ON THE GEOMETRY OF COMPLETE RICCI SOLITONS

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Abstract. In this paper we establish three basic equations for a general soliton structure on the Riemannian manifold $(M, \langle\,,\rangle)$. We then draw some geometric conclusions with the aid of the maximum principle.

1. Introduction and main results

Let $(M, \langle\,,\rangle)$ be an m-dimensional, complete, connected Riemannian manifold. A *soliton structure* $(M, \langle\,,\rangle, X)$ on M is the choice (if any) of a smooth vector field X on M and a real constant λ such that

$$\mathrm{Ricc} + \frac{1}{2}\mathcal{L}_X \langle\,,\rangle = \lambda \langle\,,\rangle\,, \tag{1.1}$$

where Ricc denotes the Ricci tensor of the metric $\langle\,,\rangle$ on M and $\mathcal{L}_X \langle\,,\rangle$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the *soliton constant*. The soliton is called *expanding*, *steady* or *shrinking* if, respectively, $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. If X is the gradient of a potential $f \in C^\infty(M)$, then (1.1) takes the form

$$\mathrm{Ricc} + \mathrm{Hess}(f) = \lambda \langle\,,\rangle\,, \tag{1.2}$$

and the Ricci soliton is called a *gradient Ricci soliton*. Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation

$$\mathrm{Ricc} = \lambda \langle\,,\rangle$$

and reduce to this latter in case X or ∇f are Killing vector fields. When $X = 0$ or f is constant we call the underlying Einstein manifold a *trivial* Ricci soliton.

Since the appearance of the seminal works of R. Hamilton, [4], and G. Perelman, [7], the study of gradient Ricci solitons has become the

subject of a rapidly increasing investigation directed mainly towards two goals, *classification* and *triviality*; among the enormous literature on the subject we only quote, as a few examples, the papers [9], [11], [10], [16], [12], [2].

In this paper we focus our attention on the more general case of *Ricci solitons*, that is, when X is not necessarily the gradient of a potential f. A first important difference is that, in the present case, we cannot make use of the weighted manifold structure $\left(M, \langle\ ,\rangle, e^{-f}\mathrm{d}vol\right)$ which naturally arises when dealing with gradient solitons. The same applies for related concepts such as the Bakry-Emery Ricci tensor, giving rise to weighted volume estimates, or the weak maximum principle for the diffusion operator Δ_f (the "f-Laplacian"), acting on $u \in C^2(M)$ by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle ,$$

that we considered in a previous investigation (see [12] for details). Thus our assumptions and techniques have to rest on the original Riemannian structure of M. This restricts the applicability of the technical tools we used in [12]; nevertheless, we are still able to obtain some stringent geometric conclusions as those we are going to describe in a shortwhile.

From now on we fix an origin $o \in M$ and let $r(x) = \mathrm{dist}(x, o)$. We set B_r and ∂B_r to denote respectively the geodesic ball of radius r centered at o and its boundary.

To state our first result, we recall that, given a Schrödinger type operator $L = \Delta + q(x)$ for some $q(x) \in C^0(M)$, the *spectral radius* of L on M is defined via the Rayleigh characterization by

$$\lambda_1^L(M) = \inf_{\substack{\varphi \in C_0^\infty(M) \\ \varphi \not\equiv 0}} \frac{\int_M |\nabla \varphi|^2 - q(x)\varphi^2}{\int_M \varphi^2}.$$

Theorem 1.1. *Let $(M, \langle\ ,\rangle)$ be a complete manifold with Ricci tensor satisfying*

$$\mathrm{Ricc} \leq \frac{1}{2} a(x) \langle\ ,\rangle \tag{1.3}$$

for some $a(x) \in C^{0,\alpha}(M)$ *with* $0 \leq \alpha < 1$. *Assume that, for some* $H \geq 1$,

$$\lambda_1^{L_H}(M) \geq 0 \tag{1.4}$$

where $L_H = \Delta + Ha(x)$. *If there exists a soliton structures* $(M, \langle\,,\rangle, X)$ *on* $(M, \langle\,,\rangle)$ *with* $X \not\equiv 0$ *satisfying*

$$\left\{\int_{\partial B_r} |X|^{4(\beta+1)}\right\}^{-1} \notin L^1(+\infty) \tag{1.5}$$

for some $0 \leq \beta \leq H-1$, *then* X *is a parallel field and* $(M, \langle\,,\rangle)$ *is Einstein. Furthermore, the simply connected universal cover of* M *is a warped product* $(\mathbb{R} \times_c P, h)$ *with* $c = |X|$, $h = dt^2 + cg$ *and* (P, g) *is Einstein.*

Remark. If $|X| \in L^{4(\beta+1)}(M)$ then (1.5) is satisfied. Furthermore, if $X \equiv 0$ then $(M, \langle\,,\rangle)$ is trivially Einstein.

The following is a "second version" of Theorem 1.1.

Theorem 1.2. *Let* $(M, \langle\,,\rangle)$ *be a complete manifold and assume, for some* $0 \leq \alpha < 1$, *the validity on* M *of the Sobolev-Poincaré inequality*

$$\int_M |\nabla\varphi|^2 \geq S(\alpha)^{-1}\left\{|\varphi|^{\frac{2}{1-\alpha}}\right\}^{1-\alpha} \tag{1.6}$$

for each $\varphi \in C_0^\infty(M)$ *with a positive constant* $S(\alpha)$. *Suppose*

$$\mathrm{Ricc} \leq \frac{1}{2}a(x)\langle\,,\rangle \tag{1.3}$$

for some $a(x) \in C^{0,\alpha}(M)$ *with* $0 \leq \alpha < 1$; *let* $\sigma > \frac{1}{2}$ *and assume that*

$$\|a_+(x)\|_{L^{\frac{1}{\alpha}}(M)} < 4\frac{2\sigma-1}{\sigma^2}\frac{1}{S(\alpha)}. \tag{1.7}$$

Then there are no Ricci solitons $(M, \langle\,,\rangle, X)$ *on* $(M, \langle\,,\rangle)$ *satisfying* $X \not\equiv 0$ *and*

$$\int_{B_r} |X|^{2\sigma} = o\big(r^2\big) \quad \text{as } r \to +\infty. \tag{1.8}$$

We then consider a general triviality result:

Theorem 1.3. *Let $(M, \langle\,,\rangle)$ be a complete manifold with Ricci tensor satisfying*

$$\mathrm{Ricc} \leq -\frac{B}{(1+r(x))^{\mu}} \tag{1.9}$$

for some constant $B > 0$ and let $\sigma, \mu \in \mathbb{R}$ satisfy

$$\sigma \geq 0, \qquad \sigma + \mu < 2. \tag{1.10}$$

Suppose that

$$\liminf_{r(x)\to+\infty} \frac{\log \mathrm{vol}(B_r)}{r(x)^{2-\sigma-\mu}} = d_0 < +\infty. \tag{1.11}$$

Then there are no Ricci solitons $(M, \langle\,,\rangle, X)$ on M satisfying $X \not\equiv 0$ and

$$\limsup_{r(x)\to+\infty} \frac{|X|^2}{r(x)^{\sigma}} \begin{cases} = 0, & 0 < \sigma < 2, \\ < +\infty, & \sigma = 0. \end{cases} \tag{1.12}$$

Our next result relates the soliton constant λ with the infimum S_* of the scalar curvature of the manifold $(M, \langle\,,\rangle)$. We recall that this latter is said to *satisfy the Omori-Yau maximum principle* if for each $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\} \subset M$ such that

$$(i)\, u(x_k) > u^* - \frac{1}{k}, \quad (ii)\, |\nabla u(x_k)| < \frac{1}{k}, \quad (iii)\, \Delta u(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$. Conditions to insure the validity of the Omori-Yau maximum principle are discussed in [13].

Theorem 1.4. *Let $(M, \langle\,,\rangle)$ be a complete manifold of dimension m with scalar curvature $S(x)$ and satisfying the Omori-Yau maximum principle. Let $(M, \langle\,,\rangle, X)$ be a Ricci soliton on $(M, \langle\,,\rangle)$ with soliton constant λ. Assume*

$$|X|^* = \sup_M |X| < +\infty. \tag{1.13}$$

Let

$$S_* = \inf_M S.$$

(i) *If* $\lambda < 0$ *then* $m\lambda \leq S_* \leq 0$. *Furthermore, if* $S(x_0) = S_* = m\lambda$ *for some* $x_0 \in M$, *then* $(M, \langle\,,\rangle)$ *is Einstein and* X *is a Killing field, while if* $S(x_0) = S_* = 0$ *for some* $x_0 \in M$, *then* $(M, \langle\,,\rangle)$ *is Ricci flat and* X *is a homothetic vector field.*

(ii) *If* $\lambda = 0$ *then* $S_* = 0$. *Furthermore, if* $S(x_0) = S_* = 0$ *for some* $x_0 \in M$, *then* $(M, \langle\,,\rangle)$ *is Ricci flat and* X *is a Killing field.*

(iii) *If* $\lambda > 0$ *then* $0 \leq S_* \leq m\lambda$. *Furthermore, if* $S(x_0) = S_* = 0$ *for some* $x_0 \in M$, *then* $(M, \langle\,,\rangle)$ *is Ricci flat and* X *is a homothetic vector field, while if* $S(x_0) = S_* = m\lambda$ *for some* $x_0 \in M$, *then* $(M, \langle\,,\rangle)$ *is compact, Einstein and* X *is a Killing field.*

Remark. In case $X = \nabla f$, that is, the soliton is a gradient Ricci soliton and $(M, \langle\,,\rangle)$ is Einstein, a complete classification is given in Theorem 1.3 of [12].

From Theorem 1.4 we immediately obtain

Corollary 1.5. *Let* $(M, \langle\,,\rangle)$ *be a complete manifold with scalar curvature* S *such that*

$$S_* = \inf_M S(x) < 0 \qquad (\textit{resp.} \ > 0)$$

and Ricci tensor satisfying

$$\mathrm{Ricc} \geq -(m-1)B^2\big(1 + r^2\big) \tag{1.14}$$

for some $B \geq 0$. *Then* $(M, \langle\,,\rangle)$ *does not support any shrinking or steady (resp., expanding or steady) soliton with*

$$|X|^* = \sup_M |X| < +\infty.$$

Note that (1.14) implies the validity of the Omori-Yau maximum principle on any complete manifold $(M, \langle\,,\rangle)$. However, the validity of this latter is guaranteed also in other circumstances such as, for instance, those of the next

Corollary 1.6. *Let* $(M, \langle\,,\rangle)$ *be a complete manifold admitting a shrinking or steady soliton structure* $(M, \langle\,,\rangle, X)$ *with* $|X|^* = \sup_M |X| < +\infty$. *Then any proper minimal immersion of* M *into* $\mathbb{R}^n$, *with* $n > m = \dim M$, *is totally geodesic.*

In our last result we consider the conformally flat case.

Theorem 1.7. *Let $(M, \langle\ ,\rangle)$ be a complete manifold of dimension $m \geq 3$ with scalar curvature $S(x)$, trace-free Ricci tensor T and satisfying the Omori-Yau maximum principle. Assume that $(M, \langle\ ,\rangle)$ is conformally flat and that*

$$S^* = \sup_M S(x) < +\infty.$$

Let $(M, \langle\ ,\rangle, X)$ be a Ricci soliton on $(M, \langle\ ,\rangle)$ with soliton constant λ and $|X|^* = \sup_M |X| < +\infty$. Then either $(M, \langle\ ,\rangle)$ is of constant sectional curvature or $|T|^* = \sup_M |T|$ satisfies

$$|T|^* \geq \frac{1}{2}\left(\sqrt{m(m-1)}\lambda - S^*\frac{m-2}{\sqrt{m(m-1)}}\right). \tag{1.15}$$

2. Preliminary results

The proof of our results rests on three interesting formulas. The first (eq. (2.8)) is due to Bochner (at least in case X is a Killing field), as reported in [8], page 191. The remaining two, that is equations (2.21) and (2.52), have been found for gradient Ricci solitons in [12] but are new and in fact unexpected, at least for us, in the present generality. In what follows, to perform computations, we shall use the method of the moving frame referring to a local orthonormal coframe $\{\theta^i\}$ for the metric and corresponding Levi-Civita connection and curvature forms, indicated respectively with $\{\theta^i_j\}$ and $\{\Theta^i_j\}$, $1 \leq i, j, \ldots \leq m = \dim M$. The Einstein summation convention will be in force throughout.

The following generalized version of the Bochner formula is probably well known; we include a proof here for the sake of completeness.

Lemma 2.1. *(Generalized Bochner formula) Let Y be a vector field on M. Then*

$$\operatorname{div}(\mathcal{L}_Y \langle\ ,\rangle)(Y) = \frac{1}{2}\Delta|Y|^2 - |\nabla Y|^2 + \operatorname{Ricc}(Y, Y) + \nabla_Y(\operatorname{div} Y), \tag{2.1}$$

where $\mathcal{L}_Y \langle\ ,\rangle$ is the Lie derivative of the metric in the direction of Y.

Proof. Let $\{e_i\}$ be the o.n. frame dual to $\{\theta^i\}$. Then

$$Y = Y^i e_i = Y_i e_i$$

and setting Y_{ij} for the coefficients of the covariant derivative ∇Y of Y we have

$$Y_{ij}\theta^j = \mathrm{d}Y_i - Y_k\theta^k_i. \tag{2.2}$$

Differentiating (2.2), using the definition of covariant derivative, the structure equations

$$\mathrm{d}\theta^i = -\theta^i_j \wedge \theta^j, \qquad \mathrm{d}\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j$$

and the components R^i_{jkt} of the Riemann curvature tensor defined by

$$\Theta^i_j = \frac{1}{2}R^i_{jkt}\theta^k \wedge \theta^t$$

we obtain

$$Y_{ikj}\theta^j \wedge \theta^k = -\frac{1}{2}Y_t R^t_{ijk}\theta^j \wedge \theta^k.$$

Thus, inverting the indexes k and j

$$Y_{ijk}\theta^k \wedge \theta^j = -\frac{1}{2}Y_t R^t_{ikj}\theta^k \wedge \theta^j.$$

Comparing these last two equations we deduce

$$Y_{ijk} - Y_{ikj} = Y_t R^t_{ijk}. \tag{2.3}$$

Since

$$\mathcal{L}_Y \langle\ ,\rangle = (Y_{ik} + Y_{ki})\theta^i \otimes \theta^k$$

we have

$$\mathrm{div}\,(\mathcal{L}_Y \langle\ ,\rangle)(Y) = Y_i Y_{ikk} + Y_i Y_{kik}. \tag{2.4}$$

From the commutation relation (2.3) tracing with respect to i and k we obtain

$$Y_{kik} = Y_{kki} + Y_t R^t_{kik} = Y_{kki} + Y_t R_{ti}, \tag{2.5}$$

where, as usual, with R_{ti} we have indicated the components of the Ricci tensor. Thus

$$Y_i Y_{kik} = \nabla_Y(\mathrm{div}\, Y) + \mathrm{Ricc}\,(Y, Y). \tag{2.6}$$

On the other hand, from $|Y|^2 = Y_i Y_i$ we deduce

$$\mathrm{d}|Y|^2 = 2Y_i Y_{ik}\theta^k$$

and

$$\Delta|Y|^2 = 2Y_{ik}Y_{ik} + 2Y_iY_{ikk},$$

or, in other words,

$$\frac{1}{2}\Delta|Y|^2 = |\nabla Y|^2 + Y_iY_{ikk}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.4) we immediately obtain (2.1). q.e.d.

Remark. If $Y = \nabla f$, $f \in C^\infty(M)$, then (2.1) can be rewritten as

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &= |\mathrm{Hess}(f)|^2 - \mathrm{Ricc}\,(\nabla f, \nabla f) - \langle \nabla\Delta f, \nabla f\rangle \\ &\quad + \mathrm{div}\,(2\,\mathrm{Hess}(f))(\nabla f) \\ &= |\mathrm{Hess}(f)|^2 + \mathrm{Ricc}\,(\nabla f, \nabla f) + \langle \nabla\Delta f, \nabla f\rangle\,,\end{aligned}$$

that is, (2.1) in this case is the classical Bochner's formula.

Proposition 2.2. *Let $(M, \langle\,,\rangle, X)$ be a Ricci soliton on $(M, \langle\,,\rangle)$. Then*

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ricc}\,(X, X). \tag{2.8}$$

Proof. We trace the soliton equation (1.1) to obtain

$$S + \mathrm{div}\,X = m\lambda,$$

and from here we deduce

$$\nabla S = -\nabla\,\mathrm{div}\,X. \tag{2.9}$$

On the other hand, contracting twice the second Bianchi's identities we have the well-known formula

$$\nabla S = 2\,\mathrm{div}\,\mathrm{Ricc}\,. \tag{2.10}$$

Thus, comparing (2.9) and (2.10),

$$\nabla\,\mathrm{div}\,X = -2\,\mathrm{div}\,\mathrm{Ricc}\,. \tag{2.11}$$

Now taking the divergence of (1.1) and using the fact that $\mathrm{div}\,(\lambda\langle\,,\rangle) = 0$ we obtain

$$\mathrm{div}\,(\mathcal{L}_X\langle\,,\rangle) = -2\,\mathrm{div}\,\mathrm{Ricc}$$

and (2.11) yields

$$\nabla \operatorname{div} X = \operatorname{div}(\mathcal{L}_X \langle\ ,\rangle).$$

In particular,

$$\nabla_X \operatorname{div} X = \operatorname{div}(\mathcal{L}_X \langle\ ,\rangle)(X). \tag{2.12}$$

Thus applying (2.1) of Lemma 2.1 we immediately get (2.8). q.e.d.

Remark. In case $X = \nabla f$, that is, the soliton is a gradient soliton, (2.8) becomes

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}(f)|^2 - \mathrm{Ricc}(\nabla f, \nabla f).$$

Then, using the identity

$$\frac{1}{2}\left\langle \nabla f, \nabla|\nabla f|^2 \right\rangle = \mathrm{Hess}(f)(\nabla f, \nabla f)$$

and the gradient Ricci soliton equation (1.2), we deduce

$$\frac{1}{2}\Delta|\nabla f|^2 = \frac{1}{2}\left\langle \nabla f, \nabla|\nabla f|^2 \right\rangle + |\mathrm{Hess}(f)|^2 - \lambda|\nabla f|^2.$$

This latter, with the aid of the diffusion operator Δ_f of the Introduction, can be written as

$$\frac{1}{2}\Delta_f|\nabla f|^2 = |\mathrm{Hess}(f)|^2 - \lambda|\nabla f|^2. \tag{2.13}$$

Formula (2.13) has often appeared in the recent literature on gradient Ricci solitons: see for instance [2], [12] and the references therein.

Before proceeding to the next proposition we need to determine some further "commutation relations".

Lemma 2.3. *Let Y be a vector field on M. Then*

$$Y_{tkkt} - Y_{kktt} = \frac{1}{2}\langle \nabla S, Y\rangle + \frac{1}{2}\operatorname{tr}(\mathcal{L}_Y \langle\ ,\rangle \circ \mathrm{Ricc}). \tag{2.14}$$

Proof. We start from the commutation relations (2.3). By taking covariant derivative we deduce

$$Y_{ijkt} - Y_{ikjt} = Y_{st}R_{sijk} + Y_s R_{sijk,\,t}. \tag{2.15}$$

Next we recall that, by definition of covariant derivative,

$$Y_{ijk}\theta^k = \mathrm{d}Y_{ij} - Y_{tj}\theta^t_i - Y_{it}\theta^t_j. \tag{2.16}$$

Thus differentiating both members of (2.16), using the structure equations and (2.16) itself, we arrive at

$$Y_{ijkl}\theta^l \wedge \theta^k = -\frac{1}{2}(Y_{tj}R_{tilk} + Y_{it}R_{tjlk})\theta^l \wedge \theta^k,$$

from which, inverting k with l and summing up, we deduce

$$Y_{ijkl} - Y_{ijlk} = Y_{tj}R_{tikl} + Y_{it}R_{tjkl}. \tag{2.17}$$

Now, (2.14) follows immediately from (2.15), (2.17), (2.10) and tracing. q.e.d.

For later use we also recall:

Lemma 2.4. *For the Ricci tensor we have*

$$R_{ij,\,k} = R_{ji,\,k}; \tag{2.18}$$

$$R_{ij,\,k} - R_{ik,\,j} = -R_{tijk,\,t}; \tag{2.19}$$

$$R_{ij,\,kl} - R_{ij,\,lk} = R_{is}R_{sjkl} + R_{js}R_{sikl}. \tag{2.20}$$

Proof. (2.18) is obvious. (2.19) follows from the second Bianchi's identities, while (2.20) can be obtained with the same methods used in the proofs of Lemmas 2.1 and 2.3. q.e.d.

We are now ready to prove

Proposition 2.5. *Let $(M, \langle\,,\rangle, X)$ be a Ricci soliton with soliton constant λ on $(M, \langle\,,\rangle)$ and let $S(x)$ be the scalar curvature. Then*

$$\begin{aligned}\frac{1}{2}\Delta S &= \frac{1}{2}\langle \nabla S, X\rangle + \lambda S - |\mathrm{Ricc}|^2 \\ &= \frac{1}{2}\langle \nabla S, X\rangle + \lambda S - \frac{S^2}{m} - \left|\mathrm{Ricc} - \frac{S}{m}\langle\,,\rangle\right|^2.\end{aligned} \tag{2.21}$$

Proof. We start from the soliton equation (1.1), which in components reads

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda\delta_{ij}. \tag{2.22}$$

Differentiating (2.22) we get

$$R_{ij,\,k} = -\frac{1}{2}(X_{ijk} + X_{jik}). \tag{2.23}$$

From (2.3) applied to X and (2.23) we obtain

$$2(R_{ik,\,j} - R_{jk,\,i}) = X_{jki} - X_{ikj} + X_t R_{tkji}. \tag{2.24}$$

Taking covariant derivatives we deduce the further commutation relation

$$2(R_{ik,\,jt} - R_{jk,\,it}) = X_{jkit} - X_{ikjt} + X_{st}R_{skji} + X_s R_{skji,\,t}. \tag{2.25}$$

Contracting (2.20) with respect to i and l we get

$$R_{lj,\,kl} - R_{lj,\,lk} = R_{ls}R_{sjkl} + R_{js}R_{sk}; \tag{2.26}$$

note that, from (2.19), we deduce

$$X_s R_{ijks,\,j} = X_s R_{skji,\,j} = X_s R_{ik,\,s} - X_s R_{is,\,k}. \tag{2.27}$$

Now, from (2.25),

(2.28)

$$\Delta R_{ik} = R_{ik,\,jj} = R_{jk,\,ij} + \frac{1}{2}(X_{jkij} - X_{ikjj} + X_{sj}R_{skji} + X_s R_{skji,\,j}).$$

Inserting (2.27) into (2.28) we get

(2.29)

$$\Delta R_{ik} = R_{jk,\,ij} + \frac{1}{2}(X_{jkij} - X_{ikjj}) + \frac{1}{2}X_{sj}R_{skji} + \frac{1}{2}X_s R_{ik,\,s} - \frac{1}{2}X_s R_{is,\,k}.$$

Next, we rewrite (2.20) in the form

$$R_{jk,\,ij} = R_{jk,\,ji} + R_{js}R_{skij} + R_{ks}R_{si} \tag{2.30}$$

and we insert (2.30) into (2.29) to get

$$\begin{aligned}\Delta R_{ik} = {} & \frac{1}{2}X_{sj}R_{skji} + \frac{1}{2}X_s R_{ik,\,s} - \frac{1}{2}X_s R_{is,\,k} \\ & + R_{jk,\,ji} + R_{js}R_{skij} + R_{ks}R_{si} + \frac{1}{2}(X_{jkij} - X_{ikjj}).\end{aligned} \tag{2.31}$$

From the second Bianchi's identities we recall that

$$2R_{ik,\,i} = S_k, \tag{2.32}$$

so that

$$R_{jk,\,ji} = \frac{1}{2}S_{ki}. \tag{2.33}$$

Using (2.33) into (2.31) yields

$$\Delta R_{ik} = \frac{1}{2}S_{ki} + \frac{1}{2}X_{sj}R_{skji} + \frac{1}{2}X_sR_{ik,\,s} - \frac{1}{2}X_sR_{is,\,k} + R_{js}R_{skij} + R_{ks}R_{si} + \frac{1}{2}(X_{jkij} - X_{ikjj}). \tag{2.34}$$

Note that from the soliton equation (1.1)

$$X_{sj} = -X_{js} + 2\lambda\delta_{sj} - 2R_{sj}, \tag{2.35}$$

and therefore

$$\frac{1}{2}X_{sj}R_{skji} = -\frac{1}{2}X_{js}R_{skji} + \lambda R_{ki} - R_{sj}R_{skji}. \tag{2.36}$$

Substituting (2.36) into (2.34) gives

$$\begin{aligned}\Delta R_{ik} = {} & \frac{1}{2}S_{ki} - \frac{1}{2}X_{js}R_{skji} + \lambda R_{ki} - 2R_{sj}R_{skji} + \\ & + \frac{1}{2}X_sR_{ik,\,s} - \frac{1}{2}X_sR_{is,\,k} + R_{ks}R_{si} + \\ & + \frac{1}{2}(X_{jkij} - X_{ikjj}).\end{aligned} \tag{2.37}$$

We trace (2.37) with respect to i and k and use the relation

$$\frac{1}{2}X_tR_{kk,\,t} - \frac{1}{2}X_tR_{kt,\,k} = \frac{1}{4}X_tR_{kk,\,t} = \frac{1}{4}X_tS_t$$

so that

$$\begin{aligned}\frac{1}{2}\Delta S = {} & \lambda S - \frac{1}{2}X_{js}R_{sj} - R_{sj}R_{sj} + \\ & + \frac{1}{2}X_tR_{kk,\,t} - \frac{1}{2}X_tR_{kt,\,k} + \frac{1}{2}(X_{jkkj} - X_{kkjj}) \\ = {} & \lambda S - |\mathrm{Ricc}|^2 - \frac{1}{2}X_{js}R_{sj} + \frac{1}{2}(X_{tkkt} - X_{kktt}) + \frac{1}{4}X_tS_t.\end{aligned} \tag{2.38}$$

Now we apply Lemma 2.3 to X and from (2.38) we immediately obtain the desired result. q.e.d.

Remark. In case $X = \nabla f$, $f \in C^\infty(M)$, that is, the soliton is a gradient soliton, we can rewrite (2.21) in the form

$$\frac{1}{2}\Delta_f S = \lambda S - |\mathrm{Ricc}|^2,$$

which is formula (2.15) (with λ constant) of Lemma 2.3 in [12].

Our aim is now to compute $\Delta|T|^2$, where T is the traceless Ricci tensor, that is,

$$T_{ij} = R_{ij} - \frac{S}{m}\delta_{ij}. \tag{2.39}$$

Thus

$$|T|^2 = |\mathrm{Ricc}|^2 - \frac{S^2}{m}, \tag{2.40}$$

and it follows that

$$\begin{aligned}\Delta|T|^2 &= \Delta|\mathrm{Ricc}|^2 - \frac{1}{m}\Delta S^2 = \\ &= 2|\nabla \mathrm{Ricc}|^2 + 2\langle \mathrm{Ricc}, \Delta \mathrm{Ricc}\rangle - \frac{2}{m}S\Delta S - \frac{2}{m}|\nabla S|^2.\end{aligned} \tag{2.41}$$

We have

Proposition 2.6. *Let $(M, \langle\,,\rangle, X)$ be a Ricci soliton with soliton constant λ on $(M, \langle\,,\rangle)$ and let $S(x)$ be the scalar curvature. Then,*

$$\begin{aligned}\frac{1}{2}\Delta|T|^2 = \frac{1}{2}\left\langle \nabla|T|^2, X\right\rangle + |\nabla T|^2 + 2\lambda|T|^2 + \\ + \frac{2}{m}\left(\frac{S^2}{m} + |T|^2\right)S + 2R_{ik}R_{sj}R_{skij}.\end{aligned} \tag{2.42}$$

Proof. Using (2.37) we have

(2.43)

$$\begin{aligned}2\langle \mathrm{Ricc}, \Delta \mathrm{Ricc}\rangle &= 2R_{ik}\Delta R_{ik} = \\ &= 2\lambda|\mathrm{Ricc}|^2 + 2\,\mathrm{Tr}(\mathrm{Ricc}^3) + \frac{1}{2}\left\langle \nabla|\mathrm{Ricc}|^2, X\right\rangle + R_{ik}S_{ik} - \\ &- X_{js}R_{skji}R_{ik} - X_s R_{ik}R_{is,k} - 4R_{ik}R_{sj}R_{skji} + \\ &+ X_{jkij}R_{ik} - X_{ikjj}R_{ik}.\end{aligned}$$

First we analyze the term $X_{jkij}R_{ik}$. Towards this aim we consider the soliton equation

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda\delta_{ij}. \tag{2.22}$$

Tracing with respect to i and j we obtain

$$S + X_{tt} = m\lambda,$$

so that, taking covariant derivatives,

$$S_i = -X_{tti} \tag{2.44}$$

and similarly from (2.44)

$$S_{ik} = -X_{ttik}.$$

It follows that

$$R_{ik}S_{ik} = -X_{ttik}R_{ik}. \tag{2.45}$$

From the commutation relations (2.17) and (2.15) we get

$$\begin{aligned} X_{jkij} &= X_{jkji} + X_{tk}R_{ti} + X_{jt}R_{tkij} = \\ &= X_{jjki} + X_{si}R_{sk} + X_sR_{sk,i} + X_{tk}R_{ti} + X_{jt}R_{tkij} \end{aligned}$$

and therefore, using (2.45) and soliton equation (2.22),

$$\begin{aligned} R_{ik}X_{jkij} &= -S_{ik}R_{ik} + R_{ik}(X_{kt} + X_{tk})R_{ti} + X_sR_{ik}R_{sk,i} + \\ &+ X_{jt}R_{ik}R_{tkij} = \\ &= -S_{ik}R_{ik} + 2\lambda|\mathrm{Ricc}|^2 - 2\,\mathrm{Tr}(\mathrm{Ricc}^3) + X_sR_{ik}R_{sk,i} + \\ &+ X_{jt}R_{ik}R_{tkij}. \end{aligned}$$

Substituting this latter into (2.43) and simplifying we obtain

$$\begin{aligned} 2R_{ik}\Delta R_{ik} = 4\lambda|\mathrm{Ricc}|^2 + \frac{1}{2}\left\langle \nabla|\mathrm{Ricc}|^2, X\right\rangle - 2X_{js}R_{ik}R_{skji} - \\ - 4R_{ik}R_{sj}R_{skji} - X_{iktt}R_{ik}. \end{aligned} \tag{2.46}$$

Now we analyze the term $X_{iktt}R_{ik}$. Towards this end we take covariant derivative of the soliton equation (2.22):

$$R_{ij,k} = -\frac{1}{2}(X_{ijk} + X_{jik}).$$

Tracing with respect to j and k we get

$$R_{ik,k} = -\frac{1}{2}(X_{ikk} + X_{kik}),$$

so that, using (2.32), (2.5) and (2.44),

$$S_k = -X_{ktt} - X_{tkt} = -X_{ktt} - X_{ttk} - X_sR_{sk} = S_k - X_{ktt} - X_sR_{sk},$$

that is,

$$X_{itt} = -X_s R_{si}.$$

Taking covariant derivative of this latter

$$X_{ittk} = -X_{sk} R_{si} - X_s R_{si,k}. \tag{2.47}$$

Next, from (2.17) and (2.15) we obtain

$$\begin{aligned} X_{ittk} &= X_{itkt} + X_{st} R_{sitk} + X_{is} R_{sttk} = \\ &= X_{iktt} + 2X_{st} R_{sitk} - X_{is} R_{sk} + X_s R_{sitk,t}. \end{aligned}$$

Hence, using (2.47) and (2.19) we deduce

(2.48)

$$\begin{aligned} R_{ik} X_{iktt} &= -X_s R_{ik} R_{si,k} - X_s R_{ik} R_{sitk,t} - 2X_{st} R_{sitk} R_{ik} = \\ &= -X_s R_{ik} R_{si,k} - 2X_{st} R_{sitk} R_{ik} + X_s R_{ik} (R_{ks,i} - R_{ki,s}) = \\ &= -\frac{1}{2} \left\langle \nabla |\mathrm{Ricc}|^2, X \right\rangle - 2X_{st} R_{sitk} R_{ik}. \end{aligned}$$

We substitute (2.48) into (2.46) to get

$$2R_{ik} \Delta R_{ik} = 4\lambda |\mathrm{Ricc}|^2 + \left\langle \nabla |\mathrm{Ricc}|^2, X \right\rangle + 4R_{ik} R_{sj} R_{skij}. \tag{2.49}$$

Thus, from (2.41), (2.49) and (2.21) we obtain

$$\begin{aligned} \Delta |T|^2 = {} & 2|\nabla \,\mathrm{Ricc}|^2 - \frac{2}{m} |\nabla S|^2 + 4\lambda |\mathrm{Ricc}|^2 + \left\langle \nabla |\mathrm{Ricc}|^2, X \right\rangle + \\ & + 4R_{ik} R_{sj} R_{skij} - 4\frac{\lambda}{m} S^2 - \frac{2}{m} S \left\langle \nabla S, X \right\rangle + \frac{4}{m^2} S^3 + \\ & + \frac{4}{m} S |T|^2. \end{aligned} \tag{2.50}$$

An immediate computation shows that

$$|\nabla T|^2 = |\nabla \,\mathrm{Ricc}|^2 - \frac{1}{m} |\nabla S|^2.$$

Using this fact and (2.40), after some algebraic manipulation from (2.50) we obtain (2.42). q.e.d.

We recall the decomposition of the curvature tensor ($m \geq 3$) into its irreducible components:

$$R_{ijks} = W_{ijks} + \frac{1}{m-2}(R_{ik}\delta_{js} - R_{is}\delta_{jk} + R_{js}\delta_{ik} - R_{jk}\delta_{is}) - \\ - \frac{S}{(m-1)(m-2)}(\delta_{ik}\delta_{js} - \delta_{is}\delta_{jk}), \tag{2.51}$$

where W_{ijks} are the component of the Weyl curvature tensor W. Note that $(M, \langle\,,\rangle)$, $m = \dim M \geq 3$, is conformally flat if and only if $W \equiv 0$.

We are now ready to prove

Corollary 2.7. *Let $(M, \langle\,,\rangle, X)$ be a Ricci soliton with soliton constant λ on $(M, \langle\,,\rangle)$ and let $S(x)$ be the scalar curvature. Assume $m \geq 3$ and that $(M, \langle\,,\rangle)$ is conformally flat. Then*

$$\frac{1}{2}\Delta|T|^2 = \frac{1}{2}\left\langle \nabla|T|^2, X\right\rangle + |\nabla T|^2 + 2\left(\lambda - \frac{m-2}{m(m-1)}S\right)|T|^2 + \\ + \frac{4}{m-2}\operatorname{Tr}(T^3). \tag{2.52}$$

Proof. A computation shows that

$$\operatorname{Tr}(T^3) = \operatorname{Tr}(\mathrm{Ricc}^3) - \frac{3}{m}S|\mathrm{Ricc}|^2 + \frac{2}{m^2}S^3. \tag{2.53}$$

Thus a simple algebraic manipulation using $W \equiv 0$, (2.51) and (2.53) gives

$$2R_{ik}R_{sj}R_{skij} = \frac{4}{m-2}\operatorname{Tr}(T^3) - 2\frac{2m-3}{m(m-1)}S|T|^2 - \frac{2}{m^2}S^3. \tag{2.54}$$

Inserting (2.54) into (2.42) immediately yields the desired equation (2.52). q.e.d.

Remark. In case $X = \nabla f$, $f \in C^\infty(M)$, that is, the soliton is a gradient soliton, we can rewrite (2.52) in the form

$$\frac{1}{2}\Delta_f|T|^2 = |\nabla T|^2 + 2\left(\lambda - \frac{m-2}{m(m-1)}S\right)|T|^2 + \frac{4}{m-2}\operatorname{Tr}(T^3),$$

which is formula (2.21) (with λ constant) of Corollary 2.7 in [12].

3. Proof of the main results

3.1. **Proof of Theorem 1.1 and a further result.** First of all, using Cauchy-Schwarz inequality we have that for any vector field Y on M

$$\frac{1}{4}\Big|\nabla|Y|^2\Big|^2 \leq |Y|^2|\nabla Y|^2. \tag{3.1}$$

We set $u = |X|^2$, we multiply (2.8) by u and use (3.1) to obtain

$$\frac{1}{2}u\Delta u + u\,\mathrm{Ricc}\,(X,X) \geq \frac{1}{4}|\nabla u|^2. \tag{3.2}$$

Next we use assumption (1.3) to deduce

$$u\Delta u + a(x)u^2 \geq \frac{1}{2}|\nabla u|^2. \tag{3.3}$$

From the work of Fisher-Colbrie and Schoen, [3], we know that assumption (1.4) implies the existence of $\varphi \in C^2(M)$, $\varphi > 0$, solution of

$$\Delta\varphi + Ha(x)\varphi = 0 \qquad \text{on } M. \tag{3.4}$$

Next, we apply the proof of Theorem 3.1 of [15] with $b(x) \equiv 0$, $K = 0$, $A = -\frac{1}{2}$ under assumption (1.5) which replaces assumption (3.6) of Theorem 3.1 with $p = 2$ and $0 \leq \beta \leq H - 1$, to arrive up to the conclusion

$$u^H = C\varphi \tag{3.5}$$

for some $C \geq 0$. Since by assumption $X \not\equiv 0$ we conclude that $C > 0$ and $u > 0$ on M. We insert the expression of φ in terms of u in (3.4) and divide by Hu^{H-2} to obtain

$$u\Delta u + a(x)u^2 = -(H-1)|\nabla u|^2.$$

Thus, from (3.3) we deduce that u and hence $|X|^2$ are constant. We then go back to (3.2) to obtain, using (1.3) and (2.8),

$$\frac{1}{2}a(x)|X|^2 \geq \mathrm{Ricc}\,(X,X) = |\nabla X|^2. \tag{3.6}$$

However, from (3.5) φ is a positive constant and (3.4) implies, since $H \geq 1$, $a(x) \equiv 0$. Thus from (3.6) $|\nabla X| \equiv 0$ on M and X is a parallel vector field. Thus X is a Killing field and going back to (1.1)

$$\mathrm{Ricc} = \lambda\,\langle\,,\rangle\,,$$

that is, $(M, \langle\,,\rangle)$ is Einstein. Now, since X is parallel, X is a closed conformal field and the final part of the Theorem follows from (c) of Proposition 2 in [5] and from Corollary 9.107 of [1]. q.e.d.

The next result is a consequence of the previous proof.

Proposition 3.1. *In the assumptions of Theorem 1.1 suppose* $\mathrm{vol}(M) = +\infty$. *Thus, there are no soliton structures* $(M, \langle\,,\rangle, X)$ *on* $(M, \langle\,,\rangle)$ *with* $X \not\equiv 0$ *and* $X \in L^p(M)$ *for some* $p > 0$.

Proof. We proceed as above up to showing that $|X|^2$ is a positive constant. Thus the result immediately follows since $\mathrm{vol}(M) = +\infty$. q.e.d.

3.2. **Proof of Theorem 1.2.** Let $|X|^2 = u$. Then u satisfies (3.3). Now apply Theorem 9.12 of [14] to contradict (1.7). q.e.d.

3.3. **Proof of Theorem 1.3.** From equation (2.8) we deduce

$$\Delta|X|^2 = 2|\nabla X|^2 - 2\,\mathrm{Ricc}(X,X) \geq \frac{2B}{(1+r(x))^\mu}|X|^2 \quad \text{on } M. \tag{3.7}$$

Assume now $|X|^2 \not\equiv 0$ and choose $\gamma > 0$ such that $\Omega_\gamma = \left\{x \in M : |X|^2 > \gamma\right\} \neq \emptyset$. On Ω_γ we have then, using (3.7),

$$(1+r(x))^\mu \Delta|X|^2 \geq 2B|X|^2 > 2B\gamma > 0,$$

which implies

$$\inf_{\Omega_\gamma} (1+r(x))^\mu \Delta|X|^2 > 0.$$

Applying now Theorem 4.1 in [13] (with $\varphi(t) = t, A = \delta = 1$) we obtain a contradiction. q.e.d.

3.4. **Proof of Theorem 1.4.** From (2.21) of Proposition 2.5 we have

$$\frac{1}{2}\Delta S \leq \frac{1}{2}\langle \nabla S, X\rangle + \lambda S - \frac{S^2}{m}. \tag{3.8}$$

Thus, $u = -S$ satisfies the differential inequality

$$\frac{1}{2}\Delta u \geq \frac{1}{2}\langle \nabla u, X\rangle + \lambda u + \frac{u^2}{m}.$$

Therefore, from (1.13) and the above we have

$$\frac{1}{2}\Delta u \geq -\frac{1}{2}|X|^*|\nabla u| + \lambda u + \frac{u^2}{m}. \tag{3.9}$$

Now the validity of the Omori-Yau maximum principle on M implies that of Theorem 1.31 on [13] that we apply with the choices $F(t) = t^2$ and

$$\varphi(u, |\nabla u|) = -\frac{1}{2}|X|^*|\nabla u| + \lambda u + \frac{u^2}{m}.$$

Then $u^* = \sup_M u < +\infty$ and

$$\lambda u^* + \frac{(u^*)^2}{m} \leq 0. \tag{3.10}$$

But $u^* = -S_*$, so that the claimed bounds on S_* in the statement of Theorem 1.4 follow immediately from (3.10).

Case (i). Suppose now $\lambda < 0$ and that for some $x_0 \in M$

$$S(x_0) = S_* = m\lambda.$$

In particular $S(x) \geq m\lambda$ on M and the function $w = S - m\lambda$ is non-negative on M. From (3.8) we immediately see that

$$\Delta w - \langle X, \nabla w\rangle + 2\lambda w \leq \Delta w - \langle X, \nabla w\rangle + 2\frac{S}{m}w \leq 0. \tag{3.11}$$

We let

$$\Omega_0 = \{x \in M : w(x) = 0\}.$$

Ω_0 is closed and not empty since $x_0 \in \Omega_0$; let now $y \in \Omega_0$. By the maximum principle applied to (3.11) $w \equiv 0$ in a neighborhood of y so that Ω_0 is open. Thus $\Omega_0 = M$ and $S(x) \equiv \lambda m$ on M. From equation (2.21) we then deduce $\left|\mathrm{Ricc} - \frac{S}{m}\langle\ ,\rangle\right| \equiv 0$, that is, $(M, \langle\ ,\rangle)$ is Einstein and from (1.1) X is a Killing field. Analogously, if $S(x_0) = S_* = 0$ for some $x_0 \in M$, we deduce that $(M, \langle\ ,\rangle)$ is Ricci flat and X is a homothetic vector field.

Case (ii). Suppose $\lambda = 0$ and that, for some $x_0 \in M$,

$$S(x_0) = S_* = 0.$$

From (3.8)

$$\Delta S - \langle X, \nabla S\rangle \leq -\frac{S^2}{m} \leq 0.$$

Since $S(x) \geq S_* = 0$, by the maximum principle we conclude $S(x) \equiv 0$, by (2.21) $(M, \langle\ ,\rangle)$ is Ricci-flat and from (1.1) X is a Killing field.

Case (iii). Finally, suppose $\lambda > 0$. Then $S(x) \geq S_* \geq 0$. From (3.8)

$$\Delta S - \langle X, \nabla S \rangle - 2\lambda S \leq 0.$$

If $S(x_0) = S_* = 0$ for some $x_0 \in M$, then again by the maximum principle $S(x) \equiv 0$. From (2.21), $(M, \langle\ ,\rangle)$ is Ricci-flat and from (1.1) $\mathcal{L}_X \langle\ ,\rangle = 2\lambda \langle\ ,\rangle$ so that X is a homothetic vector field. Suppose now $S(x_0) = S_* = m\lambda$ for some $x_0 \in M$. From (3.8)

$$\Delta S - \langle X, \nabla S \rangle \leq 2\frac{S}{m}(\lambda m - S)$$

and since $S(x) \geq S_* = m\lambda > 0$,

$$\Delta S - \langle X, \nabla S \rangle \leq 0 \qquad \text{on } M.$$

By the maximum principle $S(x) \equiv m\lambda$, from (2.21) $(M, \langle\ ,\rangle)$ is Einstein and (1.1) implies that X is a Killing field. Furthermore, since $\lambda > 0$, $(M, \langle\ ,\rangle)$ is compact by Myers's Theorem. q.e.d.

3.5. **Proof of Corollary 1.5.** Under the curvature assumption of Corollary 1.5, we have the validity of the Omori-Yau maximum principle (see [13]) and the result now follows from Theorem 1.4. q.e.d.

3.6. **Proof of Corollary 1.6.** Let $\varphi : M \to \mathbb{R}^n$ be a proper minimal immersion. Then by [13], Example 1.14, the Omori-Yau maximum principle holds on $(M, \langle\ ,\rangle)$. If φ is not totally geodesic then $S_* = \inf_M S(x) < 0$ and we contradict Theorem 1.4. q.e.d.

3.7. **Proof of Theorem 1.7.** By Okumura's lemma, [6],

$$\mathrm{Tr}(T^3) \geq -\frac{m-2}{\sqrt{m(m-1)}}|T|^3.$$

Thus, from (2.52) of Corollary 2.7 we deduce

$$\begin{aligned}\frac{1}{2}\Delta|T|^2 &\geq \frac{1}{2}\left\langle \nabla|T|^2, X\right\rangle + |\nabla T|^2 + 2\left(\lambda - \frac{m-2}{m(m-1)}S^*\right)|T|^2 - \\ &- \frac{4}{\sqrt{m(m-1)}}|T|^3 \\ &\geq -\frac{1}{2}\left|\nabla|T|^2\right||X|^* + 2\left(\lambda - \frac{m-2}{m(m-1)}S^*\right)|T|^2 - \frac{4}{\sqrt{m(m-1)}}|T|^3.\end{aligned}$$

Setting $|T|^2 = u$ we rewrite the above as

(3.12)
$$\frac{1}{2}\Delta u \geq -\frac{1}{2}|\nabla u||X|^* + 2\left(\lambda - \frac{m-2}{m(m-1)}S^*\right)u - \frac{4}{\sqrt{m(m-1)}}u^{3/2}.$$

Now if $|T|^* = +\infty$ then (1.15) is obviously satisfied. Otherwise $u^* = \sup_M u < +\infty$ and we can apply the Omori-Yau maximum principle to (3.12) to obtain

$$\frac{4}{\sqrt{m(m-1)}}u^*\left[\frac{1}{2}\left(\lambda\sqrt{m(m-1)} - \frac{m-2}{\sqrt{m(m-1)}}S^*\right) - \sqrt{u^*}\right] \leq 0,$$

from which we deduce that either $u^* = 0$, that is $T \equiv 0$ on M, or $|T|^*$ satisfies (1.15). In the first case $(M, \langle\,,\rangle)$ is Einstein, and being conformally flat it is necessarily of constant sectional curvature. q.e.d.

4. A final remark

Let Y be a smooth vector field on M. Define the associated vector field

$$W_Y = T(Y,\,)^\sharp,$$

where T is the traceless Ricci tensor and $^\sharp$ is the musical isomorphism $^\sharp : T^*M \to TM$.

Lemma 4.1. *Let $S(x)$ be the scalar curvature of $(M, \langle\,,\rangle)$. Then*

$$\operatorname{div} W_Y = \frac{1}{2}\operatorname{Tr}\left(\mathcal{L}_Y\langle\,,\rangle \circ T\right) + \frac{m-2}{2m}Y(S). \tag{4.1}$$

Proof. We give the short proof for completeness. With the notations of Section 2,

$$W_Y = Y_i T_{ij} e_j, \qquad Y = Y_i e_i,$$

where $\{e_i\}$ is the o.n. frame dual to $\{\theta^i\}$. Thus

$$\operatorname{div} W_Y = Y_{ik}T_{ik} + Y_i T_{ik,k}.$$

Using the fact that T is symmetric and (2.32) we have

$$\begin{aligned}\operatorname{div} W_Y &= \frac{1}{2}(Y_{ik} + Y_{ki})T_{ik} + Y_i\left(R_{ik,k} - \frac{S_i}{m}\right) = \\ &= \frac{1}{2}\operatorname{Tr}\left(\mathcal{L}_Y \langle\, ,\rangle \circ T\right) + \frac{m-2}{2m}S_i Y_i,\end{aligned}$$

that is, (4.1). q.e.d.

Thus,

Proposition 4.2. *Let $(M, \langle\, ,\rangle, X)$ be a soliton structure on $(M, \langle\, ,\rangle)$ and let $S(x)$ be the scalar curvature. Then*

$$\operatorname{div} W_X = \frac{m-2}{2m}X(S) - |T|^2. \tag{4.2}$$

In particular, if M is compact

$$\frac{m-2}{2m}\int_M X(S) = \int_M |T|^2. \tag{4.3}$$

Proof. We use soliton equation (1.1), (2.39) and (2.40) into (4.1) to immediately obtain (4.2). q.e.d.

Remark. Equation (4.3) can be interpreted as a kind of "Kazdan-Warner condition" for compact solitons.

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