SOME REMARKS ON BIG COHEN-MACAULAY ALGEBRAS VIA CLOSURE OPERATIONS

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Abstract. In this note we present some remarks on big Cohen-Macaulay algebras. Our methods for doing this are inspired by the notion of dagger closure and by ideas of Northcott on dropping of the Noetherian assumption of certain homological properties.

1. Introduction

The perfect closure of a reduced ring A of prime characteristic p is defined by adjoining to A all higher p-power roots of all elements of A and denote it by A^{∞}. Following [9], a ring R is called F-coherent if R^{∞} is a coherent ring. We denote the polynomial grade of an ideal $\mathfrak{a}$ on M by $\operatorname{p.grade}_A(\mathfrak{a}, M)$ (see below for definition). First we show that the Frobenius map is flat over a coherent regular ring which is of prime characteristic. Let R be a Noetherian local F-coherent domain which is either excellent or homomorphic image of a Gorenstein local ring. Then by applying some basic properties of tight closure theory, we show in Theorem 3.3 that $\operatorname{ht}(\mathfrak{a}) = \operatorname{p.grade}_{R^{\infty}}(\mathfrak{a}, R^{\infty})$ for every ideal $\mathfrak{a}$ of R^{∞}. We close Section 3 by showing that coherent big Cohen-Macaulay algebras are balanced big Cohen-Macaulay.

Let I be an ideal of a Noetherian local domain R and R^{+} be its integral closure in an algebraic closure of its fraction field. Recall from [6], an element $x \in R$ is in $I^{\dagger}$, dagger closure of I, if there are elements $\epsilon_n \in R^{+}$ of arbitrarily small order such that $\epsilon_n x \in IR^{+}$. By the main result of [6], tight closure of an ideal coincides with the dagger closure, where R is complete and of prime characteristic p. In this Section 4 we extend that notion to the submodules of finitely generated modules over R to prove that if a complete local domain is contained in an almost Cohen-Macaulay domain then there exists a balanced big Cohen-Macaulay module over it (see Corollary 4.6).

2. Preliminary Notations

Let A be an algebra equipped with a map $v : A \to \mathbb{R} \cup \{\infty\}$ satisfying (1) $v(ab) = v(a)+v(b)$ for all $a, b \in A$; (2) $v(a+b) \geq \min\{v(a), v(b)\}$ for all $a, b \in A$ and (3) $v(a) = \infty$ if and only if $a = 0$. If moreover $v(a) \geq 0$ for every $a \in T$ and $v(a) > 0$ for every non-unit $a \in A$, then we say that v is *normalized*.

2000 *Mathematics Subject Classification.* 13C14.

Key words and phrases. Almost Cohen-Macaulay, Big Cohen-Macaulay, Closure operation, Perfect algebra, Value map.

Lemma 2.1. *(see* [1, Proposition 3.2]*) Let A be an algebra equipped with a normalized value map and let $\mathfrak{a}$ be a proper and finitely generated ideal of A. Then $\bigcap_{n=1}^{\infty} \mathfrak{a}^n = 0$.*

Let A be a commutative ring of prime characteristic p. By A^0 we mean the complement of the set of all minimal primes of A. Let I be an ideal of A. By $I^{[q]}$ we mean the ideal generated by $q = p^e$-th powers of all elements of I. Then the tight closure I^* of I is the set of $x \in A$ such that there exists $c \in A^0$ such that $cx^q \in I^{[q]}$ for $q \gg 0$. Recall that a ring is coherent if each of its finitely generated ideals are finitely presented. Also recall that a ring is called regular if each of its finitely generated ideals are of finite projective dimension.

Lemma 2.2. *Let A be a coherent regular ring of prime characteristic. The following holds.*

(i) *The Frobenius map is flat.*
(ii) *If A is equipped with a normalized value map, then all finitely generated ideals of A are tightly closed.*

Proof. (i): By $F(A)$, we mean A as a group equipped with A-module structure via the Frobenius map. We show that $\operatorname{Tor}_i^A(A/\mathfrak{a}, F(A)) = 0$ for all $i > 0$ and all finitely generated ideals $\mathfrak{a} \subset A$. Note that $A/\mathfrak{a}$ has a free resolution $(F_\bullet, d_\bullet)$ consisting of finitely generated modules, since A is coherent. Such a resolution is bounded, because A is regular. Then $(F_\bullet, d_\bullet) \otimes_A F(A) = (F_\bullet, d_\bullet^p)$. Also, recall that Koszul depth is unique up to radical. In view of [3, Theorem 9.1.6], which is a beautiful theorem of Buchsbaum-Eisenbud-Northcott, we find that $(F_\bullet, d_\bullet^p)$ is exact and so $\operatorname{Tor}_i^A(A/\mathfrak{a}, F(A)) = 0$.

(ii): On perfect rings Frobenius map is bijective and so flat. By using this, [9, Lemma 4.1] yields the claim for a perfect coherent domain that is separable with respect to the proper and finitely generated ideals. In view of (i), the claim follows by the proof of [9, Lemma 4.1]. □

It is noteworthy to remark that the finitely generated assumption of the previous result is really needed.

Example 2.3. Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of prime characteristic p which is not a field. Let R^∞ be its perfect closure. It defines by adjoining to R all higher p-power roots of all elements of R. This is well-known that R^∞ is coherent and regular. Also, R^∞ is equipped with a normalized value map. Consider the ideal $\mathfrak{a} := \{x \in R^\infty : v(x) > 1/p\}$. Here, we show that $\mathfrak{a}^* \neq \mathfrak{a}$. To this end, let $x \in R^\infty$ be such that $v(x) = 1/p$. Such an element exists. Take $c \in R^\infty$ with $v(c) > 0$. Clearly, $v(c^{1/q}x) > 1/p$ and so $c^{1/q}x \in \mathfrak{a}$. Thus $cx^q = \prod_q c^{1/q}x \in \mathfrak{a}^{[q]}$ for $q \gg 0$. Therefore, $x \in \mathfrak{a}^* \setminus \mathfrak{a}$.

3. Cohen-Macaulayness of Minimal Perfect Algebra

The perfect closure of a reduced ring A of prime characteristic p is defined by adjoining to A all higher p-power roots of all elements of A and denote it by A^∞. Following [9], a ring R is called F-coherent if R^∞ is a coherent ring.

Lemma 3.1. *Let R be a Noetherian local F-coherent domain which is either excellent or homomorphic image of a Gorenstein local ring. Then R^∞ is big Cohen-Macaulay.*

Proof. This is proved in [9, Theorem 3.10] when R is homomorphic image of a Gorenstein ring. Assume that R is excellent. For each n, set $R_n := \{x \in R^\infty | x^{p^n} \in R\}$. The assignment $x \mapsto x^{p^n}$ shows that $R \simeq R_n$ and so R_n is excellent. We recall that over excellent domains one can use the colon capturing property of tight closure theory. Let $\underline{x} := x_1, \dots, x_d$ be a system of parameters for R, where $d := \dim R$ and let $r \in R^\infty$ be such that $rx_{i+1} = \sum_{j=1}^{i} r_j x_j$ for some $r_j \in R^\infty$. Then $r, r_j \in R_n$ for $n \gg 0$. So $r \in ((x_1, \dots, x_i)R_n :_{R_n} x_{i+1})$. Putting these along with Lemma 2.2,

$$\begin{aligned} ((x_1, \dots, x_i)R^\infty :_{R^\infty} x_{i+1}) &= \textstyle\bigcup_n ((x_1, \dots, x_i)R_n :_{R_n} x_{i+1}) \\ &\subseteq \textstyle\bigcup_n ((x_1, \dots, x_i)R_n)^* \\ &\subseteq ((x_1, \dots, x_i)R^\infty)^* \\ &= (x_1, \dots, x_i)R^\infty, \end{aligned}$$

which yields the claim. □

Let $\mathfrak{a}$ be an ideal of a ring A and M be an A-module. A finite sequence $\underline{x} := x_1, \dots, x_r$ of elements of A is called M-sequence if x_i is a nonzero-divisor on $M/(x_1, \dots, x_{i-1})M$ for $i = 1, \dots, r$ and $M/\underline{x}M \neq 0$. The classical grade of $\mathfrak{a}$ on M, denoted by $\operatorname{c.grade}_A(\mathfrak{a}, M)$, is defined by the supremum length of maximal M-sequences in $\mathfrak{a}$. The polynomial grade of $\mathfrak{a}$ on M is defined by

$$\operatorname{p.grade}_A(\mathfrak{a}, M) := \lim_{m\to\infty} \operatorname{c.grade}_{A[t_1,\dots,t_m]}(\mathfrak{a}A[t_1, \dots, t_m], A[t_1, \dots, t_m] \otimes_A M).$$

In what follows we will use the following well-known properties of polynomial grade.

Lemma 3.2. *(see e.g. [2]) Let $\mathfrak{a}$ be an ideal of a ring A and M an A-module. The following holds.*

(i) *If $\mathfrak{a}$ is finitely generated, then*

$$\operatorname{p.grade}_A(\mathfrak{a}, M) = \inf\{\operatorname{p.grade}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) | \mathfrak{p} \in \operatorname{V}(\mathfrak{a}) \cap \operatorname{Supp}_A M\}.$$

(ii) *Let Σ be the family of all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. Then*

$$\operatorname{p.grade}_A(\mathfrak{a}, M) = \sup\{\operatorname{p.grade}_A(\mathfrak{b}, M) : \mathfrak{b} \in \Sigma\}.$$

(iii) *We have* $\operatorname{p.grade}_A(\mathfrak{a}, M) \leq \operatorname{ht}_M(\mathfrak{a})$.

Now, we are ready to prove:

Theorem 3.3. *Let R be a Noetherian local F-coherent domain which is either excellent or homomorphic image of a Gorenstein local ring. Then* $\operatorname{ht}(\mathfrak{a}) = \operatorname{p.grade}_{R^\infty}(\mathfrak{a}, R^\infty)$ *for all ideal* $\mathfrak{a}$ *of* R^∞.

Proof. Let $\underline{x} := x_1, \ldots, x_d$ be a system of parameters for R, where $d := \dim R$. By Lemma 3.1, $\underline{x}$ is regular sequence on R^∞. So

$$d \leq \operatorname{p.grade}(\underline{x}R^\infty, R^\infty) \leq \operatorname{p.grade}(\mathfrak{m}_{R^\infty}, R^\infty) \leq \operatorname{ht}(\mathfrak{m}_{R^\infty}) = d,$$

which shows that $\operatorname{p.grade}(\mathfrak{m}_{R^\infty}, R^\infty) = \operatorname{ht}(\mathfrak{m}_{R^\infty})$ for the maximal ideal $\mathfrak{m}_{R^\infty}$ of R^∞. Now assume that $\mathfrak{a}$ is finitely generated and let P be a prime ideal of R^∞ such that $P \supseteq \mathfrak{a}$. Set $\mathfrak{p} := P \cap R$. Take $x \in (R_\mathfrak{p})^\infty$. Then $x^{p^n} \in R_\mathfrak{p}$ for some n, where p is the characteristic of R. Thus $x^{p^n} = a/b$ for some $a \in R$ and $b \in R \setminus \mathfrak{p}$. Look at $\frac{a^{1/p^n}}{b^{1/p^n}}$ as an element of R_P^∞. The assignment $x \mapsto \frac{a^{1/p^n}}{b^{1/p^n}}$ defines a well-define map between $(R_\mathfrak{p})^\infty$ and R_P^∞ which is in fact an isomorphism. By [5, Theorem 2.4.2], $R_\mathfrak{p}$ is F-coherent. Clearly, $R_\mathfrak{p}$ is either excellent or homomorphic image of a Gorenstein local ring. By the case of maximal ideals,

$$\operatorname{p.grade}(PR_P^\infty, R_P^\infty) = \operatorname{p.grade}(\mathfrak{m}_{(R_\mathfrak{p})^\infty}, (R_\mathfrak{p})^\infty) = \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(P).$$

This along with Lemma 3.2 yields that

$$\begin{aligned} \operatorname{p.grade}(\mathfrak{a}, R^\infty) &= \inf\{\operatorname{p.grade}(PR_P^\infty, R_P^\infty) | P \in \operatorname{V}(\mathfrak{a})\} \\ &= \inf\{\operatorname{ht}(P) | P \in \operatorname{V}(\mathfrak{a})\} \\ &= \operatorname{ht}(\mathfrak{a}), \end{aligned}$$

which shows that $\operatorname{p.grade}(\mathfrak{a}, R^\infty) = \operatorname{ht}(\mathfrak{a})$ for every finitely generated ideal $\mathfrak{a}$ of R^∞.

Finally we assume that $\mathfrak{a}$ is a general ideal of R^∞ and let Σ be the family of all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. We bring the following claim:

Claim: One has $\operatorname{ht}(\mathfrak{a}) = \sup\{\operatorname{ht}(\mathfrak{b}) : \mathfrak{b} \in \Sigma\}$.

To see this, let $P \in \operatorname{Spec}(R^\infty)$ be such that $\operatorname{ht}(P) = \operatorname{ht}(\mathfrak{a})$ and set $\mathfrak{p} := P \cap R$. As we saw in the above lines, $R_P^\infty = (R_\mathfrak{p})^\infty$. Set $n := \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(P)$. Due to [3, Theorem A.2], there exists a sequence $\underline{x} := x_1, \ldots, x_n$ of elements of $\mathfrak{p}$ such that $\operatorname{ht}(x_1, \ldots, x_i)R = i$ for all $1 \leq i \leq n$. Since R is catenary, $\underline{x}$ is part of a system of parameters for R. By Lemma 3.1, $\underline{x}$ is regular sequence on R^∞. So

$$\operatorname{ht}(P) \geq \operatorname{ht}(\underline{x}R^\infty) \geq \operatorname{p.grade}(\underline{x}R^\infty, R^\infty) = n = \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(P),$$

which shows that $\operatorname{ht}(\underline{x}R^\infty) = \operatorname{ht}(\mathfrak{a})$. This completes the proof of the claim.

By the case of finitely generated ideals, $\operatorname{p.grade}(\mathfrak{b}, R^\infty) = \operatorname{ht}(\mathfrak{b})$ for all $\mathfrak{b} \in \Sigma$. In light of Lemma 3.2 and the claim we see that

$$\begin{aligned} \operatorname{p.grade}(\mathfrak{a}, R^\infty) &= \sup\{\operatorname{p.grade}(\mathfrak{b}, R^\infty) : \mathfrak{b} \in \Sigma\} \\ &= \sup\{\operatorname{ht}(\mathfrak{b}) : \mathfrak{b} \in \Sigma\} \\ &= \operatorname{ht}(\mathfrak{a}), \end{aligned}$$

which is precisely what we wish to prove. □

Let M be an A-module. Recall that a prime ideal $\mathfrak{p}$ is weakly associated to M if $\mathfrak{p}$ is minimal over $(0 :_A m)$ for some $m \in M$.

Corollary 3.4. *Adopt the assumption of Theorem 3.3 and let $\mathfrak{a}$ be an ideal of R^∞ that generated by $\mathrm{ht}(\mathfrak{a})$ elements. Then all of the weak associated prime ideals of $R^\infty/\mathfrak{a}$ have the same height.*

Proof. See [2, Corollary 4.6] and its proof. □

Remark 3.5. (i): Concerning the proof of Theorem 3.3, we address the following question which is true for several classes of commutative rings such as valuation domains. Let A be a commutative ring and $\mathfrak{a}$ an ideal of A. Let Σ be the family of all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. Is $\mathrm{ht}(\mathfrak{a}) = \sup\{\mathrm{ht}(\mathfrak{b}) : \mathfrak{b} \in \Sigma\}$?

(ii): Let A be a ring with the property that $\mathrm{p.grade}_A(\mathfrak{m}; A) = \mathrm{ht}(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of A. One might ask whether $\mathrm{p.grade}_A(\mathfrak{a}; A) = \mathrm{ht}(\mathfrak{a})$ holds for all ideals $\mathfrak{a}$ of A. In view of [2, Example 3.11], this is not the case.

We close this section by the following result.

Proposition 3.6. *Let A be a quasilocal coherent big Cohen-Macaulay algebra over local ring R. Then A is balanced big Cohen-Macaulay.*

Proof. There exists a system of parameters $\underline{x} = x_1, \ldots, x_d$ of R which is an A-sequence. Let $\underline{y} = y_1, \ldots, y_d$ be a system of parameters A. Denote the Koszul grade of a finitely generated ideal $\mathfrak{a}$ by $\mathrm{K.grade}_A(\mathfrak{a}; -)$. Note that $\sqrt{\underline{x}A} = \sqrt{\underline{y}A}$. This yields that $\mathrm{K.grade}_A(\underline{y}A; A) = \mathrm{K.grade}_A(\underline{x}A; A) = d$. In view of [1, Lemma 5.1], we see that $\mathrm{K.grade}_A((y_1, \ldots, y_i)A; A) = i$, since A is coherent. By usual induction argument it turns out that $\underline{y}$ is an A-sequence. □

4. Dagger Closure and Big Cohen-Macaulay Ring

In this section we extend the notion of dagger closure to the submodules of a finitely generated module over a Noetherian local domain $(R, \mathfrak{m})$ and we present Corollary 4.6.

Definition 4.1. Let A be a local algebra with a normalized valuation $v : A \to \mathbb{R} \cup \{\infty\}$ and M be an A-module. Consider a submodule $N \subset M$. We say $x \in N_M^v$ if for every $\epsilon > 0$, there exists $a \in A$ such that $v(a) < \epsilon$ and $ax \in N$.

Definition 4.2. Let A be an algebra over a Noetherian local domain $(R, \mathfrak{m})$. Assume that A is equipped with a normalized value map $v : A \to \mathbb{R}$. Recall from [8] that A is called almost Cohen-Macaulay if each element of $((x_1, \ldots, x_{i-1})A :_A x_i)/(x_1, \ldots, x_{i-1})A$ is annihilated by elements of sufficiently small order with respect to v for all system of parameters $x_1, \ldots, x_d$ of A.

Proposition 4.3. *Let M, M' be modules over a local algebra A with a normalized valuation $v : A \to \mathbb{R} \cup \{\infty\}$. Consider the arbitrary submodules N, W of M. Then the following are true:*

(i) *N^v_M is a submodule of M containing N.*
(ii) *$(N^v_M)^v_M = N^v_M$.*
(iii) *If $N \subset W \subset M$, then $N^v_M \subset W^v_M$.*
(iv) *Let $f : M \to M'$ be a homomorphism. Then $f(N^v_M) \subset f(N)^v_{M'}$.*
(v) *If $N^v_M = N$, then $0^v_{M/N} = 0$.*

In addition to, if A is almost Cohen-Macaulay then following is true:

(vi) *Let $x_1, \dots, x_{k+1}$ be a partial system of parameters for A, and let $J = (x_1, \dots, x_k)A$. Suppose that there exists a surjective homomorphism $f : M \to A/J$ such that $f(u) = \bar{x}_{k+1}$, where $\bar{x}$ is the image of x in A/J. Then $(Au)^v_M \cap \ker f \subset (Ju)^v_M$.*

Proof. (i) Clearly $N \subset N^v_M$. For $x, y \in N^v_M$, take $\epsilon > 0$ and choose $a, b \in A$ such that $v(a), v(b) < \epsilon/2$ and $ax, by \in N$. Thus we have $v(ab) < \epsilon$ and $ab(x + y) \in N$. Thus $x + y \in N^v_M$. Consider $x \in N^v_M$ and $b \in A$. Since there exists $a \in A$ such that $v(a) < \epsilon$ and $ax \in N$ and since N is a submodule, we find $a(bx) \in N$ and $bx \in N^v_M$. Thus it is easy to see that N^v_M is a submodule containing N.

(ii) Take $x \in (N^v_M)^v_M$. For $\epsilon > 0$ and choose $a \in A$ such that $v(a) < \epsilon/2$ and $ax \in (N^v_M)$. Similarly, for $\epsilon > 0$ and choose $b \in A$ such that $v(b) < \epsilon/2$ and $(ba)x \in N$. Thus we find $ba \in A$ with $v(ba) < \epsilon$ such that $(ba)x \in N$. So $x \in (N^v_M)$, which yields the claim.

(iii) This is easy and we leave it to reader.

(iv) Consider $x \in N^v_M$, thus for every $\epsilon > 0$, there exists $a \in A$ such that $v(a) < \epsilon$ and $ax \in N$. This implies $f(ax) = af(x)$ is in $f(N)$ where $a \in A$ is of arbitrarily small positive order. So $f(x) \in f(N)^v_{M'}$.

(v) Consider $\bar{x} \in 0^v_{M/N}$ which is the image of x in M/N. This implies that $ax \in N$ for the element $a \in A$ of arbitrarily small positive order. So $x \in N^v_M = N$ and $\bar{x} \in 0$.

(vi) Take $x \in (Au)^v_M \cap \ker f$. For every $\epsilon > 0$ there exists $a \in A$ of $v(a) < \epsilon/2$ such that $ax = bu \in Au$ and $af(x) = 0 = bf(u) = b\bar{x}_{k+1}$. This implies $bx_{k+1} \in J$ i.e. $b \in (J : x_{k+1})$. Since A is almost Cohen-Macaulay, for every $\epsilon > 0$ there exists $c \in A$ of $v(c) < \epsilon/2$ such that $cb \in J$. Thus for $\epsilon > 0$ there exists $ac \in A$ of $v(ac) < \epsilon$ and $(ac)x = cbu \in Ju$. So $x \in (Ju)^v_M$.

□

Definition 4.4. Let (R, m) be a Noetherian local domain and let A be a local domain containing R with a normalized valuation $v : A \to \mathbb{R} \cup \{\infty\}$. For any finitely generated R-module M and for its submodule N we define submodule $N^{\mathbf{v}}_M$ such that $x \in N^{\mathbf{v}}_M$ if $x \otimes 1 \in \operatorname{im}(N \otimes A \to M \otimes A)^v_{M \otimes A}$.

Proposition 4.5. *Let $(R, \mathfrak{m})$ be a Noetherian local domain and let A be a local domain containing R with a normalized valuation $v : A \to \mathbb{R} \cup \{\infty\}$. Let M, M' be finitely generated R-modules. Consider the submodules N, W of M. Then the following are true:*

(i) *$N^{\mathbf{v}}_M$ is a submodule of M containing N.*

(ii) $(N^{\mathsf{v}}_M)^{\mathsf{v}}_M = N^{\mathsf{v}}_M$.
(iii) *If* $N \subset W \subset M$, *then* $N^{\mathsf{v}}_M \subset W^{\mathsf{v}}_M$.
(iv) *Let* $f : M \to M'$ *be a homomorphism. Then* $f(N^{\mathsf{v}}_M) \subset f(N)^{\mathsf{v}}_{M'}$.
(v) *If* $N^{\mathsf{v}}_M = N$, *then* $0^{\mathsf{v}}_{M/N} = 0$.
(vi) *We have* $0^{\mathsf{v}}_R = 0$ *and* $\mathfrak{m}^{\mathsf{v}}_R = \mathfrak{m}$.

In addition to, if A *is almost Cohen-Macaulay, then following is true:*

(vii) *Let* $x_1, \dots, x_{k+1}$ *be a partial system of parameters for* R, *and let* $J = (x_1, \dots, x_k)R$. *Suppose that there exists a surjective homomorphism* $f : M \to R/J$ *such that* $f(u) = \bar{x}_{k+1}$, *where* $\bar{x}$ *is the image of* x *in* R/J. *Then* $(Ru)^{\mathsf{v}}_M \cap \ker f \subset (Ju)^{\mathsf{v}}_M$.

Proof. (i) Clearly $N \subset N^{\mathsf{v}}_M$. To prove N^{v}_M is a submodule, let $x, y \in N^{\mathsf{v}}_M$ and $r \in R$. Then

$$x \otimes 1, y \otimes 1 \in \operatorname{im}(N \otimes A \to M \otimes A)^v_{M\otimes A}.$$

Thus $(x+y)\otimes 1, rx\otimes 1 \in \operatorname{im}(N\otimes A \to M\otimes A)^v_{M\otimes A}$. These yield that $x+y, rx \in N^{\mathsf{v}}_M$.

(ii) Let $x \in (N^{\mathsf{v}}_M)^{\mathsf{v}}_M$. This implies $x \otimes 1 \in \operatorname{im}(N^{\mathsf{v}}_M \otimes A \to M \otimes A)^v_{M\otimes A}$. Equivalently, for $\epsilon > 0$ there exists $a \in A$ such that $v(a) < \epsilon/2$ and $x \otimes a = \sum_{i=1}^l x_i \otimes a_i$ where $x_i \in N^{\mathsf{v}}_M$ and $a_i \in A$. Choose $c_i \in A$ such that for every i, $v(c_i) < \epsilon/2l$ and $x_i\otimes c_i \in \operatorname{im}(N\otimes A \to M\otimes A)$. Take $c = \prod_{i=1}^l c_i$. Thus $x\otimes ac \in \operatorname{im}(N\otimes A \to M\otimes A)$ and $x \in (N^{\mathsf{v}}_M)$.

(iii) This is trivial.

(iv) Let $x \in N^{\mathsf{v}}_M$ and this gives $x\otimes a = \sum_{i=1}^l x_i\otimes a_i$ for element $a \in A$ of arbitrarily small order, where $a_i \in A$ and $x_i \in N$. Applying $f\otimes 1_A$, we get $f(x)\otimes a = \sum_{i=1}^l f(x_i)\otimes a_i$ and thus we conclude.

(v) Denote the image of $x \in M$ in M/N by $\bar{x}$. If $\bar{x} \in 0^{\mathsf{v}}_{M/N}$ then $\bar{x} \otimes a \in 0$ and this implies $x \otimes a \in \operatorname{im}(N \otimes A \to M \otimes A)$ for element $a \in A$ of arbitrarily small order. Thus $x \in N^{\mathsf{v}}_M = N$ and $\bar{x} \in 0$.

(vi) Clearly, $0^v_A = 0$. So $0^{\mathsf{v}}_R = 0$. Suppose on the contrary that $s \in \mathfrak{m}^{\mathsf{v}}$ for some unit element s. It turns out that $sc \in \mathfrak{m}A$ for every element c of arbitrarily small order, i.e., $\mathfrak{m}A$ contain elements of arbitrarily small order, since s is a unit. This provides a contradiction, because $v(\mathfrak{m}A) > 0$.

(vii) Take $x \in (Ru)^{\mathsf{v}}_M \cap \ker f$. For every $\epsilon > 0$ there exists $a \in A$ of $v(a) < \epsilon/2$ such that $x \otimes a = \sum_{i=1}^n r_i u \otimes a_i$. Since $f(x) = 0$, $f(x) \otimes a = x_{k+1}(\sum_{i=1}^n r_i a_i) + JA = 0$. This implies $\sum_{i=1}^n r_i a_i \in (JA : x_{k+1}A) \subset (JA)^v$. So there exists $c \in A$ with $v(c) < \epsilon/2$ such that $c\sum_{i=1}^n r_i a_i \in JA$. Thus

$$x \otimes ac = u \otimes c \sum_{i=1}^n r_i a_i \in \operatorname{im}(Ru \otimes JA \to M \otimes A)$$

and this gives $x \otimes ac \in \operatorname{im}(Ju \otimes A \to M \otimes A)$ with $v(ac) < \epsilon$. So we conclude.

□

Corollary 4.6. *(see also* [1, Theorem 5.10]*) For a complete Noetherian local domain, if it is contained in an almost Cohen-Macaulay domain, then there exists a balanced big Cohen-Macaulay module over it.*

Proof. From above proposition we find that if a complete Noetherian local domain is contained in an almost Cohen-Macaulay domain then there exists a closure operation which satisfies [4, Axiom 1.1]. Then by [4, Theorem 3.16] we conclude. □

We end this paper by the following remark.

Remark 4.7. (i) One thing in Definition 4.4 is that it depends not only the value map but also the ring A. Thus we fixed A and v throughout the work. However, by Izumi's theorem [7], the notion of almost closure does not depend on the choice of value map over minimal perfect closure of complete domains.

(ii) Let A be a perfect domain and $\mathfrak{a}$ a nonzero radical ideal of A. Then $\mathfrak{a}^v = A$. To this end, let x be a nonzero element of $\mathfrak{a}$. It remains to note that $v(x^{1/n}) = v(x)/n$ and $x^{1/n} \in \mathfrak{a}$.

(iii) Let A be a coherent ring with a normalized valuation and let I be a finitely generated ideal A. Then $I^v = I$. Indeed, in order to show that we assume $0^v = 0$ in A/I. Take $x \in I^v$. This implies $ax \in I$ for $a \in A$ with arbitrarily small positive order. If the image of x in A/I is $\bar{x}$ then $a\bar{x} = 0$ in A/I for $a \in A$ with arbitrarily small positive order. But this means $\bar{x} \in 0^v = 0$ in A/I. So $x \in I$. This proves the first claim. In view of [5, Theorem 2.4.1], A/I is coherent. So it is sufficient to prove that $0^v = 0$ in coherent local ring A. Take $0 \neq x \in 0^v$. Consider $(0 : x) = I'$ which is finitely generated, see [5, Theorem 2.3.2]. It turns out that $v(I') > 0$ which is a contradiction.

(iv) A result similar as Corollary 4.6 is known by M. Tousi and his collaborators.

Acknowledgement . We would like to thank K. Shimomoto for his commands on the earlier version of this paper.

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