COUPLING SOLUTIONS OF BGG-EQUATIONS IN CONFORMAL SPIN GEOMETRY

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ABSTRACT. BGG-equations are geometric overdetermined systems of PDEs on parabolic geometries. Normal solutions of BGG-equations are particularly interesting and we give a simple formula for the necessary and sufficient additional integrability conditions on a solution. We then discuss a procedure for coupling known solutions of BGG-equations to produce new ones. Employing a suitable calculus for conformal spin structures this yields explicit coupling formulas and conditions between almost Einstein scales, conformal Killing forms and twistor spinors. Finally we discuss a class of generic twistor spinors that provides an invariant decomposition of conformal Killing fields.

1. INTRODUCTION

Let M be a smooth manifold and $(\mathcal{G} \to M, \omega)$ a parabolic geometry of type (G, P). Here G a is semi-simple Lie group, $P \subset G$ a parabolic subgroup and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ the Cartan connection form of the geometry with values in the Lie algebra $\mathfrak{g}$ of G. Geometries of interest could for instance be projective structures, conformal structures or CR-structures. The Cartan connection form ω generalizes the properties of the Maurer-Cartan form $\omega^{MC} \in \Omega(G, \mathfrak{g})$ to the curved setting, [ČS09].

We are interested in overdetermined operators on such geometries which appear as the first operators in the BGG-sequence

$$\mathcal{H}_0 \overset{\Theta_0}{\to} \mathcal{H}_1 \overset{\Theta_1}{\to} \cdots \overset{\Theta_{n-1}}{\to} \mathcal{H}_n$$

of natural differential operators as constructed in [ČSS01] and then presented in a simplified form in [CD01].

The study of the BGG-sequence and in particular of the first BGG-operators, and the BGG-equations these describe, has seen much interest in recent years. It has been shown that the infinitesimal symmetries of a parabolic geometry can be described by a BGG-equation, [Čap08], and that the BGG-equations are always of finite type, [BČEG06, HSSŠ10]. Moreover, solutions of BGG-equations have been shown to appear naturally as characterizing objects of Fefferman-type spaces, [ČG, HS09, HS10].

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Since the BGG-machinery that describes these equations starts from a uniform algebraic setting it is also reasonable to ask whether this construction can be used to obtain relations between solutions of different BGG-equations. An abstract formulation of this question was introduced in [CD01] via the notion of cup product. Explicit calculations and results were achieved in [GŠ08] under the name of helicity raising and lowering for conformal Killing forms. While not mentioning the BGG-machinery there, it is clear that this kind of construction is possible for certain classes of BGG-operators on parabolic geometries.

1.1. **Outline.** In section 2 we briefly review the construction of the BGG-operators and the prolongation connection. We discuss normality of a solution and give a simple formula which provides the necessary and sufficient equations. We then introduce coupling maps for solutions of BGG-equations. For $|1|$-graded parabolic geometries and coupling maps where the target space is the domain of a BGG-operator of first order we give necessary and sufficient coupling conditions.
In section 3 the coupling procedure of section 2 will be applied to conformal spin structures. We briefly introduce these structures and then discuss several interesting first BGG-operators: those governing almost Einstein scales, conformal Killing forms and generic twistor spinors. We then derive explicit coupling maps and conditions for these objects.
The formulas obtained for coupling with twistor spinors are particularly interesting when this spinor is generic in a suitable way. We show in section 3.3 that every generic twistor spinor gives rise to a natural decomposition of conformal Killing fields. For signatures $(2,3)$ and $(3,3)$, such twistor spinors have been constructed in [HS10] and we discuss this result from the viewpoint of coupling maps.

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2. BGG-equations, normality and coupling

We begin with a very brief introduction of the necessary tractor calculus for parabolic geometries. For more background we refer to [ČG02, ČS09].

2.1. **Tractor bundles.** For every irreducible G-representation V one associates the tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$. It is well known, cf. that $\mathcal{V}$ carries its canonical tractor connection, denoted by $\nabla = \nabla^V$. By forming the exterior covariant derivative d^∇ of ∇ on $\mathcal{V}$-valued differential forms this gives rise to the sequence

$$\mathcal{C}_0 \xrightarrow{\nabla} \mathcal{C}_1 \xrightarrow{d^\nabla} \mathcal{C}_2 \xrightarrow{d^\nabla} \cdots$$

on the chain spaces $\mathcal{C}_k = \Omega^k(M, \mathcal{V})$.
Moreover, one has the (algebraic) Kostant co-differential $\partial^* : \mathcal{C}_{k+1} \to \mathcal{C}_k$, $\partial^* \circ \partial^* = 0$, which yields the complex

$$\mathcal{C}_0 \overset{\partial^*}{\leftarrow} \mathcal{C}_1 \overset{\partial^*}{\leftarrow} \mathcal{C}_2 \overset{\partial^*}{\leftarrow} \cdots .$$

This complex gives rise to spaces $\mathcal{Z}_k = \ker \partial^*$ of cycles, borders $\mathcal{B}_k = \mathrm{im}\ \partial^*$ and homologies $\mathcal{H}_k = \mathcal{Z}_k/\mathcal{B}_k$. The canonical surjections are denoted $\Pi_k : \mathcal{Z}_k \to \mathcal{H}_k$.

2.2. **BGG-operators and the prolongation connection.** The BGG-machinery of [ČSS01] is based on canonical differential splitting operators $L_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{Z}_k)$: A section $s \in \Gamma(\mathcal{Z}_k)$ is of the form $L_k\sigma, \sigma \in \Gamma(\mathcal{H}_k)$ if and only if $d^\nabla s \in \ker \partial^*$. This uniquely defines the operators L_k.
Now, given a section $\sigma \in \Gamma(\mathcal{H}_k)$, one can form $\mathrm{d}^\nabla(L_k\sigma) \in \Omega^{k+1}(M, \mathcal{V})$, which by assumption on L_k is a section of $\mathcal{Z}_{k+1}$, and can therefore be projected to $\Gamma(\mathcal{H}_{k+1})$. The composition $\Theta_k := \Pi_{k+1} \circ \mathrm{d}^\nabla \circ L_k$ is the $k+1$-st BGG-operator.
For $k = 0$ one obtains the first BGG-operator $\Theta_0 = \Pi_1 \circ \nabla \circ L_0$, $\Theta_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$, which is an overdetermined operator. One does in fact have that the system $\sigma \in \Gamma(\mathcal{H}_0)$, $\Theta_0(\sigma) = 0$ is of finite type: In [HSSŠ10] a natural modification $\tilde{\nabla} = \nabla + \Psi$ with $\Psi \in \Omega^1(M, \mathrm{End}(\mathcal{V}))$ was constructed which has the following property:

Proposition 2.1 ([HSSŠ10])**.** *The solutions $\sigma \in \Gamma(\mathcal{H}_0)$ of $\Theta_0(\sigma) = 0$ are in $1:1$-correspondence with the $\tilde{\nabla}$-parallel sections of $\mathcal{V}$. This isomorphism is realized with the first BGG-splitting operator $L_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$ and the canonical projection $\Pi_0 : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{H}_0)$.*

We call $\tilde{\nabla}$ the prolongation connection of Θ_0 since the equation $s \in \Gamma(\mathcal{V}), \tilde{\nabla}s = 0$ is the prolongation of the system $\sigma \in \Gamma(\mathcal{H}_0), \Theta_0(\sigma) = 0$.

2.3. **Normal solutions.** If a section $s \in \Gamma(\mathcal{V})$ is ∇-parallel, we automatically have that $s \in \mathrm{im}\ L_0$, since $\partial^*(\nabla s)$ is vanishes trivially, and so $s = L_0\sigma$ with $\sigma = \Pi_0 s$. Then $\Theta_0\sigma = \Pi_1(\nabla L_0\sigma) = \Pi_1(\nabla s) = 0$. We say that those $\sigma \in \ker \Theta_0$ that satisfy $\nabla L_0\sigma = 0$ are the *normal solutions* of $\Theta_0(\sigma) = 0$. If the geometry is flat, all solutions are normal.
Now, if $\sigma \in \Gamma(\mathcal{H}_0)$ is an arbitrary solution of $\Theta_0(\sigma) = 0$, then equivalently, with $s = L_0\sigma$ and $\tilde{\nabla}$ the prolongation connection,

$$0 = \tilde{\nabla}s = \nabla s + \Psi s.$$

Thus $\Psi L_0\sigma$ is the obstruction against normality of σ. However, it turns out that determining normality of a solution does not need computation of Ψ, which is always possible but depends on a procedure that involves one iteration for every filtration component of the tractor bundle, [HSSŠ10].
To state the simple criteria for normality we need to introduce the curvature of the Cartan connection form ω. It is defined as $K(\xi, \xi') = d\omega(\xi, \xi') + [\omega(\xi), \omega(\xi')]$ for $\xi, \xi' \in \mathfrak{X}(\mathcal{G})$. This determines a 2-form on $\mathcal{G}$ with values in

$\mathfrak{g}$. By forming the *adjoint tractor bundle* $\mathcal{A} := \mathcal{G} \times_P \mathfrak{g}$ and using horizontality and P-equivariancy of K, we can equivalently regard it as $K \in \Omega^2(M, \mathcal{A}M)$. Now since V is a G-representation, the Lie algebraic action of $\mathfrak{g}$ on V yields action of $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$ on $\mathcal{V} = \mathcal{G} \times_P V$, which we denote via $\bullet : \mathcal{A} \to \mathrm{End}(\mathcal{V})$.

Proposition 2.2. *A solution σ of $\Theta_0(\sigma) = 0$ is normal if and only if $\partial^*\big(K\bullet(L_0\sigma)\big) = 0$.*

Proof. Since ∇ is the natural connection induced by ω on $\mathcal{V}$ the curvature of ∇ is $R = K\bullet \in \Omega^2(M, \mathrm{End}(\mathcal{V}))$.
Denote $s = L_0\sigma$. Now if $\partial^*(K\bullet s) = 0$, then since $R = d^\nabla \circ \nabla$ we have $d^\nabla(\nabla s) \in \ker \partial^*$. Thus

$$\nabla s = L_1(\Pi_1(s)) = L_1(\Theta_0\sigma) = 0.$$

The converse is clear, since if $\nabla s = 0$, also $0 = \mathrm{d}^\nabla \circ \nabla s = Rs = K\bullet s$, and then $0 = \partial^*(K\bullet s) = \partial^*(K\bullet L_0\sigma)$. $\square$

We now discuss a procedure for obtaining new solutions from known ones. This will be particularly simple for normal solutions, but milder conditions on the solutions will be sufficient for interesting classes of equations.

2.4. **Coupling maps.** Let V, V' and W be G representations and $C : V \times V' \to W$ be a G-equivariant bilinear map. The corresponding tractor map is denoted $\mathbf{C} : \mathcal{V} \times \mathcal{V}' \to \mathcal{W}$.
It induces the (differential) coupling map $\mathbf{c} : \Gamma(\mathcal{H}_0^V) \times \Gamma(\mathcal{H}_0^{V'}) \to \Gamma(\mathcal{H}_1^W)$,

$$(\sigma, \sigma') \mapsto \Pi_0^W\big(\mathbf{C}(L_0^V(\sigma), L_0^{V'}(\sigma'))\big).$$

Since $\mathbf{C} : \mathcal{V} \times \mathcal{V}' \to \mathcal{W}$ is algebraic and natural, we have that for all $s \in \Gamma(\mathcal{V}), s' \in \Gamma(\mathcal{V}')$,

$$\nabla^W \mathbf{C}(s, s') = \mathbf{C}(\nabla^V s, s') + \mathbf{C}(s, \nabla^{V'} s'). \tag{1}$$

In particular, if $\sigma \in \mathcal{H}_0^V$ and $\sigma' \in \mathcal{H}_0^{V'}$ are normal solutions of Θ_0^V resp. $\Theta_0^{V'}$, then $\eta := \mathbf{c}(\sigma, \sigma')$ is a normal solution of Θ_0^W.

2.4.1. *Coupling for $|1|$-graded parabolic geometries with Θ_0^W of first order.* The operators Θ_0^V and $\Theta_0^{V'}$ have prolongation connections $\tilde\nabla^V = \nabla^V + \Psi^V$, $\tilde\nabla^{V'} = \nabla^{V'} + \Psi^{V'}$. We write $s = L_0^V\sigma, s' = L_0^{V'}\sigma'$.
By definition, $t := \mathbf{C}(s, s')$ is a lift of $\eta = \mathbf{c}(\sigma, \sigma') = \Pi_0^W(t)$, but one doesn't necessarily have $t = L_0^W\eta$: $\nabla^W t \in \mathcal{C}_1^W = \Omega^1(M, \mathcal{W})$ need not lie in $\mathcal{Z}_1^W = \ker \partial^*$. We will circumvent this problem by building a canonical extension of $\Pi_1^W : \mathcal{Z}_1^W \to \mathcal{H}_1^W$ to a map $\Pi_{1,\odot}^W : \mathcal{C}_1^W \to \mathcal{H}_1^W$.
For this one uses the natural filtration of W that is induced by the P-representation on W. The largest filtration component is just $\overline{W} := \mathrm{im}\, \partial^* \subset W$. The parabolic group P has a Levi factor $G_0 \subset P$ and $W_0 := W/\overline{W}$ is a well defined G_0-representation. On the level of associated bundles one

obtains a semidirect composition series $\mathcal{W} = \mathcal{W}_0 \oplus \overline{\mathcal{W}}$ and this induces the semidirect composition series

$$\mathcal{C}_1^W = T^*M \otimes \mathcal{W} = T^*M \otimes \mathcal{W}_0 \oplus T^*M \otimes \overline{\mathcal{W}}.$$

In particular, we have a canonical surjection $\Pi_1^{\mathcal{W}_0} : \mathcal{C}_1^W \to T^*M \otimes \mathcal{W}_0$.
The fact that $\Theta_0^W = \Pi_1^W \circ \nabla \circ L_0^W$ is an operator of first order is equivalent to $\mathcal{H}_0$ not depending on $\Omega^1(M, \overline{\mathcal{W}})$, i.e., $\Pi_1^W(\ker \partial^* \cap T^*M \otimes \overline{\mathcal{W}}) = \{0\}$, [BČEG06]. Then $\mathcal{H}_0 \subset T^*M \otimes \mathcal{W}_0$ is the highest weight part with respect to the G_0-structure. So composing the projection to the highest weight part with $\Pi_1^{\mathcal{W}_0}$ yields a map $\Pi_{1,\odot}^W : \mathcal{C}_1^W \to \mathcal{H}_1$ with the property that its restriction to $\mathcal{Z}_1^W$ is just Π_1^W. The operator Θ_0 can now be written

$$\Theta_0^W(\eta) = \Pi_{1,\odot}^W(\nabla^W t). \tag{2}$$

Proposition 2.3. *For $\sigma \in \ker \Theta_0^V, \sigma' \in \ker \Theta_0^{V'}$ and $\eta = \mathbf{c}(\sigma, \sigma')$ one has $\Theta_0^W(\eta) = -\Pi_{1,\odot}^W\big(\mathbf{C}(\Psi^V s, s') + \mathbf{C}(s, \Psi^{V'} s')\big)$.*

Proof. For $\sigma \in \ker \Theta_0^V$ we have equivalently that $s = L_0^V$ satisfies

$$0 = \tilde{\nabla}^V s = \nabla^V s + \Psi^V s,$$

so $\nabla^V s = -\Psi^V s$, and analogously for $\sigma' \in \ker \Theta_0^{V'}$. Therefore $\nabla^W t = \nabla^W \mathbf{C}(s, s') = \mathbf{C}(\nabla^V s, s') + \mathbf{C}(s, \nabla^{V'} s') = -\mathbf{C}(\Psi^V s, s') - \mathbf{C}(s, \Psi^{V'} s')$. Thus, using (2), this proves the claim. ☐

In particular, this yields necessary and sufficient coupling conditions.

3. Coupling in conformal spin geometry

A conformal spin structure of signature (p, q) on an $n = p + q$-manifold M is a reduction of structure group of TM from $\mathrm{GL}(n)$ to $\mathrm{CSpin}(p, q) = \mathbb{R}_+ \times \mathrm{Spin}(p, q)$. This induces a conformal class $\mathcal{C}$ of pseudo-Riemannian signature (p, q)-metrics on M. The associated bundle to the spin representation $\Delta^{p,q}$ of $\mathrm{CSpin}(p, q)$ with $\mathbb{R}_+$ acting trivially is the (unweighted) conformal spin bundle $\mathcal{S}$.
We will often employ the conformal density bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$, which are associated to the 1-dimensional $\mathbb{R}_+$ representations $c \mapsto c^w$. We also employ abstract index notation $\mathcal{E}_a = \Gamma(T^*M) = \Omega^1(M), \mathcal{E}^a = \Gamma(TM) = \mathfrak{X}(M)$ with multiple indices denoting tensor products, e.g. $\mathcal{E}_{ab} = \Gamma(T^*M \otimes T^*M)$.
The curvature quantities of the conformal structure $\mathcal{C}$ are computed with respect to a $g \in \mathcal{C}$. The symmetric 2-tensor

$$\mathrm{P}_g := \frac{1}{n-2}(\mathrm{Ric}_g - \frac{\mathrm{Sc}_g}{2(n-1)} g)$$

is the *Schouten tensor*; this is a trace modification of the Ricci curvature Ric_g by a multiple of the scalar curvature Sc_g. The trace of the Schouten tensor is denoted $J_g = g^{pq}\,\mathrm{P}_{pq}$. We will omit the subscripts g hereafter when giving a formula with respect to some $g \in \mathcal{C}$.

The complete obstruction against conformal flatness of $(M, \mathcal{C})$ with, $\mathcal{C}$ having signature $p+q \geq 3$, is the *Weyl curvature*

$$C_{ab}{}^{c}{}_{d} := R_{ab}{}^{c}{}_{d} - 2\delta^{c}_{[a}\,\mathrm{P}_{b]d} + 2g_{d[a}\,\mathrm{P}^{c}_{b]},$$

where R is the Riemannian curvature tensor of D and indices between square brackets are skewed over, (cf. e.g. [Eas96].)
A conformal spin structure of signature (p,q) is equivalently described by a parabolic geometry of type $(\mathrm{Spin}(p+1,q+1),P)$, with $P \subset G = \mathrm{Spin}(p+1,q+1)$ the stabilizer of an isotropic ray in the standard representation on $\mathbb{R}^{p+1,q+1}$, cf. [Ham09].
We are now going to consider the first BGG-operators and coupling formulas for three $\mathrm{Spin}(p+1,q+1)$-representations: for the standard representation on $\mathbb{R}^{p+1,q+1}$, its exterior powers $\Lambda^{k+1}\mathbb{R}^{p+1,q+1}$, $k \geq 0$ and the spin representation $\Delta^{p+1,q+1}$.

3.1. **BGG-operators in conformal spin geometry.**

3.1.1. *The almost Einstein scale operator* $\Theta_0^{\mathbb{R}^{p+1,q+1}}$. With $T = \mathbb{R}^{p+1,q+1}$ the standard representation of $\mathrm{Spin}(p+1,q+1)$, one obtains the standard tractor bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{p+1,q+1}$ together with its tractor metric $\mathbf{h}$.
It has a semidirect composition series $\mathcal{T} = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ and with respect to any $g \in \mathcal{C}$ one obtains a decomposition $\mathcal{T} \overset{g}{\cong} \begin{pmatrix} \mathcal{E}[-1] \\ \mathcal{E}_a[1] \\ \mathcal{E}[1] \end{pmatrix}$.
With respect to the Levi-Civita connection D of $g \in \mathcal{C}$ the first BGG-operator of T is

$$\Theta_0^T : \mathcal{E}[1] \to \mathcal{E}_{(ab)}[2],\ \sigma \mapsto \mathbf{tf}(DD\sigma + \mathrm{P}\,\sigma),$$

with $\mathbf{tf}$ denoting the trace-free part and round brackets symmetrization.
If $\sigma \in \ker \Theta_0^T$ is non-trivial, then the complement of its zero set in M is open dense, and on that set σ describes a rescaling of g to an Einstein metric $\sigma^{-2}g$. One therefore says that the solutions of $\Theta_0^T(\sigma) = 0$ are the *almost Einstein scales* of $\mathcal{C}$, cf. [Gov10]. We denote $\mathrm{aEs}(\mathcal{C}) = \ker \Theta_0^T \subset \mathcal{E}[1]$.
We will need an explicit formula for the first BGG-splitting operator of $\mathcal{T}$, cf. e.g. [BEG94]:

$$L_0^{\mathcal{T}} : \mathcal{E}[1] \to \Gamma(\mathcal{T}),\ \sigma \mapsto \begin{pmatrix} -\frac{1}{n} g^{pq}(D_{pq}\sigma + \mathrm{P}_{pq}\,\sigma) \\ D\sigma \\ \sigma \end{pmatrix}. \tag{3}$$

This case is particularly simple since the standard tractor connection ∇^T is already the prolongation connection. So all solutions of $\Theta_0(\sigma) = 0$ are normal and correspond to parallel standard tractors.

3.1.2. *The conformal Killing form operator* $\Theta_0^{\Lambda^{k+1}\mathbb{R}^{p+1,q+1}}$. Now let $V = \Lambda^{k+1}\mathbb{R}^{p+1,q+1}$ for $k \geq 1$ be an exterior power of the standard representation and $\mathcal{V} = \mathcal{G} \times_P V$ the associated tractor bundle. $\mathcal{V}$ has a semidirect composition series $\mathcal{E}_{[a_1\cdots a_k]}[k+1] \mathbin{+\!\!\!\!\supset} (\mathcal{E}_{[a_1\cdots a_{k+1}]}[k+1] \oplus \mathcal{E}_{[a_1\cdots a_{k-1}]}[k-1]) \mathbin{+\!\!\!\!\supset} \mathcal{E}_{[a_1\cdots a_k]}[k-1]$.
The first BGG-operator of V is

$$\begin{aligned}
&\Theta_0^V : \mathcal{E}_{[a_1\cdots a_k]}[k+1] \to \mathcal{E}_{c[a_1\ldots a_k]}[k+1],\\
&\sigma_{a_1\cdots a_k} \mapsto D_c\sigma_{a_1\cdots a_k} - D_{[a_0}\sigma_{a_1\cdots a_k]} - \frac{k}{n-k+1} g_{c[a_1} g^{pq} D_{|p}\sigma_{q|a_2\cdots a_k]}
\end{aligned}$$

and its solutions are the conformal Killing forms.
Our coupling formulas below will employ the first BGG-splitting operator $L_0 : \mathcal{H}_0 \to \mathcal{V}$, given with respect to a $g \in \mathcal{C}$ and the corresponding splitting of the semidirect composition series. For the computation we refer to [Ham08, Ham09].

(4)

$$\sigma \mapsto \begin{pmatrix} \begin{pmatrix} -\frac{1}{n(k+1)} D^p D_p \sigma_{a_1\cdots a_k} + \frac{k}{n(k+1)} D^p D_{[a_1}\sigma_{|p|a_2\cdots a_k]} + \frac{k}{n(n-k+1)} D_{[a_1} D^p \sigma_{|p|a_2\cdots a_k]} \\ +\frac{2k}{n} \mathrm{P}^p{}_{[a_1}\sigma_{|p|a_2\cdots a_k]} - \frac{1}{n} J\sigma_{a_1\cdots a_k} \end{pmatrix} \\ D_{[a_0}\sigma_{a_1\cdots a_k]} \mid -\frac{1}{n-k+1} g^{pq} D_p \sigma_{qa_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix}.$$

Here indices between vertical bars are not skewed over.
The prolongation connection of Θ_0^V is $\tilde{\nabla}^V = \nabla^V + \Psi^V$ for $\Psi^V \in \Omega^1(M, \mathrm{End}(\mathcal{V})$ as computed in [Ham08]. For our purposes it is enough to know its part of lowest homogeneity, which is,

$$\begin{aligned}
(5)\quad &\bar{\Psi}^V \in \mathrm{Hom}(\mathcal{E}_{[a_1\cdots a_k]}[k+1], \mathcal{E}_c \otimes \big(\mathcal{E}_{[a_1\cdots a_{k+1}]}[k+1] \oplus \mathcal{E}_{[a_1\cdots a_{k-1}]}[k-1]\big)),\\
&\sigma \mapsto L(\sigma) \oplus R(\sigma)
\end{aligned}$$

with

(6)

$$\begin{aligned}
L(\sigma) &= \frac{k+1}{2} C_{[a_0 a_1}{}^{p}{}_{|c}\sigma_{p|a_2\cdots a_k]} + \frac{(k-1)(k+1)}{2n} g_{c[a_0} C_{a_1 a_2}{}^{pq}\sigma_{|pq|a_3\cdots a_k]}\\
R(\sigma) &= \frac{(k-1)(n-2)}{2(n-k)n} C_{c[a_2}{}^{pq}\sigma_{|pq|a_3\cdots a_k]} - \frac{(k-1)(k-2)}{2(n-k)n} C_{[a_2 a_3}{}^{pq}\sigma_{|cpq|a_4\ldots a_k]}.
\end{aligned}$$

3.1.3. *The twistor spinor operator* $\Theta_0^{\Delta^{p+1,q+1}}$. With $\Delta^{p+1,q+1}$ the spin representation of $\mathrm{Spin}(p+1, q+1)$ we form the associated spin tractor bundle $\Sigma := \mathcal{G} \times_P \Delta^{p+1,q+1}$. Recall the the (unweighted) spin bundle $\mathcal{S}$ of the conformal structure. Then Σ has a semidirect composition series $\Sigma = \mathcal{S}[\frac{1}{2}] \mathbin{+\!\!\!\!\supset} \mathcal{S}[-\frac{1}{2}]$. With respect to the Levi-Civita connection D of a metric $g \in \mathcal{C}$ the first BGG-operator of $\Delta^{p+1,q+1}$ is $\Theta_0^{\Delta^{p+1,q+1}} : \Gamma(\mathcal{S}[\frac{1}{2}]) \to \Gamma(\mathcal{E}_c \otimes \mathcal{S}[\frac{1}{2}])$,

$$\chi \mapsto D_c\chi + \frac{1}{n}\gamma_c \not{D}\chi,$$

where $\gamma \in \mathcal{E}_c \otimes \mathrm{End}(\mathcal{S})$ the Christoffel symbol of $\mathcal{S}$ and $\not{D}\chi = g^{pq}\gamma_p D_q \chi$. The solutions of $\Theta_0^{\Delta^{p+1,q+1}}(\chi) = 0$ are *twistor spinors.*
Using $\Sigma \overset{g}{\cong} \begin{pmatrix} \mathcal{S}[-\frac{1}{2}] \\ \mathcal{S}[\frac{1}{2}] \end{pmatrix}$ the first BGG-splitting splitting operator is

$$L_0^{\Delta^{p+1,q+1}} : \Gamma(S[\frac{1}{2}]) \to \Gamma(\Sigma),\ \chi \mapsto \begin{pmatrix} \frac{\sqrt{2}}{n}\not{D}\chi \\ \chi \end{pmatrix}. \tag{7}$$

Also in this case, the tractor connection $\nabla^{\Delta^{p+1,q+1}}$ coincides with the prolongation connection, which was employed in [Fri90, BFGK90, Bra05, Lei08, Ham09]).
We now relate Clifford multiplications and the canonical invariant pairings of the spin tractor bundle and and conformal spin bundle.

3.1.4. *Clifford multiplication and invariant pairing.* For every $g \in \mathcal{C}$ we obtain identifications $\mathcal{T} \overset{g}{\cong} \begin{pmatrix} \mathcal{E}[-1] \\ \mathcal{E}_a[1] \\ \mathcal{E}[1] \end{pmatrix}, \Sigma \overset{g}{\cong} \begin{pmatrix} \mathcal{S}[-\frac{1}{2}] \\ \mathcal{S}[\frac{1}{2}] \end{pmatrix}$. With respect to this decomposition the tractor metric is simply $\mathbf{h} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and tractor Clifford multiplication Γ is given by

$$\Gamma : \mathcal{T} \otimes \mathcal{S} \to \mathcal{S}, \qquad \begin{pmatrix} \rho \\ \sigma_a \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} -\sigma_a \cdot \tau + \sqrt{2}\rho\chi \\ \sigma_a \cdot \chi - \sqrt{2}\sigma\tau \end{pmatrix}. \tag{8}$$

One easily checks that with this definition indeed

$$t_1 \cdot (t_2 \cdot s) + t_2 \cdot (t_1 \cdot s) = -2\mathbf{h}(t_1, t_2)\ \forall\ t_1, t_2 \in \Gamma(\mathcal{T}), s \in \Gamma(\Sigma).$$

The spin bundle $\mathcal{S}$ carries a canonical pairing $\mathbf{b} : \mathcal{S} \otimes \mathcal{S} \to \mathbb{R}$ which is Clifford invariant in the sense that $\mathbf{b}(\xi \cdot \chi, \chi') + (-1)^{p+1}\mathbf{b}(\chi, xi \cdot \chi') = 0$ for all $\xi \in \mathfrak{X}(M), \chi, \chi' \in \Gamma(\mathcal{S})$, cf. [Bau81, Kat99]. The corresponding tractor spinor pairing is, [Ham09],

$$\mathbf{B} : \Sigma \otimes \Sigma \to \mathbb{R}, \qquad \mathbf{B}\left(\begin{pmatrix} \tau \\ \chi \end{pmatrix}, \begin{pmatrix} \tau' \\ \chi' \end{pmatrix}\right) = \mathbf{b}(\chi, \tau') + (-1)^{p+1}\mathbf{b}(\chi', \tau), \tag{9}$$

which then satisfies (use (8)), $\mathbf{B}(t \cdot X, X') + (-1)^p \mathbf{B}(X, t \cdot X') = 0$ for all $t \in \Gamma(\mathcal{T}), X, X' \in \Gamma(\Sigma)$.
Having this background on some basic BGG-operators in conformal geometry we can now derive a number of coupling formulas and conditions via the method of section 2.4.

3.2. **Coupling formulas.**

3.2.1. *Wedge coupling of conformal Killing forms.* Given $s \in \Gamma(\Lambda^{k+1}\mathcal{T})$ and $s' \in \Gamma(\Lambda^{k'+1}\mathcal{T})$ we form $\mathbf{C}^{\wedge}(s,s') := s \wedge s'$. Employing (4) we obtain the coupling map

(10)
$$\begin{aligned}
&\mathbf{c}^{\wedge} : \mathcal{E}_{[a_1\cdots a_k]}[k+1] \times \mathcal{E}_{[a_1\cdots a_{k'}]}[k'+1] \to \mathcal{E}_{[a_1\cdots a_{k+k'+1}]}[k+k'+2]\\
&(\sigma_{a_1\cdots a_k}, \sigma'_{a_1\cdots a_{k'}}) \mapsto (k+1)\sigma_{[a_1\cdots a_k} D_{a_{k+1}} \sigma'_{a_{k+2}\cdots a_{k+k+1}]}\\
&\qquad\qquad + (-1)^{(k+1)(k'+1)}(k'+1)\sigma'_{[a_1\cdots a_{k'}} D_{a_{k'+1}} \sigma_{a_{k'+2}\cdots a_{k+k'+1}]}.
\end{aligned}$$

Employing Proposition 2.2, the prolongation connection (5) and some simple computatons involving the symmetries of the Weyl curvature tensor C one shows:

Proposition 3.1. *Assume that $\sigma \in \ker \Theta_0^{\Lambda^{k+1}\mathbb{R}^{p+1,q}}$ and $\sigma' \ker \Theta_0^{\Lambda^{k'+1}\mathbb{R}^{p+1,q}}$. Then the coupled $(k+k'+1)$-form $\eta = \mathbf{c}^{\wedge}(\sigma,\sigma')$ is a conformal Killing form if and only if*

(11)
$$(-1)^{k+1} C_{[a_1 a_2}{}^{p}{}_{|c}\sigma_{p|a_3\cdots a_{k+1}}\sigma'_{a_{k+3}\cdots a_{k+k'+1}]} + \sigma_{[a_1\cdots a_k} C_{a_{k+1}a_{k+2}}{}^{p}{}_{|c}\sigma'_{p|a_{k+3}\cdots a_{k+k'+1}]} \overset{\odot}{=} 0.$$

Here $\odot$ denotes projection to the $\mathrm{Spin}(p,q)$*-highest weight part, which in this case are those elements in $\Gamma(T^*M \otimes \Lambda^k T^*M)$ with trivial alternation and trivial trace.*

Remark 3.2. Assume that $k < k'$. In the case where $k = 0$ $\sigma \in \ker \Theta_0^{\mathbb{R}^{p+1,q+1}}$ is an almost Einstein scale and (11) simplifies to

$$C_{[a_1 a_2}{}^{p}{}_{|c}\sigma'_{p|a_3\cdots a_{k+1}]} \overset{\odot}{=} 0,$$

since σ is non-vanishing on an open dense subset. This agrees with Theorem 5.1 of [GŠ08]. Also for the special case $k = 1$ (11) and (13) below agree with Theorem 5.4 of [GŠ08].

3.2.2. *Contraction coupling of conformal Killing forms.* Let now $k' > k$. We employ the tractor metric $\mathbf{h}$ to form a contraction map $\mathbf{C}^{\lrcorner} : \Lambda^{k+1}\mathcal{T} \times \Lambda^{k'+1}\mathcal{T} \to \Lambda^{k'-k}\mathcal{T}$. The coupling map is then

$$\begin{aligned}
(12)\qquad &\mathbf{c}^{\lrcorner} : \mathcal{E}_{[a_1\cdots a_k]}[k+1] \times \mathcal{E}_{[a_1\cdots a_{k'}]}[k'+1] \to \mathcal{E}_{[a_1\cdots a_{k'-k-1}]}[k'-k]\\
&(\sigma_{a_1\cdots a_k}, \sigma'_{a_1\cdots a_{k'}}) \mapsto (k+1)\sigma^{p_1\cdots p_k} D^q \sigma'_{qp_1\cdots p_k a_1\cdots a_{k'-k-1}}\\
&\qquad\qquad + (n-k'+1)\sigma'_{p_0\cdots p_k a_1\cdots a_{k'-k-1}} D^{p_0}\sigma^{p_1\cdots p_k}.
\end{aligned}$$

Proposition 3.3. *If σ and σ' are conformal Killing forms the coupled $(k'-k-1)$-form $\eta = \mathbf{c}(\sigma,\sigma')$ is also a conformal Killing form if and only if*

$$\begin{aligned}
(13)\qquad &(n-k')C^{p_0p_1}{}_{qc}\sigma^{qp_2\cdots p_k}\sigma'_{p_0\cdots p_k a_1\cdots a_{k'-k-1}}\\
&- (k'-1)\sigma^{p_1\cdots p_k} C_{cp_1}{}^{q_1q_2}\sigma'_{q_1q_2p_2\cdots p_k a_1\cdots a_{k'-k-1}} \overset{\odot}{=} 0.
\end{aligned}$$

Remark 3.4. In the case where $k = 0$ and σ is an almost Einstein scale Proposition 3.3 reduces to a case treated in [GŠ08], Theorem 5.1: (13) is trivially satisfied for $k' = 1$ and simplifies to

$$C_{c[a_1}{}^{q_1 q_2} \sigma'_{q_1 q_2 a_2 \cdots a_{k'-1}]} \overset{\odot}{=} 0 \tag{14}$$

for $k' \geq 2$. Since the Weyl curvature tensor is skew-symmetric in the first two slots (14) also holds automatically for $k' = 2$.

For our coupling formulas with twistor spinors below we assume that the signature (p, q) is such that the spin representation $\Delta^{p,q}$ is real, in which case also the modeling spin representation $\Delta^{p+1,q+1}$ for the spin tractor bundle is real. This avoids having to complexify the bundles $\Lambda^k T^*M$.

3.2.3. *Twistor spinor coupling.* Let $X, X' \in \Gamma(\Sigma)$ and fix a $k \geq 0$. We define an element in $\Lambda^{k+1}\mathcal{T} \cong \Lambda^{k+1}\mathcal{T}^*$ by

$$\mathbf{C}^k(X, X')(\Phi) = \mathbf{B}(\Phi \cdot X, X') \; \forall \; \Phi \in \Lambda^{k+1}\mathcal{T}.$$

This yields the invariant pairing from spinors to forms,

$$\begin{aligned} \mathbf{c}^k &: \Gamma(S[\frac{1}{2}]) \times \Gamma(S[\frac{1}{2}]) \to \mathcal{E}_{[a_1 \cdots a_k]}[k+1], \\ &(\chi, \chi') \mapsto \mathbf{b}(\chi, \gamma_{[a_1} \cdots \gamma_{a_k]} \chi'). \end{aligned} \tag{15}$$

Since the prolongation connection of Σ coincides with the tractor connection this well known map always produces a conformal Killing k-form from two given twistor spinors.

3.2.4. *Conformal Killing forms - twistor spinor coupling.* Let $k \geq 0$. The tractor Clifford multiplication provides a map $\mathbf{C}^\gamma : \Lambda^{k+1}\mathcal{T} \otimes \Sigma \to \Sigma$ and the corresponding coupling map is

$$\begin{aligned} &\mathbf{c}^\gamma : \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \times \Gamma(S[\frac{1}{2}]) \to \Gamma(S[\frac{1}{2}]), \\ &\sigma \times \chi \mapsto (-1)^{k+1} \frac{2(k+1)}{n} \sigma \cdot \not{D} \chi + (d\sigma) \cdot \chi + \frac{k(k+1)}{(n-k+1)} (\delta\sigma) \cdot \chi. \end{aligned} \tag{16}$$

Here $d\sigma = D_{[a_0} \sigma_{a_1 \cdots a_k]}$ is the exterior derivative of σ and $\delta\sigma = -g^{pq} D_p \sigma_{q a_2 \cdots a_k}$ is the divergence of σ. The divergence term is trivial for $k = 0$, in which case $\sigma \in \mathcal{E}[1]$.

Proposition 3.5. *Let $\chi \in \Gamma(\mathcal{S}[\frac{1}{2}])$ be a twistor spinor. If $\sigma \in \mathcal{E}[1]$ is an almost Einstein scale or a conformal Killing field $\sigma \in \mathcal{E}_a[2] \cong \mathfrak{X}(M)$ then $\eta = \mathbf{c}^\gamma(\sigma, \chi)$ is again a twistor spinor.*
For both cases, which correspond to $k = 0, k = 1$, one has

$$\nabla^{\Delta^{p+1,q+1}} \big((L_0^{\Lambda^{k+1}\mathbb{R}^{p+1,q+1}} \sigma) \cdot (L_0^{\Delta^{p+1,q+1}} \chi) \big) = 0,$$

which is equivalent to

$$L_0^{\Delta^{p+1,q+1}} \eta = \mathbf{C}^\gamma (L_0^{\Lambda^{k+1}\mathbb{R}^{p+1,q+1}} \sigma, L_0^{\Delta^{p+1,q+1}} \chi).$$

Proof. For $k = 0$, which is the case where $\sigma \in \mathcal{E}[1]$ is an almost Einstein scale, both statements follow immediately since the tractor connection $\nabla^{\mathbb{R}^{p+1,q+1}}$ is already the prolongation connection of $\Theta_0^{\mathbb{R}^{p+1,q+1}}$ and all solutions are normal.

For $k = 1$ one has that in formula (6) the term $R(\sigma)$ vanishes, and $L(\sigma) = C_{a_0a_1}{}^p{}_c\sigma_p$. Denote $s = L_0^{\Lambda^2\mathbb{R}^{p+1,q+1}}\sigma, X = L_0^{\Delta^{p+1,q+1}}\chi$. Since $\Psi^{\Delta^{p+1,q+1}} = 0$, one has, according to Proposition 2.3,

$$\Theta_0^{\Delta^{p+1,q+1}}\eta = -\Pi_{1,\odot}^{\Delta^{p+1,q+1}}\left(\mathbf{C}^\gamma(\Psi^{\Lambda^2\mathbb{R}^{p+1,q+1}}s, X)\right) \overset{\odot}{=} C_{a_0a_1}{}^p{}_c\sigma_p\gamma^{a_0a_1}\chi.$$

But since χ is a twistor spinor, $C_{a_0a_1}{}^p{}_c\sigma_p\gamma^{a_0a_1}\chi = 0$. This shows that in fact

$$\nabla^{\Delta^{p+1,q+1}}(s \cdot X) \in \Omega^1(M, \mathcal{S}[-\frac{1}{2}]),$$

and thus $\partial^*(\nabla^{\delta^{p+1,q+1}}(s \cdot X)) = 0$. By definition of $L_0^{\Delta^{p+1,q+1}}$ this says that $L_0^{\Delta^{p+1,q+1}}\eta = s \cdot X$, and since $\eta \in \ker\Theta_0^{\Delta^{p+1,q+1}}$ this implies already $\nabla^{\Delta^{p+1,q+1}}(s \cdot X) = 0$ since $\nabla^{\Delta^{p+1,q+1}}$ coincides with the prolongation connection. □

Proposition 3.6. *Let $\chi \in \Gamma(\mathcal{S}[\frac{1}{2}])$ be a twistor spinor. For $k \geq 2$ and $\sigma_{a_1\cdots a_k} \in \mathcal{E}_{[a_1\cdots a_k]}[k+1]$ a conformal Killing form one has that $\eta = \mathbf{c}^\gamma(\sigma, \chi)$ is a twistor spinor if and only if $C_{ca_1}{}^{pq}\sigma_{pqa_2\cdots a_k}\gamma^{a_1\cdots a_k}\chi \overset{\odot}{=} 0$.*

3.3. **Generic twistor spinors.** We start with an algebraic observation. Take a $k \geq 0$ and the map

$$C : \Delta^{p+1,q+1} \times \Delta^{p+1,q+1} \to \Lambda^{k+1}\mathbb{R}^{p+1,q+1},$$

realized with respect to the $\mathrm{Spin}(p{+}1, q{+}1)$-invariant pairing $B \in \Delta^{p+1,q+1*} \otimes \Delta^{p+1,q+1*}$.

For a fixed $X \in \Delta^{p+1,q+1}$ we can form

$$i_XC : \Delta^{p+1,q+1} \to \Lambda^{k+1}\mathbb{R}^{p+1,q+1},$$

which is $G := \mathrm{Spin}(p+1, q+1)_X$-invariant. The following lemma is then easily checked.

Lemma 3.7. *Assume that $B(X, X) \neq 0$. Then, after some suitable rescaling, one has that the map*

$$P : \Lambda^{k+1}\mathbb{R}^{p+1,q+1} \to \Lambda^{k+1}\mathbb{R}^{p+1,q+1},\ \Phi \mapsto i_XC(\Phi \cdot X)$$

satisfies $P \circ P = \pm P$.

Then $\ker P = \ker\Gamma X$ *and we obtain a G-invariant decomposition*

$$\Lambda^{k+1}\mathbb{R}^{p+1,q+1} = \ker\Gamma X \oplus \mathrm{im}\ P.$$

Definition 3.8. We say that a twistor spinor $\chi \in \Gamma(\mathcal{S}[\frac{1}{2}])$ is generic if $\mathbf{b}(\chi, \not{D}\chi) \neq 0$.

It is visible from (9) that a twistor spinor $\chi \in \Gamma(\mathcal{S}[\frac{1}{2}])$ is generic if and only if the corresponding $\nabla^{\Delta^{p+1,q+1}}$-parallel tractor $X = L_0^{\Delta^{p+1,q+1}}\chi$ satisfies $\mathbf{B}(X, X) \neq 0$.
Now for a twistor spinor χ the coupling map $\mathfrak{X}(M) \times \mathcal{S}[\frac{1}{2}] \to \mathcal{S}[\frac{1}{2}]$ can be rewritten into

$$\xi \times \chi \mapsto D_\xi \chi - \frac{1}{4}(D_{[a}\xi_{b]}) \cdot \chi + \frac{1}{2n}(D_p \xi^p)\chi.$$

For a conformal Killing field $\xi \in \mathfrak{X}(M)$ this is just the Lie derivative of the (weighted) spinor χ with respect to ξ, [Kos72, FFFG96]. Our algebraic observation from above together with Proposition 3.5 yields:

Proposition 3.9. *Every generic twistor spinor χ provides a decomposition*

$$\mathrm{cKf}(\mathcal{C}) = \mathrm{cKf}_\chi(\mathcal{C}) \oplus \mathrm{cKf}_\chi^\perp(\mathcal{C}) \tag{17}$$

of conformal Killing fields into a part which also preserves χ and a canonical complement. The projection

$$\mathrm{cKf}(\mathcal{C}) \to \mathrm{cKf}_\chi^\perp(\mathcal{C})$$

is given by

$$\xi^a \mapsto \mathbf{b}(\gamma^a \chi, D_\xi \chi - \frac{1}{4}(D_{[a}\xi_{b]}) \cdot \chi + \frac{1}{2n}(D_p \xi^p)\chi). \tag{18}$$

One should regard a generic χ on $(M, \mathcal{C})$ as a refinement of the the geometric structure $\mathcal{C}$, and (17) then says that there is a corresponding cKf-decomposition. This is motivated by the following example.

3.3.1. *Generic twistor spinors on conformal spin structures of signature* $(2, 3)$ *and* $(3, 3)$. Let $(M, \mathcal{C}, \chi)$ be a conformal spin structure of signature $(2, 3)$ with a generic twistor spinor χ. Now genericity of χ implies that $\mathcal{D}_\chi = \ker \gamma\chi$ is a generic rank 2 distribution on M, [HS10]: The subbundle $[\mathcal{D}_\chi, \mathcal{D}_\chi]$ of TM spanned by Lie brackets of sections of $\mathcal{D}_\chi$ is 3-dimensional and $TM = [\mathcal{D}_\chi, [\mathcal{D}_\chi, \mathcal{D}_\chi]]$. Similarly, if χ is a generic twistor spinor an a $(3, 3)$-signature conformal spin structure, then $\mathcal{D}_\chi = \ker \gamma\chi \subset TM$ is a generic 3-distribution on M: $[\mathcal{D}_\chi, \mathcal{D}_\chi] = TM$.
The conformal spin structure $\mathcal{C}$ together with the generic twistor spinor χ are uniquely determined by $\mathcal{D} \subset TM$. This is shown in [HS10] via a Fefferman-type construction which starts from any generic 2- resp. 3-distribution $\mathcal{D} \subset TM$ and associates $(\mathcal{C}_\mathcal{D}, \chi_\mathcal{D})$. Since there are non-flat generic distributions, this yields examples of non-flat conformal spin structures with generic twistor spinors.
It follows that the infinitesimal symmetries $\mathrm{sym}(\mathcal{D}_\chi)$ of the distribution $\mathcal{D}_\chi$ are exactly those conformal Killing fields which preserve the twistor spinor χ, and according to Proposition 3.9

$$\mathrm{cKf}(\mathcal{C}) = \mathrm{sym}(\mathcal{D}_\chi) \oplus \mathrm{cKf}_\chi^\perp(\mathcal{C}).$$

For signature $(2, 3)$ one obtains a particularly simple decomposition since in that case $\mathrm{cKf}_\chi^\perp(\mathcal{C}) = \mathrm{aEs}(\mathcal{C})$. Using compositions of the coupling maps

(15),(16) one obtains explicit formulas: An almost Einstein scale $\sigma \in \mathcal{E}[1]$ is mapped to the conformal Killing field $\xi^a = \mathbf{b}(\gamma^a\chi, -\frac{2}{5}\sigma \not{D}\chi + (D\sigma)\cdot\chi)$ and the almost Einstein scale part of a conformal Killing field $\xi \in \mathfrak{X}(M)$ is $\sigma = \mathbf{b}(\chi, D_\xi\chi - \frac{1}{4}(D_{[a}\xi_{b]})\cdot\chi) \in \mathcal{E}[1]$. The term $\frac{1}{10}(\delta\xi)\chi$ does not appear here since for signature $(2,3)$ the invariant pairing $\mathbf{b}$ is skew.

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