

Recurrence in 2D Inviscid Channel Flow

Y. Charles Li

ABSTRACT. I will prove a recurrence theorem which says that any H^s ($s > 2$) solution to the 2D inviscid channel flow returns repeatedly to an arbitrarily small H^0 neighborhood. Periodic boundary condition is imposed along the stream-wise direction. The result is an extension of an early result of the author [Li, 09] on 2D Euler equation under periodic boundary conditions along both directions.

1. Introduction

The recurrent nature of the solutions to the 2D Euler equation depends crucially upon the geometry of the domain containing the fluid. Here the specific geometry (rather than topology) is fundamental. We are interested in three types of domains:

- Periodic Cell $D_p = [0, \ell_x] \times [0, \ell_y]$, which is a period cell in \mathbb{R}^2 where the fluid flow is under periodic boundary conditions in both x and y directions with periods ℓ_x and ℓ_y .
- Channel Cell $D_c = [0, \ell_x] \times [0, a]$, which is a channel cell of the channel $\mathbb{R} \times [0, a]$ where the fluid flow is under periodic boundary condition in x direction with period ℓ_x , and non-penetrating condition along the walls $y = 0, a$.
- Annulus $D_a = \{(r, \theta) \mid R_1 \leq r \leq R_2\}$ in polar coordinates, where the fluid flow is under non-penetrating condition along the walls $r = R_1, R_2$.

For the fluid domain D_p , the solutions to the 2D Euler equation are recurrent somewhere along the orbit in the L^2 norm of velocity [3]. Below we will show that this is also true for the fluid domain D_c . For the fluid domain D_a , we believe that this is no longer true. In the D_a case, the enstrophy is not equivalent to the $W^{1,2}$ norm of velocity. For example,

$$(1.1) \quad (u, v) = \frac{1}{x^2 + y^2}(-y, x)$$

is an irrotational steady solution of the 2D Euler equation. Its vorticity is zero, while its $W^{1,2}$ norm is non-zero. The argument of [3] for recurrence depends crucially upon the equivalence of the enstrophy to the $W^{1,2}$ norm of velocity.

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On the other hand, for the fluid domain D_a , there is an initial condition such that for any other initial condition in a C^1 vorticity neighborhood, the solution never returns to the C^1 vorticity neighborhood [4] [5]. The inner and outer circles of the annulus play a fundamental role in the the construction of this example. It is the multiconnectness nature of the annulus that makes the example possible. The irrotational steady solution (1.1) is also fundamental in the construction of the example. For the D_p and D_c cases, we do not believe the non-recurrent example of the D_a case exists, since the irrotational solution (1.1) does not exist.

The recurrence in L^2 norm of velocity is a result of two main ingredients: (1). the conservation of energy and enstrophy, (2). the equivalence of the enstrophy to the $W^{1,2}$ norm of velocity. The first ingredient is also credited for generating the well-studied phenomenon, the so-called forward energy cascade and inverse enstrophy cascade, see e.g. [6].

2. Recurrence in 2D Inviscid Channel Flow

The 2D inviscid channel flow is governed by

$$(2.1) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x},$$

$$(2.2) \quad \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y},$$

$$(2.3) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;$$

subject to the boundary condition

$$(2.4) \quad v = 0 \quad \text{at } y = a, b.$$

Here we also pose periodic boundary condition along x -direction (of period L_x). Thus

$$(2.5) \quad \begin{aligned} u &= \sum_{n=-\infty}^{+\infty} u_n(y) e^{in\alpha x}, & v &= \sum_{n=-\infty}^{+\infty} v_n(y) e^{in\alpha x}, \\ p &= \sum_{n=-\infty}^{+\infty} p_n(y) e^{in\alpha x}; \end{aligned}$$

where $\alpha = 2\pi/L_x$. Our first goal here is to reach a setting that the spatial averages of u and v are zero. This setting is not necessary for proving the recurrence, but it is an interesting fact itself. By the incompressibility condition (2.3),

$$\frac{\partial v_0}{\partial y} = 0.$$

Using the boundary condition (2.4), one gets

$$(2.6) \quad v_0(y) = 0.$$

Thus the spatial average of v is zero. Denote by $\langle u \rangle$ the spatial average of u

$$\langle u \rangle = \frac{1}{(b-a)L_x} \int_a^b \int_0^{L_x} u \, dx dy = \frac{1}{b-a} \int_a^b u_0(y) \, dy.$$

The inviscid channel flow (2.1)-(2.3) together with its boundary conditions are invariant under the transformation:

$$u = \tilde{u} + \langle u \rangle, \quad \tilde{x} = x - \langle u \rangle t.$$

Together with (2.6), without loss of generality, we can assume that the spatial averages of u and v are zero. A simple calculation shows that the kinetic energy and enstrophy

$$E = \int_a^b \int_0^{L_x} (u^2 + v^2) \, dx dy, \quad G = \int_a^b \int_0^{L_x} \omega^2 \, dx dy$$

where $\omega = \partial_x v - \partial_y u$ is the vorticity, are invariant under the inviscid channel flow. Next we will show that G is equivalent to the $W^{1,2}$ norm of velocity.

Lemma 2.1.

$$\int_a^b \int_0^{L_x} [(\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2] \, dx dy = G.$$

PROOF. By the incompressibility condition (2.3),

$$\int_a^b \int_0^{L_x} [(\partial_x u)^2 + (\partial_y v)^2] \, dx dy = -2 \int_a^b \int_0^{L_x} (\partial_x u)(\partial_y v) \, dx dy.$$

Notice that

$$\begin{aligned} & \int_a^b \int_0^{L_x} [(\partial_x u)(\partial_y v) - (\partial_y u)(\partial_x v)] \, dx dy \\ &= \int_a^b \int_0^{L_x} [\partial_y(v \partial_x u) - \partial_x(v \partial_y u)] \, dx dy = 0, \end{aligned}$$

where the first term vanishes due to the boundary condition (2.4) and the second term vanishes due to the periodicity along x -direction. Using the above two facts, we have

$$\begin{aligned} & \int_a^b \int_0^{L_x} [(\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2] \, dx dy \\ &= \int_a^b \int_0^{L_x} [(\partial_y u)^2 + (\partial_x v)^2 - 2(\partial_x u)(\partial_y v)] \, dx dy \\ &= \int_a^b \int_0^{L_x} [(\partial_y u)^2 + (\partial_x v)^2 - 2(\partial_y u)(\partial_x v)] \, dx dy = G. \end{aligned}$$

□

Lemma 2.2. For any $C > 0$, the set

$$S = \left\{ \vec{v} = (u, v) \mid G = \int_a^b \int_0^{L_x} \omega^2 \, dx dy \leq C \right\}$$

is compactly embedded in $L^2(D_c)$ of \vec{v} , $D_c = [0, L_x] \times [a, b]$. That is, the closure of S in $L^2(D_c)$ is a compact subset of $L^2(D_c)$.

Remark 2.3. By the fact (2.6),

$$\|v\|_{L^2(D_c)} \leq C\|\nabla v\|_{L^2(D_c)}.$$

By the Poincaré inequality for $u_0(y)$ and the fact that the y -average of $u_0(y)$ is zero,

$$\|u\|_{L^2(D_c)} \leq C\|\nabla u\|_{L^2(D_c)}.$$

Without the facts of (2.6) and the vanishing of the y -average of $u_0(y)$, we can still carry out the rest argument of the paper by replacing G with $E + G$ in the above definition of S . The above lemma is the well-known Rellich lemma. In a simpler language, an enstrophy ball is compactly embedded in the kinetic energy space. The proof below follows that of [1]. \square

PROOF. By Lemma 2.1,

$$\int_a^b \int_0^{L_x} |\nabla \vec{v}|^2 dx dy \leq C.$$

Starting from the representation (2.5), we expand $\vec{v}_n(y)$ into a Fourier integral

$$\vec{v}_n(y) = \int_{-\infty}^{+\infty} \vec{v}_{n\xi} e^{i\xi y} d\xi,$$

where (ignoring a constant factor)

$$\vec{v}_{n\xi} = \int_a^b \vec{v}_n(y) e^{-i\xi y} dy.$$

Let $\{\vec{v}^k\}$ be a sequence in S .

$$\partial_\xi \vec{v}_{n\xi}^k = \int_a^b (-iy) \vec{v}_n^k(y) e^{-i\xi y} dy.$$

Thus

$$|\vec{v}_{n\xi}^k|, |\partial_\xi \vec{v}_{n\xi}^k| \leq C_1 \|\vec{v}^k\|_{L^2(D_c)},$$

where C_1 is a constant only dependent on a and b . For any n , by the Arzelà-Ascoli theorem, there is a subsequence $\{\vec{v}_n^{k_j}\}$ which converges uniformly on compact sets of ξ . For a different n , we can start from the subsequence $\{\vec{v}_n^{k_j}\}$ and find a further uniformly convergent subsequence. Thus there is a uniformly convergent subsequence $\{\vec{v}_{n\xi}^{k_j}\}$ on $|\xi| \leq A$, $|n| \leq N$ for any finite A and N . Next we show that

$\{\vec{v}^k\}$ forms a Cauchy sequence in $L^2(D_c)$.

$$\begin{aligned}
 \|\vec{v}^{k_j} - \vec{v}^{k_m}\|_{L^2(D_c)}^2 &= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 d\xi \\
 &= \sum_{|n| \leq N} \int_{-A}^{+A} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 d\xi + \sum_{|n| \leq N} \int_{|\xi| > A} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 d\xi \\
 &+ \sum_{|n| > N} \int_{-\infty}^{+\infty} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 d\xi \leq \sup_{|n| \leq N, |\xi| \leq A} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 4NA \\
 &+ A^{-2} \sum_{|n| \leq N} \int_{|\xi| > A} |\xi|^2 |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 d\xi \\
 &+ N^{-2} \sum_{|n| > N} n^2 \int_{-\infty}^{+\infty} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 d\xi \\
 &\leq 4NA \sup_{|n| \leq N, |\xi| \leq A} |\vec{v}_{n\xi}^{k_j} - \vec{v}_{n\xi}^{k_m}|^2 + (A^{-2} + N^{-2}) \|\vec{v}^{k_j} - \vec{v}^{k_m}\|_{H^1(D)}.
 \end{aligned}$$

For any $\epsilon > 0$, choose A and N large enough so that the second term is less than $\epsilon/2$ for all j and m . Then choose j and m large enough so that the first term is less than $\epsilon/2$. Thus, $\{\vec{v}^k\}$ forms a Cauchy sequence in $L^2(D_c)$. Finally let $\{\vec{v}^k\}$ be a sequence of the accumulation points of S in $L^2(D_c)$. Then we can find a sequence $\{\vec{u}^k\}$ in S such that

$$\|\vec{v}^k - \vec{u}^k\|_{L^2(D_c)} < 1/k.$$

Let $\{\vec{u}^{k_j}\}$ be the convergent subsequence, then $\{\vec{v}^{k_j}\}$ is also a convergent subsequence. Thus the closure of S in $L^2(D_c)$ is a compact subset of $L^2(D_c)$. \square

The inviscid channel flow (2.1)-(2.3), together with the boundary condition (2.4) and periodicity along x -direction, is globally well-posed in $H^s(D_c)$ ($s > 2$) [8] [7] [2]. We have the following recurrence theorem.

Theorem 2.4. *For any $\vec{v}_0 \in H^s(D_c)$ ($s > 2$), any $\delta > 0$, and any $T > 0$; there is a $\vec{v}_* \in H^s(D_c)$ such that*

$$F^{m_j T}(\vec{v}_0) \in B_\delta^0(\vec{v}_*) = \{\vec{v} \in H^s(D_c) \mid \|\vec{v} - \vec{v}_*\|_{H^0(D_c)} < \delta\}$$

where $\{m_j\}$ is an infinite sequence of positive integers, and F^t is the evolution operator of the inviscid channel flow.

PROOF. Choose the C in Lemma 2.2 to be

$$2 \int_a^b \int_0^{L_x} |\nabla \vec{v}_0|^2 dx dy = 2 \int_a^b \int_0^{L_x} \omega_0^2 dx dy.$$

Define two sets:

$$\begin{aligned}
 S &= \left\{ \vec{v} \mid \int_a^b \int_0^{L_x} \omega^2 dx dy \leq 2 \int_a^b \int_0^{L_x} \omega_0^2 dx dy \right\}, \\
 S_1 &= \left\{ \vec{v} \in H^s(D_c) \mid \int_a^b \int_0^{L_x} \omega^2 dx dy \leq 2 \int_a^b \int_0^{L_x} \omega_0^2 dx dy \right\}.
 \end{aligned}$$

Notice that S_1 is invariant under the 2D inviscid channel flow, and S_1 is a dense subset of S , $S_1 = S \cap H^s(D_c)$. By Lemma 2.2, the closure of S in $L^2(D_c) = H^0(D_c)$ is a compact subset. For any $\vec{v} \in S$, denote by

$$B_{\delta/2}(\vec{v}) = \{\vec{u} \in H^0(D_c) \mid \|\vec{u} - \vec{v}\|_{H^0(D_c)} < \delta/2\}.$$

All these balls $\{B_{\delta/2}(\vec{v})\}_{\vec{v} \in S}$ is an open cover of the closure of S in $H^0(D_c)$. Thus there is a finite subset $\{\vec{v}_1, \dots, \vec{v}_N\} \subset S$ such that $\{B_{\delta/2}(\vec{v}_n)\}_{n=1, \dots, N}$ is a finite cover. Since S_1 is dense in S , for each such \vec{v}_n , one can find a $\vec{v}_n^* \in S_1$ such that

$$\|\vec{v}_n - \vec{v}_n^*\|_{H^0(D_c)} \leq \|\vec{v}_n - \vec{v}_n^*\|_{W^{1,2}(D_c)} < \delta/4,$$

by the Poincaré inequality, where again the $W^{1,2}(D)$ norm is equivalent to the vorticity L^2 norm in S , by Lemma 2.1. All the balls

$$B_\delta(\vec{v}_n^*) = \{\vec{v} \in H^0(D_c) \mid \|\vec{v} - \vec{v}_n^*\|_{H^0(D_c)} < \delta\}$$

still covers S , thus covers $S_1 = S \cap H^s(D_c)$. Let $B_\delta^0(\vec{v}_n^*) = B_\delta(\vec{v}_n^*) \cap H^s(D_c)$,

$$B_\delta^0(\vec{v}_n^*) = \{\vec{v} \in H^s(D_c) \mid \|\vec{v} - \vec{v}_n^*\|_{H^0(D_c)} < \delta\}.$$

Then

$$S_1 \subset \bigcup_{n=1}^N B_\delta^0(\vec{v}_n^*).$$

By the invariance of S_1 under the 2D inviscid channel flow F^t , there is at least one n such that an infinite subsequence of $\{F^{mT}(\vec{v}_0)\}_{m=0,1,\dots}$ is included in $B_\delta^0(\vec{v}_n^*)$. The theorem is proved. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA
E-mail address: liyan@missouri.edu