DEPTH OF EDGE RINGS ARISING FROM FINITE GRAPHS

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ABSTRACT. Let G be a finite graph and $K[G]$ the edge ring of G. Based on the technique of Gröbner bases and initial ideals, it will be proved that, given integers f and d with $7 \leq f \leq d$, there exists a finite graph G on $[d] = \{1, \ldots, d\}$ with depth $K[G] = f$ and with Krull-dim $K[G] = d$.

INTRODUCTION

The edge ring [3] and its toric ideal [4] arising from a finite graph have been studied from viewpoints of both commutative algebra and combinatorics. Especially, the normality of the edge ring as well as Gröbner bases of its toric ideal is extensively investigated. However, the fundamental question when an edge ring is Cohen–Macaulay is presumably open.

Let G be a finite simple graph, i.e., a finite graph with no loop and no multiple edge, on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_r\}$ its edge set. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ be the polynomial ring in d variables over a field K and write $K[G]$ for the subring of $K[\mathbf{t}]$ generated by those squarefree quadratic monomials $\mathbf{t}^e = t_i t_j$ with $e = \{i, j\} \in E(G)$. The semigroup ring $K[G]$ is called the *edge ring* of G. Let Krull-dim $K[G]$ denote the Krull dimension of $K[G]$ and depth $K[G]$ the depth of $K[G]$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_r]$ be the polynomial ring in r variables over a field K. The kernel I_G of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, r$ is called the *toric ideal* of G. One has $K[G] \cong K[\mathbf{x}]/I_G$. If G is connected and is nonbipartite (resp. bipartite), then Krull-dim $K[G] = d$ (resp. Krull-dim $K[G] = d - 1$).

The criterion of normality [3, Corollary 2.3] of edge rings guarantees that $K[G]$ is normal if either G is bipartite or $d \leq 6$. If $d = 7$, then there exists a finite graph G for which $K[G]$ is nonnormal. However, it follows easily that $K[G]$ is Cohen–Macaulay whenever $d \leq 7$. Computing the depth of the edge rings of all connected nonbipartite graphs G with 7 vertices shows that the depth of $K[G]$ is at least 7. Moreover, our computational experiment would naturally lead the authors into the temptation to give the following

Conjecture 0.1. Let G be a finite graph on $[d]$ with $d \geq 7$. Then depth $K[G] \geq 7$.

Now, even though Conjecture 0.1 is completely open, by taking Conjecture 0.1 into consideration, this paper will be devoted to proving the following

Theorem 0.2. *Given integers f and d with $7 \leq f \leq d$, there exists a finite graph G on $[d]$ with depth $K[G] = f$ and with Krull-dim $K[G] = d$.*

Let $k \geq 1$ be an arbitrary integer and G_{k+6} the finite graph on $[k+6]$ of Figure 0.1. The essential part of a proof of Theorem 0.2 is to show that

$$(0.1) \qquad \text{depth} \, K[G_{k+6}] = \text{depth} \, K[\mathbf{x}]/I_{G_{k+6}} = 7.$$

In Section 1, by virtue of the formula [1, Theorem 2.1], the inequality depth $K[G_{k+6}] \leq 7$ will be proved. In Section 2, we compute a Gröbner basis of $I_{G_{k+6}}$ and an initial ideal $\text{in}(I_{G_{k+6}})$ of $I_{G_{k+6}}$, and show the inequality depth $K[\mathbf{x}]/\text{in}(I_{G_{k+6}}) \geq 7$. In general, one has depth $K[\mathbf{x}]/I_{G_{k+6}} \geq$ depth $K[\mathbf{x}]/\text{in}(I_{G_{k+6}})$ (e.g., [2, Theorem 3.3.4 (d)]). Thus the desired equality (0.1) follows.

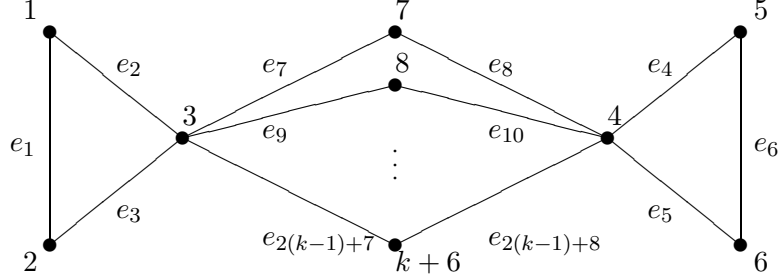**Figure 0.1.** (finite graph G_{k+6})

Once we know that depth $K[G_{k+6}] = 7$, to prove Theorem 0.2 is straightforward. In fact, given integers f and d with $7 \leq f \leq d$, let Γ denote the finite graph G_{d-f+7} on $[d - f + 7]$ and write G for the finite graph on $[d]$ obtained from Γ by adding $f - 7$ edges

$$\{1, d - f + 8\}, \{1, d - f + 9\}, \dots, \{1, d\}$$

to Γ. It then follows that depth $K[G] =$ depth $K[\Gamma] + f - 7$. Since depth $K[\Gamma] = 7$, one has depth $K[G] = f$, as required.

1. PROOF OF depth $K[G_{k+6}] \leq 7$

Let $G = G_{k+6}$ of Figure 0.1. In this section, we prove that depth $K[G] \leq 7$. Since the number of edges of G is $r = 2(k - 1) + 8$, Auslander–Buchsbaum formula implies that we may prove pd $K[G] \geq r - 7 = 2k - 1$.

Let S_G be the semigroup arising from G. Let $\mathcal{A}_G = \{\underline{a}_1, \dots, \underline{a}_r\}$ be the set of columns of the incidence matrix of G where $\underline{a}_l$ corresponds to the edge e_l (which corresponds to the variable x_l). Therefore, $S_G = \mathbb{N}\mathcal{A}_G$.

To prove pd $K[G] \geq 2k - 1$, we use the following theorem due to Briales, Campillo, Marijuán, and Pisón [1]. For $\underline{s} \in S_G$, we define the simplicial complex

$$\Delta_{\underline{s}} = \{F \subset [r] \; : \; \underline{s} - \underline{n}_F \in S_G\},$$

where $\underline{n}_F = \sum_{l \in F} \underline{a}_l$. We denote by $\beta_{i,\underline{s}}(K[G])$, the ith multigraded Betti number of $K[G]$ in degree $\underline{s}$.

Lemma 1.1 ([1, Theorem 2.1]). *Let G be a finite simple graph. Then*

$$\beta_{j+1,\underline{s}}(K[G]) = \dim_K \tilde{H}_j(\Delta_{\underline{s}}; K).$$

We consider the case where

$$\underline{s} = (1, 1, k + 1, k + 1, 1, 1, 2, 2, \dots, 2).$$

By Lemma 1.1, it is sufficient to prove the following lemma:

Lemma 1.2. *Set $\underline{s} = (1, 1, k + 1, k + 1, 1, 1, 2, 2, \dots, 2)$. Then*

$$\dim_K \tilde{H}_{2k-2}(\Delta_{\underline{s}}; K) \neq 0.$$

We set $\Delta = \Delta_{\underline{s}}$. Before proving Lemma 1.2, we compute the simplicial complex Δ.

Lemma 1.3. *Set $\underline{s} = (1, 1, k+1, k+1, 1, 1, 2, 2, \ldots, 2)$. Then facets of $\Delta_{\underline{s}}$ are the following subsets of $[r]$:*

$$F_{1,i} = \{1, 4, 5, 7, 8, \ldots, 2(k-1)+8\} \setminus \{2(i-1)+8\}, \qquad i = 1, \ldots, k;$$
$$F_{2,j} = \{2, 3, 6, 7, 8, \ldots, 2(k-1)+8\} \setminus \{2(j-1)+7\}, \qquad j = 1, \ldots, k.$$

Proof. Since $\underline{s} - \underline{n}_{F_{1,i}} = \underline{a}_{2(i-1)+7} \in S_G$, we have $F_{1,i} \in \Delta_{\underline{s}} = \Delta$. (It follows that $\underline{s} \in S_G$.) Similarly, we have $F_{2,j} \in \Delta$.

To prove that there are no facet other than $F_{1,i}, F_{2,j}$, it is enough to show that

- $\{1,2\}, \{1,3\}, \{4,6\}, \{5,6\} \notin \Delta$;
- $\{1,6\} \notin \Delta$;
- $\{2,4\}, \{2,5\}, \{3,4\}, \{3,5\} \notin \Delta$;
- $F_0 = \{7, 8, \ldots, 2(k-1)+8\} \notin \Delta$.

Since the first entry of $\underline{s} - \underline{n}_{\{1,2\}}$ is $-1 < 0$, it follows that $\underline{s} - \underline{n}_{\{1,2\}} \notin S_G$. Therefore $\{1,2\} \notin \Delta$. By the symmetry, we also have $\{1,3\}, \{4,6\}, \{5,6\} \notin \Delta$.

Second we show that $\{1,6\} \notin \Delta$. Suppose, on the contrary, that $\{1,6\} \in \Delta$, i.e.,

$$\underline{s} - \underline{n}_{\{1,6\}} = (0, 0, k+1, k+1, 0, 0, 2, 2, \ldots, 2) \in S_G.$$

Then we can write $\underline{s} - \underline{n}_{\{1,6\}} = \sum_{l=1}^{r} c_l \underline{a}_l$, where $c_l \in \mathbb{N}$. Since $(\underline{s} - \underline{n}_{\{1,6\}})_1 = (\underline{s} - \underline{n}_{\{1,6\}})_2 = 0$ and $(\underline{s} - \underline{n}_{\{1,6\}})_3 = k+1$, we have $c_1 = c_2 = c_3 = 0$ and $\sum_{i=1}^{k} c_{2(i-1)+7} = k+1$. Similarly, we have $c_4 = c_5 = c_6 = 0$ and $\sum_{j=1}^{k} c_{2(j-1)+8} = k+1$. Then $\sum_{i=1}^{k} c_{2(i-1)+7} + \sum_{j=1}^{k} c_{2(j-1)+8} = 2(k+1)$, but it must be $2k$. This is a contradiction.

Next we show that $\{2,4\}, \{2,5\}, \{3,4\}, \{3,5\} \notin \Delta$. Suppose that $\{2,4\} \in \Delta$, i.e.,

$$\underline{s} - \underline{n}_{\{2,4\}} = (0, 1, k, k, 0, 1, 2, 2, \ldots, 2) \in S_G.$$

Then we can write $\underline{s} - \underline{n}_{\{2,4\}} = \sum_{l=1}^{r} c_l \underline{a}_l$, where $c_l \in \mathbb{N}$. Since $(\underline{s} - \underline{n}_{\{2,4\}})_1 = 0$ and $(\underline{s} - \underline{n}_{\{2,4\}})_2 = 1$, we have $c_3 = 1$. Similarly, we have $c_5 = 1$. Thus

$$(0, 0, k-1, k-1, 0, 0, 2, 2, \ldots, 2) \in S_G.$$

Then the similar argument on the proof of $\{1,6\} \notin \Delta$ yields a contradiction. Therefore $\{2,4\} \notin \Delta$. By the symmetry, we also have $\{2,5\}, \{3,4\}, \{3,5\} \notin \Delta$.

Last, we show $F_0 \notin \Delta$. It follows from

$$\underline{s} - \underline{n}_{F_0} = (1, 1, 1, 1, 1, 1, 0, 0, \ldots, 0) \notin S_G.$$

$\square$

Now we prove Lemma 1.2.

Proof of Lemma 1.2. Let Δ_1 be the subcomplex of Δ whose facets are $F_{1,i}$, $i = 1, \ldots, k$, and Δ_2 the subcomplex of Δ whose facets are $F_{2,j}$, $j = 1, \ldots, k$. Then $\Delta = \Delta_1 \cup \Delta_2$. Also facets of the simplicial complex $\Delta_1 \cap \Delta_2$ are

$$\{7, 8, \ldots, 2(k-1)+8\} \setminus \{2(j-1)+7, 2(i-1)+8\}, \qquad i, j = 1, \ldots, k.$$

In particular, $\dim \Delta_1 \cap \Delta_2 = 2k - 3$. Note that both of Δ_1 and Δ_2 are cones over some simplicial complexes and so the reduced homologies of these all vanish. Therefore the Mayer–Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_i(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \longrightarrow \tilde{H}_i(\Delta; K)$$
$$\longrightarrow \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_{i-1}(\Delta_1; K) \oplus \tilde{H}_{i-1}(\Delta_2; K) \longrightarrow \cdots$$

yields

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \qquad \text{for all } i.$$

We can see $\tilde{H}_{2k-3}(\Delta_1 \cap \Delta_2; K) \neq 0$ by considering the alternating sum of all facets of $\Delta_1 \cap \Delta_2$:

$$\sum_{1 \leq i,j \leq k} (-1)^{i+j} \{7, 8, \dots, 2(k-1)+8\} \setminus \{2(j-1)+7, 2(i-1)+8\}.$$

Therefore we have $\tilde{H}_{2k-2}(\Delta; K) \neq 0$.

$\square$

2. PROOF OF depth $K[G_{k+6}] \geq 7$

Let, as before, $G = G_{k+6}$ of Figure 0.1. In this section we prove that depth $K[G] \geq 7$.

We set $C_1 = (e_2, e_1, e_3)$ and $C_2 = (e_4, e_6, e_5)$, both of which are 3-cycles of G. By [4, Lemma 3.2], there are 3 kinds of primitive even closed walks Γ of G up to the way:

(I) a 4-cycle: $\Gamma = (e_{2(i-1)+7}, e_{2(i-1)+8}, e_{2(j-1)+8}, e_{2(j-1)+7})$, where $i < j$;

(II) a walk on two 3-cycles C_1, C_2 and a single path connecting C_1 and C_2: $\Gamma = (C_1, e_{2(i-1)+7}, e_{2(i-1)+8}, C_2, e_{2(i-1)+8}, e_{2(i-1)+7})$, where $i = 1, \dots, k$;

(III) a walk on two 3-cycles C_1, C_2 and two different paths combining C_1 and C_2: $\Gamma = (C_1, e_{2(i-1)+7}, e_{2(i-1)+8}, C_2, e_{2(j-1)+8}, e_{2(j-1)+7})$, where $i < j$.

It was proved in [4, Lemma 3.1] that binomials corresponding to these primitive even closed walks generate the toric ideal I_G. Let us consider the lexicographic order $< = <_{\text{lex}}$ with $x_1 > x_2 > x_3 > \cdots > x_{2(k-1)+8}$.

Lemma 2.1. *The set of binomials corresponding to primitive even closed walks (I), (II), (III) is a Gröbner basis of I_G with respect to $<_{\text{lex}}$.*

Proof. The result follows from a straightforward application of Buchberger's algorithm to the set of generators of I_G corresponding to the primitive even closed walks listed above. Let f and g be two such generators. We will prove that the S-polynomial, $S(f, g)$, yielding from Buchberger's algorithm will reduce to 0 by generators of type (I), (II) and (III). For convenience of notation, we will assume that i, j, p, and q are all odd integers such that $7 \leq i < j, 7 \leq p < q$.

Case 1: Let $f = x_i x_{j+1} - x_{i+1} x_j$ and $g = x_p x_{q+1} - x_{p+1} x_q$ be generators of type (I). If $i \neq p$ and $j \neq q$, then the leading terms of f and g are relatively prime and thus the S-polynomial $S(f, g)$ will reduce to 0 (e.g., [2, Lemma 2.3.1]). Suppose $i = p$, then

$$
\begin{aligned}
S(f, g) &= \frac{\text{lcm}(f, g)}{LT_{<_{\text{lex}}}(f)} f - \frac{\text{lcm}(f, g)}{LT_{<_{\text{lex}}}(g)} g \\
&= x_{q+1}(x_i x_{j+1} - x_{i+1} x_j) - x_{j+1}(x_i x_{q+1} - x_{i+1} x_q) \\
&= x_{i+1} x_{j+1} x_q - x_{i+1} x_j x_{q+1} \\
&= x_{i+1}(x_{j+1} x_q - x_j x_{q+1}).
\end{aligned}
$$

Note that, up to sign, $x_{j+1} x_q - x_j x_{q+1}$ is a generator of I_G of type (I) and therefore $S(f, g)$ will reduce to 0. The case of $j = q$ is similar.

Case 2: Let f be the same as above and $g = x_1 x_4 x_5 x_p^2 - x_2 x_3 x_6 x_{p+1}^2$ a generator of type (II). If $i \neq p$ then the leading terms of f and g are relatively prime and therefore

negligible. If $i = p$ then

$$\begin{aligned} S(f,g) &= x_1x_4x_5x_i(x_ix_{j+1} - x_{i+1}x_j) - x_{j+1}(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) \\ &= x_2x_3x_6x_{i+1}^2x_{j+1} - x_1x_4x_5x_ix_{i+1}x_j \\ &= -x_{i+1}(x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}) \end{aligned}$$

where $x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}$ is a generator of type (III).

  Case 3: Again, we assume that f is the same as above. Now assume g is of type (III), $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$. If $i \neq p, q$ then the leading terms of f and g will be relatively prime. Suppose $i = p$, then

$$\begin{aligned} S(f,g) &= x_1x_4x_5x_q(x_ix_{j+1} - x_{i+1}x_j) - x_{j+1}(x_1x_4x_5x_ix_q - x_2x_3x_6x_{i+1}x_{q+1}) \\ &= -x_{i+1}(x_1x_4x_5x_qx_j - x_2x_3x_6x_{q+1}x_{j+1}) \end{aligned}$$

and again we have that $x_1x_4x_5x_qx_j - x_2x_3x_6x_{q+1}x_{j+1}$ is either a type (II) or type (III) generator of I_G. The case of $i = q$ is similar.

  Case 4: Now let f and g both be generators of type (II), $f = x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2, g = x_1x_4x_5x_j^2 - x_2x_3x_6x_{j+1}^2$. Then the S-polynomial

$$\begin{aligned} S(f,g) &= x_j^2(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i^2(x_1x_4x_5x_j^2 - x_2x_3x_6x_{j+1}^2) \\ &= x_2x_3x_6(x_i^2x_{j+1}^2 - x_{i+1}^2x_j^2) \\ &= x_2x_3x_6(x_ix_{j+1} + x_{i+1}x_j)(x_ix_{j+1} - x_{i+1}x_j) \end{aligned}$$

is a multiple of a type (I) generator.

  Case 5: Let f be the same as in Case 4 and $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$ of type (III). First suppose that $i \neq p, q$. Let us consider the case of $i < p$. Then

$$\begin{aligned} S(f,g) &= x_px_q(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i^2(x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}) \\ &= x_2x_3x_6(x_i^2x_{p+1}x_{q+1} - x_{i+1}^2x_px_q) \\ &= x_2x_3x_6[x_ix_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_ix_{i+1}x_px_{q+1} - x_{i+1}^2x_px_q] \\ &= x_2x_3x_6[x_ix_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_{i+1}x_p(x_ix_{q+1} - x_{i+1}x_q)]. \end{aligned}$$

And so $S(f,g)$ reduce to 0 by two type (I) generators. The cases of $p < i < q$ and $q < i$ are similar.

  Now suppose $i = p$, then the S-polynomial,

$$\begin{aligned} S(f,g) &= x_q(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i(x_1x_4x_5x_ix_q - x_2x_3x_6x_{i+1}x_{q+1}) \\ &= x_2x_3x_6x_{i+1}(x_ix_{q+1} - x_{i+1}x_q). \end{aligned}$$

is a multiple of a type (I) generator. The case of $i = q$ is similar.

  Case 6: Finally, we let consider the case that both f and g are of type (III): $f = x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}$, $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$. We may assume that $i \leq p$. Let us first suppose that $i, j \neq p, q$, then

$$\begin{aligned} S(f,g) &= x_px_q(x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}) - x_ix_j(x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}) \\ &= x_2x_3x_6(x_ix_jx_{p+1}x_{q+1} - x_{i+1}x_{j+1}x_px_q) \\ &= x_2x_3x_6[x_jx_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_{i+1}x_p(x_jx_{q+1} - x_{j+1}x_q)]. \end{aligned}$$

  Now let $i = p$. We then have

$$\begin{aligned} S(f,g) &= x_qf - x_jg = -x_qx_2x_3x_6x_{i+1}x_{j+1} + x_jx_2x_3x_6x_{i+1}x_{q+1} \\ &= x_2x_3x_6x_{i+1}(x_jx_{q+1} - x_{j+1}x_q). \end{aligned}$$

The cases of $j = p$ and $j = q$ are similar.

$$\square$$

Now we prove that $\operatorname{depth} K[G] \geq 7$. We denote by $\operatorname{in}(I_G)$, the initial ideal of I_G with respect to $<_{\mathrm{lex}}$. Since

$$\operatorname{depth} K[G] = \operatorname{depth} K[\mathbf{x}]/I_G \geq \operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_G),$$

it is sufficient to prove that $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_G) \geq 7$. By Auslander–Buchsbaum formula, it is enough to prove the following lemma:

Lemma 2.2.

$$\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \leq 2k - 1.$$

Proof. First we compute $\operatorname{in}(I_G)$.

The binomials corresponding to type (I) are

$$x_{2(i-1)+7}x_{2(j-1)+8} - x_{2(i-1)+8}x_{2(j-1)+7}, \qquad \text{where } i < j.$$

The initial term of this binomial is $x_{2(i-1)+7}x_{2(j-1)+8}$ $(i < j)$. We denote by I', the ideal generated by these monomials. Note that x_8 and $x_{2(k-1)+7}$ do not appear in the minimal system of monomial generators of I'.

The binomials corresponding to types (II), (III) are

$$x_2 x_3 x_6 x_{2(i-1)+8} x_{2(j-1)+8} - x_1 x_4 x_5 x_{2(i-1)+7} x_{2(j-1)+7}, \qquad \text{where } i \leq j.$$

The initial term of this binomial is $-x_1 x_4 x_5 x_{2(i-1)+7} x_{2(j-1)+7}$ $(i \leq j)$.

Therefore

$$\operatorname{in}(I_G) = x_1 x_4 x_5 (x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I'$$
$$= ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I') \cap ((x_1 x_4 x_5) + I').$$

We set

$$I_1 = (x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I'$$
$$I_2 = (x_1 x_4 x_5) + I'.$$

By the short exact sequence $0 \to K[\mathbf{x}]/I_1 \cap I_2 \to K[\mathbf{x}]/I_1 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(I_1 + I_2) \to 0$, we have
(2.1)
$$\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \leq \max\{\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(I_1 + I_2) - 1\}.$$

Now we investigate each of $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(I_1 + I_2)$.

First we consider the ideal I_1. Note that $x_1, \ldots, x_6$ and x_8 do not appear in the minimal system of monomial generators of I_1. Let $K[\mathbf{x}']$ be the polynomial ring over K with variables $x_7, x_9, x_{10}, \ldots, x_{2(k-1)+8}$. Then $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1 = \operatorname{pd}_{K[\mathbf{x}']} K[\mathbf{x}']/(I_1 \cap K[\mathbf{x}'])$. By Hilbert's syzygy theorem, we have $\operatorname{pd}_{K[\mathbf{x}']} K[\mathbf{x}']/(I_1 \cap K[\mathbf{x}']) \leq 2k - 1$.

Next we consider the ideal $I_2 = (x_1 x_4 x_5) + I'$. Since the variables x_1, x_4, x_5 do not appear in the minimal systems of generators of I', we have

$$\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2 = \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I' + \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(x_1 x_4 x_5) = \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I' + 1.$$

Then similarly to the case of I_1, we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I' \leq 2k - 2$. Thus we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2 \leq 2k - 1$.

Last, we consider the ideal $I_1 + I_2 = (x_1 x_4 x_5) + I_1$. The same reason as the case of I_2, we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(I_1 + I_2) = \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1 + 1 \leq 2k$.

Combining these results with (2.1), we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \leq 2k-1$, as desired. $\square$

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