIAL THEORY OF RR VARIETIES

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ABSTRACT. We construct the RR varieties as the fiber products of Bott-Samelson varieties over Richardson varieties. We study their homogeneous coordinate rings and standard monomial theory.

1. INTRODUCTION

The main object of our investigation is a fiber product $Z_w \times_{X_w^v} Z^v$, which we call an *RR variety*, over a Richardson variety X_w^v, of two Bott-Samelson varieties Z_w and Z^v associated with a Schubert variety X_w and an opposite Schubert variety X^v in the flag variety.

We study its standard monomial theory based on an explicit description of its homogeneous coordinate ring.

2. SCHUBERT AND OPPOSITE SCHUBERT VARIETIES

In this section, we recall definitions and properties of Schubert and Richardson varieties, and fix our notation.

2.1. **Bruhat-Chevalley order.** Let $G = GL_n(\mathbb{C})$ be the general linear group over the complex numbers $\mathbb{C}$, and B be its Borel subgroup consisting of upper triangular matrices. We write T for the maximal torus of G consisting of diagonal matrices. Note that the symmetric group $\mathfrak{S}_n$ is the Weyl group $W = N(T)/T$ of G, where $N(T)$ is the normalizer of T in G, and there are finitely many T-fixed points e_w in G/B labeled by elements w of $\mathfrak{S}_n$. The B-orbits $C_w = B \cdot e_w$ in G/B are called the *Schubert cells*. The Zariski closure

of C_w is called the *Schubert variety* associated with w and denoted by X_w. There is a partial order, called the *Bruhat-Chevalley order*, on the elements of $\mathfrak{S}_n$:

$$w_1 \geq w_2 \text{ if and only if } X_{w_1} \supseteq X_{w_2}.$$

The Grassmannian $\mathrm{Gr}(d, n)$ of d dimensional subspaces in $\mathbb{C}^n$ can be realized as the quotient of the space of $n \times d$ matrices of rank d by $\mathrm{GL}_d(\mathbb{C})$, and then the Schubert varieties in $\mathrm{Gr}(d, n)$ can be described explicitly. For $w \in \mathfrak{S}_n$, let i_k be the k-th smallest element in $\{w(1), \cdots, w(d)\}$, then define

$$(w(1), \cdots, w(d)) \uparrow = (i_1, \cdots, i_d),$$

i.e., the rearrangement of $(w(1), \cdots, w(d))$ in increasing order. For the elementary basis $\{e_i\}$ of $\mathbb{C}^n$, if we let E_w be an $n \times d$ matrix whose k-th column represents e_{i_k} for $1 \leq k \leq d$, then the Schubert variety X_w in $\mathrm{Gr}(d, n)$ is the Zariski closure of its B-orbit, $\overline{B \cdot E_w}$.

The flag variety G/B can be embedded in the product of Grassmannians:

$$(1) \qquad G/B \hookrightarrow \mathrm{Gr}(1, n) \times \mathrm{Gr}(2, n) \times \cdots \times \mathrm{Gr}(n-1, n),$$

and the Grassmannian $\mathrm{Gr}(d, n)$ can be identified with $G/\widehat{P}_d$ for a maximal parabolic subgroup $\widehat{P}_d$ of G containing B. Then with respect to the projections $\pi_d : G/B \to G/\widehat{P}_d$, the Bruhat-Chevalley order can be realized as follows: $w_1 \geq w_2$ if and only if

$$\pi_d(X_{w_1}) \supseteq \pi_d(X_{w_2}) \text{ for } 1 \leq d \leq n-1$$

or more explicitly, $w_1 \geq w_2$ if and only if, for $1 \leq d \leq n-1$,

$$(w_1(1), \cdots, w_1(d)) \uparrow \succeq (w_2(1), \cdots, w_2(d)) \uparrow.$$

where for $a_1 < \cdots < a_d$ and $b_1 < \cdots < b_d$,

$$(2) \qquad\qquad (a_1, \cdots, a_d) \succeq (b_1, \cdots, b_d),$$

if $a_i \geq b_i$ for all i.

With respect to the diagonal embedding (1), this condition is compatible with the inclusion order of the Schubert varieties in each $\mathrm{Gr}(d, n)$. We refer to [BL00][LG01] for further details on the Schubert varieties and the Bruhat-Chevalley order.

2.2. **Flag variety G/B.** Recall that for $\mathbf{m} = (m_1, \cdots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$, we have a line bundle $\mathcal{O}_{\mathbf{m}}$ over G/B induced from the Plücker bundles of the Grassmannians:

$$\mathcal{O}_{\mathrm{Gr}(1,n)}(m_1) \otimes \cdots \otimes \mathcal{O}_{\mathrm{Gr}(n-1,n)}(m_{n-1})$$

Standard monomial theory lets us describe the section ring of G/B:

$$\bigoplus_{p \geq 0} H^0(G/B, \mathcal{O}_{\mathbf{m}}^{\otimes p})$$

explicitly in terms of the Plücker coordinates or determinant functions over the space M_n of $n \times n$ complex matrices.

To be more precise, let $\mathbb{C}[M_n]$ be the coordinate ring of M_n. For $d \leq n$, consider subsets $R = \{r_1, \cdots, r_d\}$ and $C = \{c_1, \cdots, c_d\}$ of $\{1, \cdots, n\}$. With $r_1 < \cdots < r_d$ and $c_1 < \cdots < c_d$, we will also write R and C as $(r_1, \cdots, r_d)$ and $(c_1, \cdots, c_d)$.

We let $[R : C]$ or $[r_1, \cdots, r_d | c_1, \cdots, c_d]$ denote the map from M_n to $\mathbb{C}$ by assigning to a matrix $X \in M_n$ the determinant of the $d \times d$ minor of X formed by taking rows R and columns C:

(3)
$$[R : C] = [r_1, \cdots, r_d | c_1, \cdots, c_d]$$
$$= \det \begin{bmatrix} x_{r_1 c_1} & \cdots & x_{r_1 c_d} \\ \vdots & \ddots & \vdots \\ x_{r_d c_1} & \cdots & x_{r_d c_d} \end{bmatrix}$$

We shall identify this determinant with a tableaux obtained by filling in the c_k-th box of the $1 \times n$ diagram with r_k for $1 \leq k \leq d$. Then a product of them will be denoted by a filling of a rectangular diagram with multiple rows. We place the i-th factor in the i-th row counting from the bottom row. For example, if $n = 6$, then $[1, 2, 4, 5 | 1, 3, 4, 6] \times [2, 3, 5, 6 | 1, 2, 3, 6]$ can be denoted by

(4)

2	3	5			6
1		2	4		5

For $\mathbf{m} = (m_1, \cdots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$, let $C^{(d)} = (1, 2, \cdots, d)$ and $R_{t,d} \subset \{1, \cdots, n\}$ with $|R_{t,d}| = d$ for $1 \leq d \leq n - 1$. Then the product

(5)
$$\prod_{1 \leq t \leq m_1} [R_{t,1} : C^{(1)}] \times \cdots \times \prod_{1 \leq t \leq m_{n-1}} [R_{t,n-1} : C^{(n-1)}]$$

can be identified with a filling of $|\sum_i m_i| \times n$ rectangular diagram.

With these fixed column indices $\{C^{(d)}\}$, if the entries in each row are strictly increasing from left to right and the entries in each column is weakly increasing from top to bottom, then it can be identified with a semistandard Young tableaux of shape $\lambda = (\lambda_1, \cdots, \lambda_{n-1})$ with entries from $\{1, \cdots, n\}$ in the literature (e.g., [Sta99]) where $\lambda_i = m_i + \cdots + m_r$ for $1 \leq i \leq n - 1$. Recall that they form a $\mathbb{C}$-basis of the section space $H^0(G/B, \mathcal{O}_\mathbf{m})$, and called the *standard monomials* for $H^0(G/B, \mathcal{O}_\mathbf{m})$. See [LG01][Se07] for further detail.

2.3. **Schubert variety** X_w. Moreover, standard monomial theory for G/B descends to its Schubert varieties X_w in a way compatible with the embedding (1). Note that we can compare Plücker coordinates in the Grassmannians in terms of the partial order given in (2). Then for $X_w \subset G/B$, the kernel of the restriction map

$$H^0(G/B, \mathcal{O}_\mathbf{m}) \to H^0(X_w, \mathcal{O}_\mathbf{m})$$

is spanned by $\prod_d \prod_t [R_{t,d} : C^{(d)}]$ such that $(w(1), \cdots, w(d)) \uparrow\not\succeq R_{t,d}$; and the following standard monomials in $H^0(G/B, \mathcal{O}_\mathbf{m})$ project to a $\mathbb{C}$-basis of $H^0(X_w, \mathcal{O}_\mathbf{m})$:

(6)
$$\left\{ \prod_{1 \leq d \leq n-1} \prod_{1 \leq t \leq m_d} [R_{t,d} : C^{(d)}] : (w(1), \cdots, w(d)) \uparrow\succeq R_{t,d} \right\}$$

See [LG01] for further details.

2.4. **Richardson variety** X_w^v. For $v \in \mathfrak{S}_n$, the opposite Schubert variety X^v is the Zariski closure of the B^--orbit $B^- \cdot e_v$ where B^- is the opposite Borel subgroup. Then for $w, v \in \mathfrak{S}_n$, the Richardson variety X_w^v is defined as

$$X_w^v = X_w \cap X^v.$$

In particular, this is non-empty if and only if $w \geq v$ with respect to the Bruhat-Chevalley order.

By applying standard monomial theory to the opposite Schubert variety, we can obtain standard monomials for X_w^v. This, via (1), follows directly from the case of the Richardson varieties in the Grassmannian. For the restriction map

$$H^0(G/B, \mathcal{O}_{\mathbf{m}}) \to H^0(X_w^v, \mathcal{O}_{\mathbf{m}}),$$

the following standard monomials in $H^0(G/B, \mathcal{O}_{\mathbf{m}})$ project to a $\mathbb{C}$-basis of $H^0(X_w^v, \mathcal{O}_{\mathbf{m}})$:

(7)
$$\left\{ \prod_{1 \leq d \leq n-1} \prod_{1 \leq t \leq m_d} [R_{t,d} : C^{(d)}] : (w(1), \cdots, w(d)) \succeq R_{t,d} \succeq (v(1), \cdots, v(d)) \right\}$$

See [LG01] for further details.

3. Bott-Samelson varieties

Let us review and generalize the results on the Bott-Samelson varieties given in [FK09].

3.1. **Reduced word i.** To obtain explicit descriptions, once and for all we fix the following reduced decomposition of the longest element in $\mathfrak{S}_n$:

$$(s_1)(s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_1)$$

where s_i is the simple reflection $(i, i+1)$ for $1 \leq i \leq n-1$, and fix the following word

(8)
$$\mathbf{i} = (i_1, \cdots, i_\ell)$$

associated with the above expression $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of the longest element. We write $w_{\mathbf{i}}$ for the longest element in $\mathfrak{S}_n$ and ℓ for the length of $w_{\mathbf{i}}$, which is $n(n-1)/2$.

3.2. **Bott-Samelson varieties.** Let us consider a word $\mathbf{j} = (j_1, \cdots, j_{\ell'})$ whose corresponding expression of the element $w_{\mathbf{j}} = s_{j_1} s_{j_2} \cdots s_{j_{\ell'}}$ in $\mathfrak{S}_n$ is reduced. The *Bott-Samelson variety* is the quotient space

$$Z_{\mathbf{j}} = P_{j_1} \times P_{j_2} \times \cdots \times P_{j_{\ell'}} / B^{\ell'}$$

where P_{j_i} is the minimal parabolic subgroup of G associated with the simple reflection s_{j_i}, and $B^{\ell'}$ acts on the product of P_{j_i}'s by

$$(p_1, \cdots, p_{\ell'}).(b_1, \cdots, b_{\ell'}) = (p_1 b_1, b_1^{-1} p_2 b_2, \cdots, b_{\ell'-1}^{-1} p_{\ell'} b_{\ell'}).$$

We can also realize the Bott-Samelson variety $Z_{\mathbf{j}}$ as a configuration variety [Ma98]:

(9)
$$Z_{\mathbf{j}} \subset Gr(\mathbf{j}) = Gr(j_1, n) \times \cdots \times Gr(j_{\ell'}, n)$$

More precisely, after realizing $\mathrm{Gr}(j_d, n)$ via $G/\widehat{P}_{j_d}$, the Bott-Samelson variety $Z_{\mathbf{j}}$ is the closure of the B-orbit of

$$z_{\mathbf{j}} = (s_{j_1}\widehat{P}_{j_1}, s_{j_1}s_{j_2}\widehat{P}_{j_2}, ..., s_{j_1}s_{j_2}\cdots s_{j_{\ell'}}\widehat{P}_{j_{\ell'}})$$

Then for $\mathbf{m} = (m_1, \cdots, m_{\ell'}) \in \mathbb{Z}_{>0}^{\ell'}$, we can consider a natural line bundle $L_{\mathbf{j},\mathbf{m}}$ induced from the Plücker bundles over $\mathrm{Gr}(j_i, n)$.

3.3. **Homogeneous coordinate ring of $Z_{\mathbf{j}}$.** For any $\mathbf{m} = (m_1, \cdots, m_{\ell'}) \in \mathbb{Z}_{>0}^{\ell'}$, with the realization of $Z_{\mathbf{j}}$ given in (9), the section ring of $Z_{\mathbf{j}}$ with respect to $L_{\mathbf{j},\mathbf{m}}$:

$$\mathcal{R}_{\mathbf{j},\mathbf{m}} = \bigoplus_{p \geq 0} H^0(Z_{\mathbf{j}}, L_{\mathbf{j},\mathbf{m}}^{\otimes p})$$

can be identified with a ring generated by products of determinants defined by $\mathbf{j}$ and $\mathbf{m}$.

Definition 3.1. *The column sets attached to $\mathbf{j}$ are*

$$K_{\mathbf{j}}^{(r)} = s_{j_1}s_{j_2}\cdots s_{j_r}\{1, 2, \cdots, i_r\}$$

for $1 \leq r \leq \ell'$.

With the notation we set in (3), let us consider a multiset of determinants $[R_{t,r} : K_{\mathbf{j}}^{(r)}]$ whose column indices are given by the column sets $K_{\mathbf{j}}^{(r)}$. Then, by repeating $K_{\mathbf{j}}^{(r)}$'s m_r times, the product $\mathcal{T}$ of determinants $[R_{t,r} : K_{\mathbf{j}}^{(r)}]$ for $1 \leq t \leq m_r$ can be encoded by a filling of a $|\mathbf{m}| \times n$ rectangular diagram having $[R_{t,r} : K_{\mathbf{j}}^{(r)}]$ as its $(m_1 + \cdots + m_{r-1} + t)$-th row counting from the bottom, where $|\mathbf{m}| = \sum_i m_i$.

Definition 3.2. *A tableau $\mathcal{T}$ of shape $(\mathbf{j}, \mathbf{m})$ is*

$$(10) \qquad \mathcal{T} = \prod_{1 \leq t \leq m_1} [R_{t,1} : K_{\mathbf{j}}^{(1)}] \cdot \prod_{1 \leq t \leq m_2} [R_{t,2} : K_{\mathbf{j}}^{(2)}] \cdot ... \cdot \prod_{1 \leq t \leq m_{\ell'}} [R_{t,\ell'} : K_{\mathbf{j}}^{(\ell')}]$$

where for each r, all the row indexing sets satisfy $K_{\mathbf{j}}^{(r)} \succeq R_{t,r}$ for $1 \leq t \leq m_r$. Let $M(\mathbf{j}, \mathbf{m})$ be the space spanned by the tableaux of shape $(\mathbf{j}, \mathbf{m})$.

As given in (4), we will identify every tableau of shape $(\mathbf{j}, \mathbf{m})$ with a filling of a rectangular diagram of size $|\mathbf{m}| \times n$. Hence the entry in the cell (a, b) of a tableau $\mathcal{T}$ means the entry in the a-th row and b-th column in the diagram realization of $\mathcal{T}$ counting from bottom to top and left to right respectively.

Proposition 3.3 (§3 [Ma98]). *For $\mathbf{m} = (m_1, \cdots, m_{\ell'}) \in \mathbb{Z}_{>0}^{\ell'}$, the section space $H^0(Z_{\mathbf{j}}, L_{\mathbf{j},\mathbf{m}})$ of $Z_{\mathbf{j}}$ is isomorphic to the space spanned by the tableaux of shape $(\mathbf{j}, \mathbf{m})$, i.e.,*

$$H^0(Z_{\mathbf{j}}, L_{\mathbf{j},\mathbf{m}}) \cong M(\mathbf{j}, \mathbf{m}).$$

3.4. **Straight tableaux for** $\mathcal{R}_{\mathbf{i},\mathbf{m}}$. In the special case of $\mathbf{j} = \mathbf{i}$, each column set $K_{\mathbf{i}}^{(r)}$ contains consecutive integers, and this fact lets us realize tableaux of shape $(\mathbf{i}, \mathbf{m})$ in the context of tableaux of a *row-convex shape* which is defined in [Ta01] as a generalized skew Young tableaux. Using this observation, [FK09] gives a presentation of the section ring

$$\mathcal{R}_{\mathbf{i},\mathbf{m}} = \bigoplus_{p \geq 0} M(\mathbf{i},p\mathbf{m})$$

in terms of tableaux, and then identified an explicit basis. Note that up to sign, we can always assume that the entries in each row of $\mathcal{T}$ are increasing from left to right. If such is the case, then $\mathcal{T}$ is called a *row-standard tableau*.

Definition 3.4. *A row-standard tableau $\mathcal{T}$ of shape $(\mathbf{i}, \mathbf{m})$ is called a straight tableau, if $\mathcal{T}$ as a $|\mathbf{m}| \times n$ tableau (10) satisfies the following condition: for two cells (i, k) and (j, k) with $i < j$ in the same column, the entry in the cell (i, k) may be strictly larger than the entry in (j, k) only if the cell $(i, k-1)$ exists and contains an entry weakly larger than the one in the cell (j, k).*

Theorem 3.5 ([FK09]). *Straight tableaux form a $\mathbb{C}$-basis for the $\mathbb{Z}$-graded algebra $\mathcal{R}_{\mathbf{i},\mathbf{m}}$. In particular, straight tableaux of shape $(\mathbf{i},p\mathbf{m})$ form a $\mathbb{C}$-basis for the section space $M(\mathbf{i},p\mathbf{m})$.*

Our next task is to extend the above result to $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ for a subword $\mathbf{j}$ of $\mathbf{i}$.

3.5. **Relative description of** $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ **to** $\mathcal{R}_{\mathbf{i},\mathbf{m}}$. For a subword $\mathbf{j}$ of $\mathbf{i}$, we can study a relative description of $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ to $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ by using the canonical embedding $Z_{\mathbf{j}} \subset Z_{\mathbf{i}}$:

$$\begin{array}{ccc} \mathrm{Gr}(\mathbf{j}) & \longrightarrow & \mathrm{Gr}(\mathbf{i}) \\ \uparrow & & \uparrow \\ Z_{\mathbf{j}} & \longrightarrow & Z_{\mathbf{i}} \end{array}$$

See (9) for notation.

For this purpose, in what follows, we write a subword $\mathbf{j}$ of $\mathbf{i}$ as $(j_1, \cdots, j_\ell)$ by adopting the convention of using 0 for the omitted letters. Its associated element $w_{\mathbf{j}}$ in $\mathfrak{S}_n$ is $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ with s_0 being the identity in $\mathfrak{S}_n$. We further assume that $\mathbf{j}$ is reduced. For the rest of our discussion, if $j_r = 0$, then we assume the corresponding object is considered to be omitted or a trivial one. For example, in a product of the Grassmannians $\mathrm{Gr}(j_1, n) \times \mathrm{Gr}(j_2, n) \times \cdots \times \mathrm{Gr}(j_\ell, n)$, if $j_r = 0$ then we omit $\mathrm{Gr}(j_r, n)$ and therefore the column set $K_{\mathbf{j}}^{(r)} = \emptyset$. Also, in $\mathbf{m} = (m_1, \cdots, m_\ell)$ attached to $\mathbf{j}$, we assume $m_r = 0$ if $j_r = 0$.

Let us consider the restriction map $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Z_{\mathbf{j}}, L_{\mathbf{j},\mathbf{m}})$, or more explicitly the following map $\phi : M(\mathbf{i}, \mathbf{m}) \to M(\mathbf{j}, \mathbf{m})$

(11)
$$\phi \left(\prod_{1 \le t \le m_1} [R_{t,1} : K_{\mathbf{i}}^{(1)}] \cdot \ldots \cdot \prod_{1 \le t \le m_\ell} [R_{t,\ell} : K_{\mathbf{i}}^{(\ell)}] \right)$$
$$= \prod_{1 \le t \le m_1} \phi_1([R_{t,1} : K_{\mathbf{i}}^{(1)}]) \cdot \ldots \cdot \prod_{1 \le t \le m_\ell} \phi_\ell([R_{t,\ell} : K_{\mathbf{i}}^{(\ell)}])$$

where $\phi_r([R_{t,r} : K_{\mathbf{i}}^{(r)}]) = [R_{t,r} : K_{\mathbf{j}}^{(r)}]$ for r such that $j_r = i_r$; and $\phi_r([R_{t,r} : K_{\mathbf{i}}^{(r)}]) = 1$ for r such that $j_r = 0$. Then ϕ is surjective and its kernel is the union of the kernels of ϕ_r's.

Proposition 3.6. *i) The kernel of the restriction map* $\phi : H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Z_{\mathbf{j}}, L_{\mathbf{j},\mathbf{m}})$ *is spanned by*

$$\left\{ \prod_{1 \le t \le m_1} [R_{t,1} : K_{\mathbf{i}}^{(1)}] \cdot \ldots \cdot \prod_{1 \le t \le m_\ell} [R_{t,\ell} : K_{\mathbf{i}}^{(\ell)}] \in M(\mathbf{i}, \mathbf{m}) : R_{t,r} \not\preceq K_{\mathbf{j}}^{(r)} \text{ for } r \text{ such that } j_r = i_r \right\}$$

ii) The straight tableaux of shape $(\mathbf{i}, \mathbf{m})$ *such that* $R_{t,r} \preceq K_{\mathbf{j}}^{(r)}$ *for* r *such that* $j_r = i_r$ *project to a* $\mathbb{C}$*-basis of the space* $H^0(Z_{\mathbf{j}}, L_{\mathbf{j},\mathbf{m}})$.

Proof. From the embedding $Z_{\mathbf{j}} \subset Gr(\mathbf{j})$ in (9), it is enough to check the statements for the individual factors of $Gr(\mathbf{j})$. To each factor $G/\widehat{P}_{j_d}$ of $Gr(\mathbf{j})$, the Bott-Samelson variety $Z_{\mathbf{j}}$ projects to a Schubert variety, because it is the closure of the B orbit of the T-invariant element $s_{j_1} s_{j_2} \cdots s_{j_r} \widehat{P}_{j_r}$ in $G/\widehat{P}_{j_r}$. Then the statements follow from the standard monomial description for the section space of Schubert varieties given in §2.3 and Theorem 3.5. $\square$

The map ϕ naturally extends to Φ from $\mathcal{R}_{\mathbf{i},\mathbf{m}} = \bigoplus_{p \ge 0} M(\mathbf{i}, p\mathbf{m})$ to $\mathcal{R}_{\mathbf{j},\mathbf{m}}$. Therefore, the above description gives the relative description of $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ to $\mathcal{R}_{\mathbf{i},\mathbf{m}}$. From now on, we identify the space $M(\mathbf{j}, \mathbf{m})$ with the quotient space

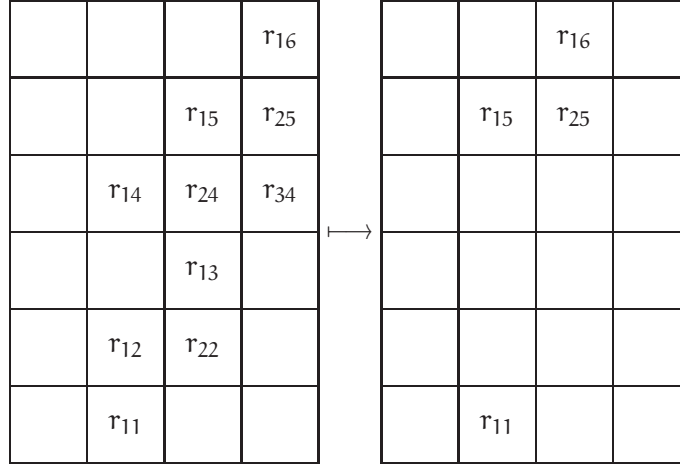$$M(\mathbf{j}, \mathbf{m}) = M(\mathbf{i}, \mathbf{m}) / \ker \phi.$$

Example 3.7. *For* $n = 4$, $\mathbf{i} = (1, 2, 1, 3, 2, 1)$ *and the column sets* $K_{\mathbf{i}}^{(r)}$ *are*

$$K_{\mathbf{i}}^{(1)} = \{2\}, K_{\mathbf{i}}^{(2)} = \{2, 3\}, K_{\mathbf{i}}^{(3)} = \{3\},$$
$$K_{\mathbf{i}}^{(4)} = \{2, 3, 4\}, K_{\mathbf{i}}^{(5)} = \{3, 4\}, K_{\mathbf{i}}^{(6)} = \{4\}.$$

For the subword $\mathbf{j} = (1, 0, 0, 0, 2, 1)$, *its column sets* $K_{\mathbf{j}}^{(r)}$ *are*

$$K_{\mathbf{j}}^{(1)} = \{2\}, K_{\mathbf{j}}^{(2)} = \emptyset, K_{\mathbf{j}}^{(3)} = \emptyset,$$
$$K_{\mathbf{j}}^{(4)} = \emptyset, K_{\mathbf{j}}^{(5)} = \{2, 3\}, K_{\mathbf{j}}^{(6)} = \{3\}.$$

With $\mathbf{m} = (1,1,1,1,1,1)$, the surjection Φ sends tableaux shape of $(\mathbf{i}, \mathbf{m})$ to tableaux of shape $(\mathbf{j}, \mathbf{m})$ as follows:

			r_{16}
		r_{15}	r_{25}
	r_{14}	r_{24}	r_{34}
		r_{13}	
	r_{12}	r_{22}	
	r_{11}		

$\longmapsto$

		r_{16}	
	r_{15}	r_{25}	
	r_{11}		

Then the kernel is spanned by the tableaux with $(r_{15}, r_{25}) \nleq (2,3)$ and $(r_{11}) \nleq (2)$. The second condition in this case is void.

4. RR Varieties

4.1. Involution $w_{\mathbf{i}}$. Let us state parallel results for opposite Schubert varieties and the corresponding Bott-Samelson varieties. Fix a subword $\mathbf{j} = (j_1, \cdots, j_\ell)$ of $\mathbf{i}$ such that the corresponding expression of the element $w_{\mathbf{j}} = s_{j_1} s_{j_2} \cdots s_{j_\ell}$ in $\mathfrak{S}_n$ is reduced. Then we define the corresponding opposite Schubert variety as

$$X^{\mathbf{j}} = \overline{B^- \cdot e_v}$$

where $v = w_{\mathbf{i}} w_{\mathbf{j}}$, and its corresponding Bott-Samelson variety $Z^{\mathbf{j}}$ as the closure of B^- orbit in the product of the Grassmannians $\mathrm{Gr}(j_r, n)$ as in (9).

The section space of the line bundle $\widetilde{L}_{\mathbf{j},\mathbf{m}}$ over $Z^{\mathbf{j}}$ can be obtained by applying the involution $w_{\mathbf{i}}$ to the line bundle $L_{\mathbf{j},\mathbf{m}}$ over $Z_{\mathbf{j}}$. A row-standard tableau $\mathcal{T}$ of the form (10) is sent to

$$w_{\mathbf{i}}(\mathcal{T}) = \left(\prod_{m_\ell \geq t \geq 1} [\widetilde{R}_{t,\ell} : \widetilde{K}_{\mathbf{j}}^{(\ell)}] \right) \cdot \left(\prod_{m_{\ell-1} \geq t \geq 1} [\widetilde{R}_{t,\ell-1} : \widetilde{K}_{\mathbf{j}}^{(\ell-1)}] \right) \cdot \ldots \cdot \left(\prod_{m_1 \geq t \geq 1} [\widetilde{R}_{t,1} : \widetilde{K}_{\mathbf{j}}^{(1)}] \right)$$

where for a subset $X = \{x_1 < \cdots < x_r\}$ of $\{1, \cdots, n\}$,

$$\widetilde{X} = \{n + 1 - x_r, \cdots, n + 1 - x_1\}.$$

Note that we reverse the order of multiplication. In terms of our diagram notation (4), $w_{\mathbf{i}}(\mathcal{T})$ is obtained by rotating $\mathcal{T}$ by $180°$ and then replacing row indexing entries r_{ij} by $n + 1 - r_{ij}$.

Example 4.1. *The surjection in Example 3.7, via the involution* $w_{\mathbf{i}}$, *corresponds the restriction map from* $H^0(Z^{\mathbf{i}}, \widetilde{L}_{\mathbf{i},\mathbf{m}})$ *to* $H^0(Z^{\mathbf{j}}, \widetilde{L}_{\mathbf{j},\mathbf{m}})$:

		r'_{11}		
	r'_{22}	r'_{12}		
	r'_{13}			
r'_{34}	r'_{24}	r'_{14}		
r'_{25}	r'_{15}			
r'_{16}				

$\longmapsto$

		r'_{11}	
r'_{25}	r'_{15}		
r'_{16}			

where $r'_{ij} = n + 1 - r_{ij}$.

Then it follows from Proposition 3.6 that

Corollary 4.2. *For* $\mathbf{m} = (m_1, \cdots, m_\ell) \in \mathbb{Z}^\ell_{>0}$ *attached to* $\mathbf{j}$, *we have*

$$H^0(Z^{\mathbf{j}}, \widetilde{L}_{\mathbf{j},\mathbf{m}}) \cong w_{\mathbf{i}}(M(\mathbf{j}, \mathbf{m})),$$

and for the straight tableaux $\mathcal{T}$ *of shape* $(\mathbf{j}, \mathbf{m})$, $w_{\mathbf{i}}(\mathcal{T})$ *project to a* $\mathbb{C}$-*basis of* $H^0(Z^{\mathbf{j}}, \widetilde{L}_{\mathbf{j},\mathbf{m}})$.

4.2. Fiber Product. For $\mathbf{m} = (m_1, \cdots, m_\ell) \in \mathbb{Z}^\ell_{>0}$, let us consider the homogeneous coordinate rings $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ and $\mathcal{R}^{\mathbf{k},\mathbf{m}}$ of $Z_{\mathbf{j}}$ and $Z^{\mathbf{k}}$ respectively

$$\mathcal{R}_{\mathbf{j},\mathbf{m}} = \bigoplus_{p \geq 0} M(\mathbf{j}, p\mathbf{m})$$

$$\mathcal{R}^{\mathbf{k},\mathbf{m}} = \bigoplus_{p \geq 0} w_{\mathbf{i}}(M(\mathbf{k}, p\mathbf{m}))$$

Then by taking the last $(n-1)$ entries of $\mathbf{m}$, we set $\mathbf{m}_0 = (m_{\ell-n+2}, \cdots, m_\ell)$. Then let $\mathcal{A} = \mathcal{A}^v_w = \bigoplus_{p \geq 0} H^0(X^v_w, \mathcal{O}^{\otimes p}_{\mathbf{m}_0})$ be the homogeneous coordinate ring, given in (7), of the Richardson variety $X^v_w = X_w \cap X^v$ where $w = w_{\mathbf{j}}$ and $v = w_{\mathbf{i}} w_{\mathbf{k}}$.

We define the coproduct of $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ and $\mathcal{R}^{\mathbf{k},\mathbf{m}}$ over $\mathcal{A}$

$$\mathcal{R}^{\mathbf{k}}_{\mathbf{j}} = \mathcal{R}_{\mathbf{j},\mathbf{m}} \otimes_{\mathcal{A}} \mathcal{R}^{\mathbf{k},\mathbf{m}}$$

with respect to the following injective maps: $\varphi_{\mathbf{j}} : H^0(X^v_w, \mathcal{O}_{\mathbf{m}_0}) \to M(\mathbf{j}, \mathbf{m})$ sending $\mathcal{T}$ to

$$\varphi_{\mathbf{j}}(\mathcal{T}) = \left(\prod_{1 \leq t \leq m_1} [R^0_{t,1} : K^{(1)}_{\mathbf{j}}] \cdot \dots \cdot \prod_{1 \leq t \leq m_{\ell+1-n}} [R^0_{t,\ell+1-n} : K^{(\ell+1-n)}_{\mathbf{j}}] \right) \cdot \mathcal{T}$$

in the quotient $M(\mathbf{j}, \mathbf{m}) = M(\mathbf{i}, \mathbf{m})/\ker \phi$ where $R_{t,r}^0 = \{1, 2, \cdots, |K_{\mathbf{j}}^{(r)}|\}$ for $1 \leq r \leq \ell + 1 - n$, and

$$\varphi^{\mathbf{k}} : H^0(X_w^v, \mathcal{O}_{\mathbf{m}_0}) \to w_{\mathbf{i}}(M(\mathbf{k}, \mathbf{m}))$$

defined by $\varphi^{\mathbf{k}}(\mathcal{T}) = w_{\mathbf{i}}(\varphi_{\mathbf{j}}(\mathcal{T}))$. Note that these maps correspond to the projections from the Bott-Samelson varieties to the flag varieties (cf. [FK09, §4.3]).

With $(\mathcal{R}_{\mathbf{j}, \mathbf{m}}, \mathcal{R}^{\mathbf{k}, \mathbf{m}}, \mathcal{A}_w^v, \varphi_{\mathbf{j}}, \varphi^{\mathbf{k}})$, we define the *RR variety* as the fiber product of $Z_{\mathbf{j}}$ and $Z^{\mathbf{k}}$ over X_w^v.

4.3. Toric Degenerations.

A monomial order on the polynomial ring $\mathbb{C}[M_n]$ is called a *diagonal term order* if the leading monomial of a determinant of any minor over M_n is equal to the product of the diagonal elements. For a subring $\mathcal{R}$ of the polynomial ring we let $\text{in}(\mathcal{R})$ denote the algebra generated by the leading monomials $\text{in}(f)$ of all $f \in \mathcal{R}$ with respect to a given monomial order. The leading monomials for all $f \in \mathcal{R}$ form an affine semigroup, therefore $\text{in}(\mathcal{R})$ is a semigroup algebra representing a toric variety in the sense of [Stu95].

In [Ta01], it is shown that for a row-convex shape $(\mathbf{h}, \mathbf{m})$, straight tableaux of shape $(\mathbf{h}, \mathbf{m})$ form a SAGBI basis of the graded algebra $\mathcal{R} \subset \mathbb{C}[M_n]$ generated by tableaux of shape $(\mathbf{h}, \mathbf{m})$ with respect to any diagonal term order. A finite SAGBI basis for $\mathcal{R}$ provides a toric degeneration of $\text{Spec}(\mathcal{R})$ to $\text{Spec}(\text{in}(\mathcal{R}))$ or $\text{Proj}(\mathcal{R})$ to $\text{Proj}(\text{in}(\mathcal{R}))$ if there is a $\mathbb{Z}$-grading. This is called a *SAGBI-degeneration* (cf. [MS05, p.281][CHV96, Theorem 1.2]). Using this method, [FK09] shows $Z_{\mathbf{i}}$ can be flatly deformed into a toric variety. For $Z_{\mathbf{j}}$, we will apply an analogous method to the quotient algebra $\mathcal{R}_{\mathbf{j}, \mathbf{m}} = \mathcal{R}_{\mathbf{i}, \mathbf{m}}/\ker \Phi$.

Theorem 4.3. *The Bott-Samelson variety $Z_{\mathbf{j}}$ can be flatly deformed into a toric variety.*

Proof. We show that there is a flat $\mathbb{C}[t]$ module $\mathcal{R}_{\mathbf{j}, \mathbf{m}}^t$ whose general fiber is isomorphic to $\mathcal{R}_{\mathbf{j}, \mathbf{m}}$ and special fiber is isomorphic to a semigroup ring corresponding to an initial object of $\mathcal{R}_{\mathbf{j}, \mathbf{m}}$. To specify this initial object of the quotient algebra, we will use the fact that every element of $\mathcal{R}_{\mathbf{j}, \mathbf{m}}$ has a cannonical representative. From Proposition 3.6, every homogeneous element H of the quotient $\mathcal{R}_{\mathbf{i}, \mathbf{m}}/\ker \Phi$ can be expressed as a linear combination of straight tableaux of the same shape

$$H = \sum c_i \mathcal{T}_i$$

From the fact that leading monomials of straight tableaux of a fixed shape are distinct ([Ta01]), the leading monomial of f should be equal to the leading monomial of $\mathcal{T}_i$ for some i. Therefore, we have a well defined notion of the leading monomials $\text{in}(H)$ for H in $\mathcal{R}_{\mathbf{i}, \mathbf{m}}/\ker \Phi$ and it is equal to $\text{in}(\mathcal{T})$ for a straight tableau $\mathcal{T}$. In this sense, straight tableaux form a SAGBI basis for $\mathcal{R}_{\mathbf{i}, \mathbf{m}}/\ker \Phi$, i.e., the semigroup ring $\text{in}(\mathcal{R}_{\mathbf{i}, \mathbf{m}}/\ker \Phi)$ is generated by the leading monomials of straight tableaux. Since for any shape $(\mathbf{j}, \mathbf{m})$ we only have a finite number of straight tableaux, straight tableaux form a finite SAGBI basis for $\mathcal{R}_{\mathbf{i}, \mathbf{m}}/\ker \Phi$. Then, by the same argument given in [ST99, Proposition 1], we

have a $\mathbb{Z}_{\geq 0}$-filtration $\{\mathsf{F}_\alpha\}$ on $\mathcal{R}_{\mathbf{j},\mathbf{m}} \cong \mathcal{R}_{\mathbf{i},\mathbf{m}}/\ker \Phi$ such that the Rees algebra $\mathcal{R}^{\mathsf{t}}_{\mathbf{j},\mathbf{m}}$ of $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ with respect to $\{\mathsf{F}_\alpha\}$:

$$\mathcal{R}^{\mathsf{t}}_{\mathbf{j},\mathbf{m}} = \bigoplus_{\alpha \geq 0} \mathsf{F}_\alpha(\mathcal{R}_{\mathbf{j},\mathbf{m}}) \mathsf{t}^\alpha$$

which is flat over $\mathbb{C}[\mathsf{t}]$, has a general fiber isomorphic to $\mathcal{R}_{\mathbf{j},\mathbf{m}}$ and the special fiber isomorphic to the semigroup ring $\mathrm{in}(\mathcal{R}_{\mathbf{i},\mathbf{m}}/\ker \Phi)$. $\square$

Corollary 4.4. *The RR variety can be flatly deformed into a fiber product of toric varieties.*

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