Derivative Formula and Applications for Hyperdissipative Stochastic Navier-Stokes/Burgers Equations*

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Abstract

By using coupling method, a Bismut type derivative formula is established for the Markov semigroup associated to a class of hyperdissipative stochastic Navier-Stokes/Burgers equations. As applications, gradient estimates, dimension-free Harnack inequality, strong Feller property, heat kernel estimates and some properties of the invariant probability measure are derived.

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1 Introduction

Let H be the divergence free sub-space of $L^2(\mathbb{T}^d;\mathbb{R}^d)$, where $\mathbb{T}^d := (\mathbb{R}/[0,2\pi])^d$ is the d-dimensional torus. The d-dimensional Navier-Stokes equation (for $d \geq 2$) reads

$$\mathrm{d}X_t = \{\nu\Delta X_t - B(X_t,X_t)\}\mathrm{d}t,$$

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where $\nu > 0$ is the viscosity constant and $B(u,v) := \mathbf{P}(u\cdot\nabla)v$ for $\mathbf{P} : L^2(\mathbb{T}^d;\mathbb{R}^d) \to H$ the orthogonal projection (see e.g. [13]). When $d = 1$ and $H = L^2(\mathbb{T}^d;\mathbb{R}^d)$, this equation reduces to the Burgers equation. In recent years, the stochastic Navier-Stokes equations have been investigated intensively, see e.g. [6] for the ergodicity of 2D Navier-Stokes equations with degenerate noise, and see [3, 5, 12] for the study of 3D stochastic Navier-Stokes equations. The main purpose of this paper is to establish the Bismut type derivative formula for the Markov semigroup associated to stochastic Navier-Stokes type equations, and as applications, to derive gradient estimates, Harnack inequality, and strong Feller property for the semigroup.

We shall work with a more general framework as in [8], which will be reduced to a class of hyperdissipative (i.e. the Laplacian has a power larger than 1) stochastic Navier-Stokes/Burgers equations in Section 2.

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|_H)$ be a separable real Hilbert space, and $(L, \mathscr{D}(L))$ a positively definite self-adjoint operator on H with $\lambda_0 := \inf\sigma(L) > 0$, where $\sigma(L)$ is the spectrum of L. Let $V = \mathscr{D}(L^{1/2})$, which is a Banach space with norm $\|\cdot\|_V := \|L^{1/2}\cdot\|$. Let Q be a Hilbert-Schmidt linear operator on H with Ker Q $= \{0\}$. Then $\mathscr{D}(Q^{-1}) := Q(H)$ is a Banach space with norm $\|x\|_Q := \|Q^{-1}x\|_H$. In general, for $\theta > 0$, let $V_\theta = \mathscr{D}(L^{\theta/2})$ with norm $\|L^{\theta/2}\cdot\|_H$. We assume that there exist two constants $\theta \in (0,1]$ and $K_1 > 0$ such that $V_\theta \subset \mathscr{D}(Q^{-1})$ and

(A0) $\|u\|_Q^2 \le K_1\|u\|_{V_\theta}^2, \quad u \in V_\theta.$

Moreover, let

$$B : V \times V \to H$$

be a bilinear map such that

(A1) $\langle v, B(v,v)\rangle = 0, \quad v \in V;$

(A2) There exists a constant $C > 0$ such that $\|B(u,v)\|_H^2 \le C\|u\|_H^2\|v\|_V^2,\ u,v \in V;$

(A3) There exists a constant $K_2 > 0$ such that $\|B(u,v)\|_Q^2 \le K_2\|u\|_{V_\theta}^2\|v\|_{V_\theta}^2, \quad u,v \in V.$

Finally, let W_t be the cylindrical Brownian motion on H. We consider the following stochastic differential equation on H:

$$\mathrm{d}X_t = QdW_t - \{LX_t + B(X_t)\}\mathrm{d}t, \tag{1.1}$$

where $B(X_t) := B(X_t, X_t)$. According to [8], for any initial value $X_0 \in H$ the equation (1.1) has a unique strong solution, which gives rise to a Markov process on H (see Appendix for details). For any $x \in H$, let X_t^x be the solution starting at x. Let $\mathscr{B}_b(H)$ be the set of all bounded measurable functions on H. Then

$$P_tf(x) := \mathbb{E}f(X_t^x), \quad x \in H, t \geq 0, f \in \mathscr{B}_b(H)$$

defines a Markov semigroup $(P_t)_{t\geq 0}$.

We shall adopt a coupling argument to establish a Bismut type derivative formula for P_t, which will imply explicit gradient estimates and the dimension-free Harnack inequality in the sense of [14]. This type of Harnack inequality has been applied to the study of several models of SDEs and SPDEs, see e.g. [4, 7, 9, 11, 10, 15] and references within.

For $f \in \mathscr{B}_b(H), h \in V_\theta, x \in H$ and $t > 0$, let

$$D_hP_tf(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big\{P_tf(x+\varepsilon h) - P_tf(x)\big\}$$

provided the limit in the right-hand side exists. Let $\tilde{B}(u,v) = B(u,v) + B(v,u)$.

Theorem 1.1. *Assume that* **(A0)**-**(A3)** *hold for some constants* $\theta \in (0,1], K_1, K_2, C > 0$. *Then for any* $t > 0, h \in V_\theta$ *and* $f \in \mathscr{B}_b(H)$, D_hP_tf *exists on* H *and satisfies*

$$D_hP_tf(x) = \mathbb{E}\bigg\{f(X_t^x)\int_0^t \Big\langle Q^{-1}\Big(\frac{1}{t}\mathrm{e}^{-sL}h - \frac{t-s}{t}\tilde{B}(X_s^x, \mathrm{e}^{-sL}h)\Big), \mathrm{d}W_s\Big\rangle\bigg\}, \quad x \in H. \tag{1.2}$$

Let V_θ^* be the dual space of V_θ. According to Theorem 1.1, under assumptions **(A0)**-**(A3)** we may define the gradient $DP_tf : H \to V_\theta^*$ by letting

$${}_{V_\theta^*}\langle DP_tf(x), h\rangle_{V_\theta} = D_hP_tf(x), \quad x \in H, h \in V_\theta.$$

We shall estimate

$$\|DP_tf(x)\|_{V_\theta^*} := \sup_{\|h\|_{V_\theta}\leq 1}|D_hP_tf(x)|, \quad x \in H.$$

To this end, let $\|Q\|$ and $\|Q\|_{HS}$ be the operator norm and the Hilbert-Schmidt norm of $Q : H \to H$ respectively.

Corollary 1.2. *Under assumptions of Theorem 1.1.*

(1) *For any* $t > 0, x \in H$ *and* $f \in \mathscr{B}_b(H)$,

$$\|DP_tf(x)\|_{V_\theta^*}^2 \leq (P_tf^2(x))\Big\{\frac{2K_1}{t} + \frac{4K_2}{\lambda_0^{2-\theta}}\big(\|x\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}.$$

(2) *Let* $f \in \mathscr{B}_b(H)$ *be positive. For any* $x \in H, t > 0$ *and* $\delta \geq 4\sqrt{K_2}\,\|Q\|\lambda_0^{(\theta-3)/2}$,

$$\begin{aligned}\|DP_tf(x)\|_{V_\theta^*} \leq &\delta\big\{P_t(f\log f) - (P_tf)\log P_tf\big\}(x)\\ &+ \frac{2}{\delta}\Big\{\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}P_tf(x).\end{aligned}$$

(3) *Let* $\alpha > 1, t > 0$ *and* $f \geq 0$. *The Harnack inequality*

$$(P_t f(x))^\alpha \leq (P_t f^\alpha(y)) \exp\left[\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha-1}\Big\{\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}\right]$$

holds for $x, y \in H$ *such that*

$$\|x-y\|_{V_\theta} \leq \frac{(\alpha-1)\lambda_0^{(3-\theta)/2}}{4\alpha\|Q\|\sqrt{K_2}}.$$

In particular, P_t *is* V_θ*-strong Feller, i.e.* $\lim_{\|y-x\|_{V_\theta}\to 0} P_t f(y) = P_t f(x)$ *holds for* $f \in \mathscr{B}_b(H), t > 0, x \in H$.

As applications of the Harnack inequality derived above, we have the following result.

Corollary 1.3. *Under assumptions of Theorem 1.1.* P_t *has an invariant probability measure* μ *such that* $\mu(V) = 1$ *and hence,* $\mu(V_\theta) = 1$. *If moreover* $\theta \in (0,1)$, *then:*

(1) P_t *has a unique invariant probability measure* μ, *and the measure has full support on* V_θ.

(2) P_t *has a density* $p_t(x,y)$ *on* V_θ *w.r.t.* μ. *Moreover, let* $r_0 = \frac{(\alpha-1)\lambda_0^{(3-\theta)/2}}{4\alpha\|Q\|\sqrt{K_2}}$ *and* $B_\theta(x,r_0) = \{y : \|y-x\|_{V_\theta} \leq r_0\}$,

$$\begin{aligned}&\left(\int_{V_\theta} p_t(x,y)^{(\alpha+1)/\alpha}\mu(\mathrm{d}y)\right)^\alpha\\ &\leq \frac{1}{\int_{B_\theta(x,r_0)} \exp\big[-\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha-1}\big\{\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}(\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t)\big\}\big]\mu(\mathrm{d}y)} < \infty\end{aligned}$$

holds for any $t > 0, \alpha > 1$ *and* $x \in V_\theta$.

Note that the Harnack inequality presented in Corollary 1.2 is local in the sense that $\|x-y\|_{V_\theta}$ has to be bounded above by a constant. To derive a global Harnack inequality, we need to extend the gradient-entropy inequality in Corollary 1.2 (2) to all $\delta > 0$. In this spirit, we have the following result.

Theorem 1.4. *Under assumptions of Theorem 1.1.*

(1) *For any* $\delta > 0$ *and any positive* $f \in \mathscr{B}_b(H)$,

$$\begin{aligned}\|DP_t f(x)\|_{V_\theta^*} \leq &\delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}(x)\\ &+ \frac{2}{\delta}\Big\{\frac{K_1}{t\wedge t_\delta} + \frac{2K_2\mathrm{e}}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}P_t f(x), \quad x \in H, t > 0\end{aligned}$$

holds for $t_\delta := \frac{\delta^2\lambda_0^{3-\theta}}{4\|Q\|^2\mathrm{e}K_2}$.

(2) *Let $\alpha > 1, t > 0$ and $f \geq 0$. Then*

$$(P_t f(x))^\alpha \leq (P_t f^\alpha(y)) \exp\left[\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha-1}\Big\{K_1\Big(\frac{1}{t} \vee \frac{4\alpha^2\|Q\|^2 \mathrm{e} K_2 \|x-y\|_{V_\theta}^2}{(\alpha-1)^2\lambda_0^{3-\theta}}\Big) + \frac{2K_2\mathrm{e}}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}\right]$$

holds for all $x, y \in \mathbb{H}$.

The remainder of the paper is organized as follows. We first consider in Section 2 a class of stochastic Navier-Stokes type equations to illustrate our results, then prove these results in Section 3.

2 Stochastic hyperdissipative Navier-Stokes/Burgers equations

Let $\mathbb{T}^d = (\mathbb{R}/[0, 2\pi])^d$ for $d \geq 1$. Let Δ be the Laplace operator on $\mathbb{T}^d$. To formulate Δ using spectral representation, we first consider the complex L^2 space $L^2(\mathbb{T}^d; \mathbb{C}^d)$. Recall that for $a = (a_1, \cdots, a_d), b = (b_1, \cdots, b_d) \in \mathbb{C}^d$, we have $a \cdot b = \sum_{i=1}^d a_i \bar{b}_i$. Let

$$e_k(x) = (2\pi)^{-d/2} \mathrm{e}^{\mathrm{i}(k \cdot x)}, \quad k \in \mathbb{Z}^d, x \in \mathbb{T}^d.$$

Then $\{e_k : k \in \mathbb{Z}^d\}$ is an ONB of $L^2(\mathbb{T}^d; \mathbb{C})$. Obviously, for a sequence $\{u_k\}_{k \in \mathbb{Z}^d} \subset \mathbb{C}^d$,

$$u := \sum_{k \in \mathbb{Z}^d} u_k e_k \in L^2(\mathbb{T}^d; \mathbb{R}^d)$$

if and only if $\bar{u}_k = u_{-k}$ holds for any $k \in \mathbb{Z}^d$ and $\sum_{k \in \mathbb{Z}^d} |u_k|^2 < \infty$. By spectral representation, we may characterize $(\Delta, \mathscr{D}(\Delta))$ on $L^2(\mathbb{T}^d; \mathbb{R}^d)$ as follows:

$$\Delta u = -\sum_{k \in \mathbb{Z}^d} |k|^2 u_k e_k, \quad u := \sum_{k \in \mathbb{Z}^d} u_k e_k \in \mathscr{D}(\Delta),$$
$$\mathscr{D}(\Delta) := \Big\{\sum_{k \in \mathbb{Z}^d} u_k \mathrm{e}_k : \ u_k \in \mathbb{C}^d, \bar{u}_k = u_{-k}, \sum_{k \in \mathbb{Z}^d} |u_k|^2 |k|^4 < \infty\Big\}.$$

To formulate the Navier-Stokes/Burgers type equation, when $d \geq 2$ we consider the sub-space divergence free elements of $L^2(\mathbb{T}^d; \mathbb{R}^d)$. It is easy to see that a smooth vector field

$$u = \sum_{k \in \mathbb{Z}^d} u_k e_k$$

is divergence free if and only if $u_k \cdot k = 0$ holds for all $k \in \mathbb{Z}^d$. Moreover, to make the spectrum of $-\Delta$ strictly positive, we shall not consider non-zero constant vector fields. Therefore, the Hilbert space we are working on becomes

$$H := \Big\{ \sum_{k\in\hat{\mathbb{Z}}^d} u_k e_k : \, u_k \in \mathbb{C}^d, (d-1)(u_k\cdot k) = 0, \bar{u}_k = u_{-k}, \sum_{k\in\hat{\mathbb{Z}}^d} |u_k|^2 < \infty \Big\},$$

where $\hat{\mathbb{Z}}^d = \mathbb{Z}^d \setminus \{0\}$. Since when $d = 1$ the condition $(d-1)(u_k\cdot k) = 0$ is trivial, the divergence free restriction does not apply for the one-dimensional case.

Let $(A, \mathscr{D}(A)) = (-\Delta, \mathscr{D}(\Delta))|_H$, the restriction of $(\Delta, \mathscr{D}(\Delta))$ on H, and let $\mathbf{P} : L^2(\mathbb{T}^d; \mathbb{R}^d) \to H$ be the orthogonal projection. Let

$$L = \lambda_0 A^{\delta+1}$$

for some constants $\lambda_0, \delta > 0$. As in Section 1, define $V = \mathscr{D}(L^{1/2})$ and $V_\theta = \mathscr{D}(L^{\theta/2})$. Then

$$B : V \times V \to H; \quad B(u, v) = \mathbf{P}(u\cdot\nabla)v$$

is a continuous bilinear (see the (b) in the proof of Theorem 2.1 below). Let $Q = A^{-\sigma}$ for some $\sigma > 0$, and let W_t be the cylindrical Brownian motion on H. Obviously, $\|Q\| \le 1$ and when $\sigma > \frac{d}{4}$,

$$\|Q\|_{HS}^2 \le \sum_{k\in\hat{\mathbb{Z}}^d} |k|^{-4\sigma} < \infty.$$

We consider the stochastic differential equation

$$\mathrm{d}X_t = Q\mathrm{d}W_t - (LX_t + B(X_t))\mathrm{d}t, \tag{2.1}$$

where $B(u) := B(u, u)$ for $u \in V$. Thus, we are working on the stochastic hyperdissipative Navier-Stokes (for $d \ge 2$) and Burgers (for $d = 1$) equations.

Theorem 2.1. *Let $\delta > \frac{d}{2}, \sigma \in (\frac{d}{4}, \frac{\delta}{2}]$ and $\theta \in [\frac{2\sigma+1}{\delta+1}, 1]$. Then all assertions in Section 1 hold for $K_1 = \frac{1}{\lambda_0^\theta}$ and*

$$K_2 = \frac{4^{2\delta\theta+1}}{\lambda_0^{2\theta}} \sum_{k\in\hat{\mathbb{Z}}^d} |k|^{-2(\delta+1)\theta} < \infty.$$

Proof. Since $\sigma > \frac{d}{4}$, $Q : H \to H$ is Hilbert-Schmidt. By Theorem 1.1 and its consequences, it suffices to verify assumptions **(A0)**-**(A3)**. Since **(A1)** is trivial for $d = 1$ and follows from the divergence free property for $d \ge 2$, we only have to prove **(A0)**, **(A2)** and **(A3)**. Let

$$u = \sum_{k\in\hat{\mathbb{Z}}^d} u_k e_k, \quad v = \sum_{k\in\hat{\mathbb{Z}}^d} v_k e_k$$

be two elements in V_θ.

(a) Since $\theta \in [\frac{2\sigma+1}{\delta+1}, 1]$ implies $4\sigma \le 2\theta(\delta+1)$, we have

$$\|u\|_Q^2 = \sum_{k\in\hat{\mathbb{Z}}^d} |u_k|^2|k|^{4\sigma} \le \frac{1}{\lambda_0^\theta} \sum_{k\in\hat{\mathbb{Z}}^d} \lambda_0^\theta |u_k|^2|k|^{2\theta(\delta+1)} = \frac{1}{\lambda_0^\theta}\|u\|_{V_\theta}^2.$$

Thus, **(A0)** holds for $K_1 = \frac{1}{\lambda_0^\theta}$.

(b) It is easy to see that

$$B(u,v) = \mathbf{P} \sum_{l,m\in\hat{\mathbb{Z}}^d, m\ne l} \mathrm{i}(u_{l-m}\cdot m)v_m \mathrm{e}_l. \tag{2.2}$$

By Hölder inequality,

$$\begin{aligned}\|B(u,v)\|_H^2 &\le \sum_{l\in\hat{\mathbb{Z}}^d}\left(\sum_{m\in\hat{\mathbb{Z}}^d\setminus\{l\}} |u_{l-m}|\cdot|m|\cdot|v_m|\right)^2 \\ &\le \sum_{l\in\hat{\mathbb{Z}}^d}\left(\sum_{m\in\hat{\mathbb{Z}}^d\setminus\{l\}} |u_{l-m}|^2|m|^{-2\delta}\right)\sum_{m\in\hat{\mathbb{Z}}^d}|v_m|^2|m|^{2(\delta+1)} \\ &\le \frac{1}{\lambda_0}\left(\sum_{m\in\hat{\mathbb{Z}}^d}|m|^{-2\delta}\right)\|u\|_H^2\|v\|_V^2.\end{aligned}$$

Since $\delta > \frac{d}{2}$, we have $\sum_{m\in\hat{\mathbb{Z}}^d}|m|^{-2\delta} < \infty$. Thus, **(A2)** holds for some constant C.

(c) By (2.2), we have

$$\begin{aligned}\|B(u,v)\|_Q^2 &:= \|A^\sigma B(u,v)\|_H^2 \le \sum_{l\in\hat{\mathbb{Z}}^d}|l|^{4\sigma}\left(\sum_{m\in\hat{\mathbb{Z}}^d}|u_{l-m}|\cdot|m|\cdot|v_m|\right)^2 \\ &\le 2\sum_{l\in\hat{\mathbb{Z}}^d}|l|^{4\sigma}\left(\sum_{|m|>\frac{|l|}{2}, m\ne l}|u_{l-m}|\cdot|m|\cdot|v_m|\right)^2 \\ &\quad + 2\sum_{l\in\hat{\mathbb{Z}}^d}|l|^{4\sigma}\left(\sum_{|m|\le\frac{|l|}{2}, m\in\hat{\mathbb{Z}}^d}|u_{l-m}|\cdot|m|\cdot|v_m|\right)^2 := 2I_1 + 2I_2.\end{aligned} \tag{2.3}$$

By the Schwartz inequality,

$$I_1 \le \sum_{l\in\hat{\mathbb{Z}}^d}|l|^{4\sigma}\left(\sum_{|m|>\frac{|l|}{2}, m\ne l}|u_{l-m}|^2|l-m|^{2(\delta+1)\theta}|m|^{2-2(\delta+1)\theta}\right)\sum_{|m|>\frac{|l|}{2}, m\ne l}|v_m|^2|m|^{2(\delta+1)\theta}|l-m|^{-2(\delta+1)\theta}.$$

Since $\theta \ge \frac{2\sigma+1}{\delta+1}$ implies that $4\sigma - 2(\delta+1)\theta + 2 \le 0$, if $|m| > \frac{|l|}{2}$ and $|l| \ge 1$ we have

$$|l|^{4\sigma}|m|^{-2(\delta+1)\theta+2} \le 4^{(\delta+1)\theta-1}|l|^{4\sigma-2(\delta+1)\theta+2} \le 4^{(\delta+1)\theta-1}.$$

Therefore,

$$\begin{aligned} I_1 &\le \frac{1}{\lambda_0^\theta}4^{(\delta+1)\theta-1}\|u\|_{V_\theta}^2\sum_{l\in\hat{\mathbb{Z}}^d}\sum_{|m|>\frac{|l|}{2},m\ne l}|v_m|^2|m|^{2(\delta+1)\theta}|l-m|^{-2(\delta+1)\theta} \\ &\le \frac{1}{\lambda_0^{2\theta}}4^{(\delta+1)\theta-1}\bigg(\sum_{m\in\hat{\mathbb{Z}}^d}|m|^{-2(\delta+1)\theta}\bigg)\|u\|_{V_\theta}^2\|v\|_{V_\theta}^2. \end{aligned} \tag{2.4}$$

Similarly, when $|m|\le\frac{|l|}{2}$ we have $|l-m|\ge\frac{|l|}{2}$ and thus, due to $4\sigma-2(\delta+1)\theta\le 0$,

$$|l|^{4\sigma}|l-m|^{-2(\delta+1)\theta} \le 4^{(\delta+1)\theta}|l|^{4\sigma-2(\delta+1)\theta} \le 4^{(\delta+1)\theta}|m|^{4\sigma-2(\delta+1)\theta}.$$

Therefore,

$$\begin{aligned} I_2 &\le \sum_{l\in\hat{\mathbb{Z}}^d}|l|^{4\sigma}\bigg(\sum_{1\le|m|\le\frac{|l|}{2}}|u_{l-m}|^2|l-m|^{2(\delta+1)\theta}|m|^{2-2(\delta+1)\theta}\bigg)\sum_{1\le|m|\le\frac{|l|}{2}}|v_m|^2|m|^{2(\delta+1)\theta}|l-m|^{-2(\delta+1)\theta} \\ &\le \frac{4^{(\delta+1)\theta}}{\lambda_0^{2\theta}}\bigg(\sum_{m\in\hat{\mathbb{Z}}^d}|m|^{4\sigma-4(\delta+1)\theta+2}\bigg)\|u\|_{V_\theta}^2\|v\|_{V_\theta}^2 \le \frac{4^{(\delta+1)\theta}}{\lambda_0^{2\theta}}\bigg(\sum_{m\in\hat{\mathbb{Z}}^d}|m|^{-2(\delta+1)\theta}\bigg)\|u\|_{V_\theta}^2\|v\|_{V_\theta}^2, \end{aligned}$$

where the last step is due to $4\sigma-2(\delta+1)\theta+2\le 0$ mentioned above. Combining this with (2.3) and (2.4), we prove **(A3)** for the desired K_2 which is finite since $\theta\ge\frac{2\sigma+1}{\delta+1}$ and $\sigma>\frac{d}{4}$ imply that $2(\delta+1)\theta\ge 4\sigma+1>d$. □

3 Proofs of Theorem 1.1 and consequences

We first present an exponential estimate of the solution, which will be used in the proof of Theorem 1.1.

Lemma 3.1. *In the situation of Theorem 1.1, we have*

$$\mathbb{E}\exp\bigg[\frac{\lambda_0^2}{2\|Q\|^2}\int_0^t\|X_s^x\|_V^2\mathrm{d}s\bigg] \le \exp\bigg[\frac{\lambda_0^2}{2\|Q\|^2}(\|x\|_H^2+\|Q\|_{HS}^2t)\bigg],\quad x\in H, t\ge 0.$$

Moreover, for any $t>0$ and $x\in H$,

$$\mathbb{E}\exp\bigg[\frac{2}{\|Q\|^2\mathrm{e}t}\int_0^t\|X_s^x\|_V^2\mathrm{d}s\bigg] \le \exp\bigg[\frac{2}{\|Q\|^2t}(\|x\|_H^2+\|Q\|_{HS}^2t)\bigg].$$

Proof. (a) Since $\langle B(u,v),v\rangle = 0$, by the Itô formula we have

$$\mathrm{d}\|X_t^x\|_H^2 \le -2\|X_t^x\|_V^2\mathrm{d}t + \|Q\|_{HS}^2\mathrm{d}t + 2\langle X_t^x, Q\mathrm{d}W_t\rangle. \tag{3.1}$$

Let

$$\tau_n := \inf\{t\ge 0 : \|X_t^x\|_H \ge n\}.$$

By Theorem 4.1 below we have $\tau_n \to \infty$ as $n\to\infty$. So, for any $\lambda>0$ and $n\ge 1$,

$$\begin{aligned}\mathbb{E}\exp\bigg[\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\mathrm{d}s\bigg] &\le \mathbb{E}\exp\bigg[\frac{\lambda}{2}(\|x\|_H^2+\|Q\|_{HS}^2t) + \lambda\int_0^{t\wedge\tau_n}\langle X_s^x, Q\mathrm{d}W_s\rangle\bigg]\\ &\le \exp\bigg[\frac{\lambda}{2}(\|x\|_H^2+\|Q\|_{HS}^2t)\bigg]\bigg(\mathbb{E}\exp\bigg[2\lambda^2\|Q\|^2\int_0^{t\wedge\tau_n}\|X_s^x\|_H^2\mathrm{d}s\bigg]\bigg)^{1/2} < \infty.\end{aligned}$$

Since $\|\cdot\|_H^2 \le \frac{1}{\lambda_0}\|\cdot\|_V^2$, this implies that

$$\mathbb{E}\exp\bigg[\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\mathrm{d}s\bigg] \le \mathrm{e}^{\frac{\lambda}{2}(\|x\|_H^2+\|Q\|_{HS}^2t)}\bigg(\mathbb{E}\exp\bigg[\frac{2\lambda^2\|Q\|^2}{\lambda_0}\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\mathrm{d}s\bigg]\bigg)^{1/2}.$$

Letting $\lambda = \frac{\lambda_0^2}{2\|Q\|^2}$, we obtain

$$\mathbb{E}\exp\bigg[\frac{\lambda_0^2}{2\|Q\|^2}\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\mathrm{d}s\bigg] \le \exp\bigg[\frac{\lambda_0^2}{2\|Q\|^2}(\|x\|_H^2+\|Q\|_{HS}^2t)\bigg].$$

This proves the first inequality by letting $n\to\infty$.

(b) Next, due to the first inequality and the Jensen inequality, we only have to prove the second one for $t\le\lambda_0^{-2}$. In this case, let

$$\beta(s) = \mathrm{e}^{(\lambda_0^2 - t^{-1})s}, \quad s\in[0,t].$$

By the Itô formula, we have

$$\mathrm{d}\|X_s^x\|_H^2\beta(s) = \big\{-2\|X_s^x\|_V^2\beta(s) + \beta'(s)\|X_s^x\|_H^2 + \beta(s)\|Q\|_{HS}^2\big\}\mathrm{d}s + 2\beta(s)\langle X_s^x, Q\mathrm{d}W_s\rangle.$$

Thus, for any $\lambda>0$,

$$\begin{aligned}&\mathbb{E}\exp\bigg[2\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\beta(s)\mathrm{d}s - \lambda\|x\|_H^2 - \lambda\|Q\|_{HS}^2t\bigg]\\ &\le \mathbb{E}\exp\bigg[2\lambda\int_0^{t\wedge\tau_n}\beta(s)\langle X_s^x, Q\mathrm{d}W_s\rangle + \lambda\int_0^{t\wedge\tau_n}\beta'(s)\|X_s^x\|_H^2\mathrm{d}s\bigg]\\ &\le \bigg(\mathbb{E}\exp\bigg[2\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\beta(s)\mathrm{d}s\bigg]\bigg)^{1/2}\bigg(\mathbb{E}\exp\bigg[4\lambda\int_0^{t\wedge\tau_n}\beta(s)\langle X_s^x, Q\mathrm{d}W_s\rangle\\ &\qquad - 2\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_H^2\big(\lambda_0^2\beta(s)-\beta'(s)\big)\mathrm{d}s\bigg]\bigg)^{1/2}.\end{aligned} \tag{3.2}$$

Note that the first inequality in the above display implies that

$$\mathbb{E}\exp\left[2\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\beta(s)\mathrm{d}s\right]<\infty,\quad n\geq 1.$$

Let

$$\lambda=\frac{1}{t\|Q\|^2}.$$

By our choice of $\beta(s)$ and noting that $t\leq\lambda_0^{-2}$ so that $\beta(s)\leq 1$, we have

$$\frac{1}{2}(4\lambda)^2\beta(s)^2\|Q\|^2\leq 2\lambda^2\beta(s)\|Q\|^2\leq 2\lambda\big(\lambda_0^2\beta(s)-\beta'(s)\big).$$

Therefore,

$$\mathbb{E}\exp\left[4\lambda\int_0^{t\wedge\tau_n}\beta(s)\langle X_s^x,Q\mathrm{d}W_s\rangle-2\lambda\int_0^{t\wedge\tau_n}\|X_s^x\|_H^2\big(\lambda_0^2\beta(s)-\beta'(s)\big)\mathrm{d}s\right]\leq 1.$$

Combining this with (3.2) for $\lambda=(t\|Q\|^2)^{-1}$, we obtain

$$\mathbb{E}\exp\left[\frac{2}{\|Q\|^2\mathrm{e}t}\int_0^{t\wedge\tau_n}\|X_s^x\|_V^2\mathrm{d}s\right]\leq\exp\left[\frac{2}{\|Q\|^2t}(\|x\|_H^2+\|Q\|_{HS}^2t)\right].$$

This completes the proof by letting $n\to\infty$. □

Proof of Theorem 1.1. Simply denote $X_s=X_s^x$, which solves (2.1) for $X_0=x$. For given $h\in V_\theta$ and $\varepsilon>0$, by Theorem 4.1 below the equation

$$\mathrm{d}Y_s=Q\mathrm{d}W_s-\Big\{LY_s+B(X_s)+\frac{\varepsilon}{t}\mathrm{e}^{-Ls}h\Big\}\mathrm{d}s,\quad Y_0=x+\varepsilon h \tag{3.3}$$

has a unique solution. So,

$$\mathrm{d}(X_s-Y_s)=-L(X_s-Y_s)\mathrm{d}s+\frac{\varepsilon}{t}\mathrm{e}^{-Ls}h\mathrm{d}s.$$

This implies that

$$\begin{aligned}X_s-Y_s&=\mathrm{e}^{-Ls}(X_0-Y_0)+\frac{\varepsilon}{t}\int_0^s\mathrm{e}^{-L(s-r)}\mathrm{e}^{-Lr}h\mathrm{d}r\\&=\frac{\varepsilon(t-s)}{t}\mathrm{e}^{-Ls}h=:Z_s,\quad s\in[0,t].\end{aligned}\tag{3.4}$$

Let

$$\eta_s=B(X_s+Z_s)-B(X_s)-\frac{\varepsilon}{t}\mathrm{e}^{-Ls}h,$$

which is well-defined since according to Lemma3.1, $X \in V$ holds $\mathbb{P} \times \mathrm{d}s$-a.e. Then, by (3.4) the equation (3.3) reduces to

$$\mathrm{d}Y_s = Q\mathrm{d}W_s - \{LY_s + B(Y_s)\}\mathrm{d}s + \eta_s\mathrm{d}s = Q\mathrm{d}\tilde{W}_s - \{LY_s + B(Y_s)\}\mathrm{d}s, \tag{3.5}$$

where

$$\tilde{W}_s := W_s + \int_0^s Q^{-1}\eta_r\mathrm{d}r, \quad s \in [0,t].$$

By **(A0)** and **(A3)** we have

$$\begin{aligned} \|Q^{-1}\eta_s\|_H^2 &\le \frac{2\varepsilon^2 K_1^2}{t^2}\|h\|_{V_\theta}^2 + 2\|\tilde{B}(X_s, Z_s) + B(z_s, z_s)\|_Q^2 \\ &\le \varepsilon^2 C(t)\big(\|h\|_{V_\theta}^2 + \varepsilon^2\|h\|_{V_\theta}^4 + \|h\|_{V_\theta}^2\|X_s\|_{V_\theta}^2\big). \end{aligned} \tag{3.6}$$

Since $\theta \le 1$ so that $\|\cdot\|_{V_\theta} \le c\|\cdot\|_V$ holds for some constant $c > 0$, combining (3.6) with Lemma 3.1 we concluded that

$$\mathbb{E}\mathrm{e}^{\int_0^t \|\eta_s\|_Q^2\mathrm{d}s} < \infty$$

holds for small enough $\varepsilon > 0$. By the Girsanov theorem, in this case

$$R_s := \exp\bigg[-\int_0^s \langle Q^{-1}\eta_r, \mathrm{d}W_r\rangle - \frac{1}{2}\int_0^s \|\eta_r\|_Q^2\mathrm{d}r\bigg], \quad s \in [0,t]$$

is a martingale and $\{\tilde{W}_s\}_{s\in[0,t]}$ is the cylindrical Brrownian motion on H under the probability measure $\mathbb{R}_t\mathbb{P}$. Combining this with (3.5) and the fact that $Y_t = X_t$ due to (3.4), for small $\varepsilon > 0$ we have

$$P_t f(x + \varepsilon h) = \mathbb{E}[R_t f(Y_t)] = \mathbb{E}[R_t f(X_t)].$$

Therefore, by the dominated convergence theorem due to Lemma 3.1 and (3.6), we conclude that

$$\begin{aligned} D_h P_t f(x) &:= \lim_{\varepsilon\to 0}\frac{P_t f(x+\varepsilon h) - P_t f(x)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\mathbb{E}\Big[\frac{R_t - 1}{\varepsilon}f(X_t)\Big] = -\mathbb{E}\bigg\{f(X_t)\lim_{\varepsilon\to 0}\int_0^t \Big\langle Q^{-1}\frac{\eta_s}{\varepsilon}, \mathrm{d}W_s\Big\rangle\bigg\} \\ &= -\mathbb{E}\bigg\{f(X_t)\int_0^t \Big\langle Q^{-1}\Big(\frac{t-s}{t}\tilde{B}(\mathrm{e}^{-Ls}h, X_s) - \frac{1}{t}\mathrm{e}^{-Ls}h\Big), \mathrm{d}W_s\Big\rangle\bigg\}, \end{aligned}$$

where the last step is due to the bilinear property of B, which implies that

$$\begin{aligned}\frac{\eta_s}{\varepsilon} &= \frac{1}{\varepsilon}\tilde{B}(X_s, z_s) + \frac{1}{\varepsilon}B(Z_\varepsilon) - \frac{1}{t}\mathrm{e}^{-Ls}h \\ &= \frac{t-s}{t}\tilde{B}(X_s, \mathrm{e}^{-Ls}h) - \frac{1}{t}\mathrm{e}^{-Ls}h + \frac{\varepsilon(t-s)}{t}B(\mathrm{e}^{-Ls}h, \mathrm{e}^{-Ls}h).\end{aligned}$$

□

Proof of Corollary 1.2. (1) By (1.2) and the Schwartz inequality, for any h with $\|h\|_{V_\theta} \le 1$, we have

$$(3.7)\qquad \begin{aligned}|D_h P_t f(x)|^2 &\le (P_t f(x))^2 \mathbb{E}\int_0^t \Big\|\frac{1}{t}\mathrm{e}^{-Ls}h - \frac{t-s}{t}\tilde{B}(X_s^x, h)\Big\|_Q^2 \mathrm{d}s \\ &\le 2(P_t f^2(x))\Big\{\frac{K_1}{t} + \mathbb{E}\int_0^t \|\tilde{B}(X_s^x, h)\|_Q^2 \mathrm{d}s\Big\},\end{aligned}$$

where the last step is due to the fact that **(A0)** implies

$$(3.8)\qquad \|\mathrm{e}^{-Ls}h\|_Q^2 \le K_1\|\mathrm{e}^{-Ls}h\|_{V_\theta}^2 \le K_1\|h\|_{V_\theta}^2.$$

Next, by **(A3)** and $\theta \le 1$ we have

$$(3.9)\qquad \|\tilde{B}(X_s^x, h)\|_Q^2 \le 4K_2\|h\|_{V_\theta}^2\|X_s^x\|_{V_\theta}^2 \le \frac{4K_2}{\lambda_0^{1-\theta}}\|X_s^x\|_V^2.$$

Combining this with (3.1) we obtain

$$\mathbb{E}\int_0^t \|\tilde{B}(X_s^x, h)\|_Q^2 \mathrm{d}s \le \frac{2K_2}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 + \|Q\|_{HS}^2 t\big).$$

The proof of (1) is completed by this and (3.7).

(2) Let $f \ge 0$ and h be such that $\|h\|_{V_\theta} \le 1$. Let

$$M_t = \int_0^t \Big\langle Q^{-1}\Big(\frac{t-s}{t}\tilde{B}(\mathrm{e}^{-Ls}h, X_s) - \frac{1}{t}\mathrm{e}^{-Ls}h\Big), \mathrm{d}W_s\Big\rangle.$$

By (1.2) and the Young inequality (see e.g. [2, Lemma 2.4]),

$$(3.10)\qquad |D_h P_t f(x)| \le \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}(x) + \big\{\delta \log \mathbb{E}\mathrm{e}^{\frac{1}{\delta}M_t}\big\}P_t f(x), \quad \delta > 0.$$

Since by (3.8) and (3.9) we have

$$
(3.11)\qquad \begin{aligned} \langle M\rangle_t &= \int_0^t \Big\|\frac{1}{t}\mathrm{e}^{-Ls}h - \frac{t-s}{t}\tilde B(X_s^x,h)\Big\|_Q^2\mathrm{d}s\\ &\le \frac{2K_1}{t} + \frac{4K_2}{\lambda_0^{1-\theta}}\int_0^t \|X_s^x\|_V^2\mathrm{d}s, \end{aligned}
$$

it follows from Lemma 3.1 that for any $\delta\ge\delta_0 := 4\sqrt{K_2}\,\|Q\|\lambda_0^{(\theta-3)/2}$,

$$
\begin{aligned} \mathbb{E}\exp\Big[\frac{1}{\delta}M_t\Big] &\le \Big(\mathbb{E}\exp\Big[\frac{2}{\delta^2}\langle M\rangle_t\Big]\Big)^{1/2} \le \Big(\mathbb{E}\exp\Big[\frac{2}{\delta_0^2}\langle M\rangle_t\Big]\Big)^{\delta_0^2/(2\delta^2)}\\ &\le \exp\Big[\frac{2K_1}{\delta^2 t}\Big]\Big(\mathbb{E}\exp\Big[\frac{8K_2}{\delta_0^2\lambda_0^{1-\theta}}\int_0^t\|X_s^x\|_V^2\mathrm{d}s\Big]\Big)^{\delta_0^2/(2\delta^2)}\\ &= \exp\Big[\frac{2K_1}{\delta^2 t}\Big]\Big(\mathbb{E}\exp\Big[\frac{\lambda_0^2}{2\|Q\|^2}\int_0^t\|X_s^x\|_V^2\mathrm{d}s\Big]\Big)^{\delta_0^2/(2\delta^2)}\\ &\le \exp\Big\{\frac{2K_1}{\delta^2 t} + \frac{\lambda_0^2\delta_0^2}{4\delta^2\|Q\|^2}(\|x\|_H^2 + \|Q\|_{HS}^2 t)\Big\}\\ &= \exp\Big\{\frac{2}{\delta^2}\Big(\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}(\|x\|_H^2+\|Q\|_{HS}^2 t)\Big)\Big\}. \end{aligned}
$$

Combining this with (3.10) we prove (2).

(3) According to e.g. [4, proof of Proposition 4.1]), the V_θ-strong Feller property of P_t follows from the claimed Harnack inequality, which we prove below by using an argument in [2, Proof of Theorem 1.2]. Let $x\ne y$ be such that

$$
(3.12)\qquad \|x-y\|_{V_\theta} \le \frac{\alpha-1}{\alpha\delta_0}\ \text{ for }\ \delta_0 := \frac{4\|Q\|\sqrt{K_2}}{\lambda_0^{(3-\theta)/2}}.
$$

Let

$$
\beta_s = 1+s(\alpha-1),\quad \gamma_s = x+s(y-x),\quad s\in[0,1].
$$

We have

$$
\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\log(P_t f^{\beta(s)})^{\alpha/\beta(s)}(\gamma_s)\\ &= \frac{\alpha(\alpha-1)}{\beta(s)^2}\cdot\frac{P_t(f^{\beta(s)}\log f^{\beta(s)}) - (P_t f^{\beta(s)})\log P_t f^{\beta(s)}}{P_t f^{\beta(s)}}(\gamma_s) + \frac{\alpha D_{y-x}P_t f^{\beta(s)}}{\beta(s)P_t f^{\beta(s)}}(\gamma_s)\\ &\ge \frac{\alpha\|x-y\|_{V_\theta}}{\beta(s)P_t f^{\beta(s)}(\gamma_s)}\Big\{\frac{\alpha-1}{\beta(s)\|x-y\|_{V_\theta}}\Big(P_t(f^{\beta(s)}\log f^{\beta(s)}) - (P_t f^{\beta(s)})\log P_t f^{\beta(s)}\Big)(\gamma_s)\\ &\qquad - \|DP_t f^{\beta(s)}(\gamma_s)\|_{V_\theta}^*\Big\}. \end{aligned}
$$

Therefore, applying (2) to

$$\delta := \frac{\alpha - 1}{\beta(s)\|x - y\|_{V_\theta}}$$

which is larger than δ_0 according to (3.12), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log(P_t f^{\beta(s)})^{\alpha/\beta(s)}(\gamma_s) &\geq -\frac{2\alpha\|x-y\|_{V_\theta}}{\delta\beta(s)}\left\{\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}(\|\gamma_s\|_H^2 + \|Q\|_{HS}^2 t)\right\} \\ &\geq -\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha - 1}\left\{\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}(\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t)\right\}.\end{aligned}$$

Integrating over $[0,1]$ w.r.t. $\mathrm{d}s$, we derive the desired Harnack inequality. □

Proof of Corollary 1.3. Since $u \mapsto \|u\|_V^2$ is a compact function on H, i.e. for any $r > 0$ the set $\{u \in H : \|u\|_V \leq r\}$ is relatively compact in H, (3.1) implies the existence of the invariant probability measure satisfying (1) by a standard argument (see e.g. [15, Proof of Theorem 1.2]). Moreover, any invariant probability measure μ satisfies $\mu(\|\cdot\|_V^2) < \infty$, hence, $\mu(V) = 1$. Below, we assume $\theta \in (0,1)$ and prove (1) and (2) repsectively.

(1) Let μ be an invariant probability measure, we first prove it has full support on μ.

$$r_0 = \frac{\lambda_0^{(3-\theta)/2}}{8\|Q\|\sqrt{K_2}}.$$

By Corollary 1.2(3) for $\alpha = 2$, for any fixed $t > 0$ there exists a constant $C(t) > 0$ such that

$$(P_t f(x))^2 \leq (P_t f^2(y))\mathrm{e}^{C(t)(\|x\|_H^2 + \|y\|_H^2)}, \quad \|x - y\|_{V_\theta} \leq r_0.$$

Applying this inequality n times, we may find a constant $c(t,n) > 0$ such that

$$(P_t f(x))^{2n} \leq (P_t f^{2n}(y))\mathrm{e}^{C(t,n)(\|x\|_H^2 + \|y\|_H^2)}, \quad \|x - y\|_{V_\theta} \leq nr_0. \tag{3.13}$$

Since V is dense in V_θ, to prove that μ has full support on V_θ, it suffices to show that

$$\mu(B_\theta(x,\varepsilon)) > 0, \quad x \in V, \varepsilon > 0 \tag{3.14}$$

holds for $B_\theta(x,\varepsilon) := \{y : \|y - x\|_{V_\theta} < \varepsilon\}$. Since $\mu(V_\theta) = 1$, there exists $n \geq 1$ such that $\mu(B_\theta(x, nr_0)) > 0$. Applying (3.13) to $f = 1_{B_\theta(x,\varepsilon)}$ we obtain

$$\mathbb{P}(\|X_t^x - x\|_{V_\theta} < \varepsilon)^{2n} \int_{B_\theta(x,nr_0)} \mathrm{e}^{-C(t,n)(\|x\|_H^2 + \|y\|_H^2)}\mu(\mathrm{d}y) \leq \mu(B_\theta(x,\varepsilon)).$$

So, if $\mu(B_\theta(x,\varepsilon)) = 0$ then

(3.15) $$\mathbb{P}(\|X_t^x - x\|_{V_\theta} \geq \varepsilon) = 1, \quad t > 0.$$

To see that this is impossible, let us observe that for any $m \geq 1$ there exists a constant $c(m) > 0$ such that

(3.16) $$\|\cdot\|_{V_\theta}^2 \leq c(m)\|\cdot\|_H^2 + \frac{1}{(\lambda_0 m)^{1-\theta}}\|\cdot\|_V^2$$

holds. Moreover, using $\langle\cdot,\cdot\rangle$ to denote the duality w.r.t H, we have

$$\begin{aligned} 2\langle X_t^x - x, LX_t^x\rangle &= 2\|X_t^x - x\|_V^2 + 2\langle X_t^x - x, Lx\rangle \\ &\geq 2\|X_t^x - x\|_V^2 - 2\|X_t^x - x\|_V\|x\|_V \geq \|X_t^x - x\|_V^2 - \|x\|_V^2 \end{aligned}$$

and due to **(A1)** and **(A2)**,

$$2\langle X_t^x - x, B(X_t^x)\rangle = -2\langle x, B(X_t^x)\rangle \leq 2C\|x\|_H\|X_t^x\|_V\|X_t^x\|_H \leq \frac{1}{2}\|X_t^x - x\|_V^2 + c_1 + c_2\|X_t^x\|_H^2$$

holds for some constants c_1, c_2 depending on x. Therefore, by the Itô formula for $\|X_t^x - x\|_H^2$, we arrive at

$$\begin{aligned} \mathrm{d}\|X_t^x - x\|_H^2 &= \big\{\|Q\|_{HS}^2 - 2\langle X_t^x - x, LX_t^x\rangle + 2\langle X_t^x - x, B(X_t^x)\rangle\big\}\mathrm{d}t + 2\langle X_t^x - x, Q\mathrm{d}W_t\rangle \\ &\leq -\frac{1}{2}\|X_t^x - x\|_V^2\mathrm{d}t + (c_3 + c_2\|X_t^x\|_H^2)\mathrm{d}t + 2\langle X_t^x - x, Q\mathrm{d}W_t\rangle \end{aligned}$$

for some constant $c_3 > 0$. Since by Theorem 4.1 below $\mathbb{E}\sup_{t\in[0,1]}\|X_t\|_H^2 < \infty$, this and the continuity of X_s^x in s imply

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \|X_s^x - x\|_H^2\mathrm{d}s = 0$$

and

$$\mathbb{E}\int_0^t \|X_t^x - x\|_V^2\mathrm{d}s \leq c_0 t, \quad t \in [0,1]$$

for some constant $c_0 > 0$. Combining these with (3.16), we conclude that

$$\limsup_{t\to 0}\frac{1}{t}\int_0^t \mathbb{E}\|X_s^x - x\|_{V_\theta}^2\mathrm{d}s \leq \frac{c_0}{(\lambda_0 m)^{1-\theta}}, \quad m \geq 1.$$

Letting $m \to \infty$ we obtain

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \mathbb{E}\|X_s^x - x\|_{V_\theta}^2\mathrm{d}s = 0.$$

this is contractive to (3.15).

Next, if the invariant probability measure is not unique, we may take two different extreme elements μ_1, μ_2 of the set of all invariant probability measures. It is well-known that μ_1 and μ_2 are singular with each other. Let D be a μ_1-null set, since μ_1 has full support on V_θ and $P_t 1_D$ is continuous and $\mu_1(P_t 1_D) = \mu_1(D) = 0$, we have $P_t 1_D \equiv 0$. Thus, $\mu_2(D) = \mu_2(P_t 1_D) = 0$. This means that μ_2 has to be absolutely continuous w.r.t. μ_1, which is contradictive to the singularity of μ_1 and μ_2.

(2) As observe above that $P_t 1_D \equiv 0$ for any μ-null set D. So, P_t has a transition density $p_t(x, y)$ w.r.t. μ on V_θ. Next, let $f \geq 0$ such that $\mu(f^\alpha) \leq 1$. By the Harnack inequality in Corollary 1.2(3), we have

$$(P_t f(x))^\alpha \int_{B_\theta(x,r_0)} \exp\bigg[-\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha-1}\Big\{\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}\bigg]\mu(\mathrm{d}y) \leq 1.$$

Then the desired estimate on $\int p_t(x,z)^{(\alpha+1)/\alpha a}\mu(\mathrm{d}z)$ follows by taking

$$f(\cdot) = p_t(x, \cdot).$$

Proof of Theorem 1.4. (1) Let M_t be in the proof of Corollary 1.2 (2). By (3.11), for $\delta > 0$ we have

$$\begin{aligned}
\mathbb{E}\exp\Big[\frac{M_t}{\delta}\Big] &\leq \Big(\mathbb{E}\exp\Big[\frac{2\langle M\rangle_t}{\delta^2}\Big]\Big)^{1/2} \\
&\leq \exp\Big[\frac{2K_1}{\delta^2 t}\Big]\bigg(\exp\Big[\frac{8K_2}{\lambda_0^{1-\theta}\delta^2}\int_0^t \|X_s^x\|_V^2 \mathrm{d}s\Big]\bigg)^{1/2}.
\end{aligned}$$

If $t \leq t_\delta$ then

$$\frac{8K_2}{\lambda_0^{1-\theta}\delta^2} \leq \frac{2\lambda_0^2}{\|Q\|^2 \mathrm{e}t},$$

so that by the Jensen inequality and the second inequality in Lemma 3.1,

$$\begin{aligned}
\mathbb{E}\exp\Big[\frac{M_t}{\delta}\Big] &\leq \exp\Big[\frac{2K_1}{\delta^2 t}\Big]\bigg(\exp\Big[\frac{2\lambda_0^2}{\|Q\|^2 \mathrm{e}t}\int_0^t \|X_s^x\|_V^2 \mathrm{d}s\Big]\bigg)^{\frac{2K_2\|Q\|^2 \mathrm{e}t}{\delta^2\lambda_0^{3-\theta}}} \\
&\leq \exp\Big[\frac{2K_1}{\delta^2 t} + \frac{4K_2\mathrm{e}}{\delta^2\lambda_0^{1-\theta}}\Big], \quad t \leq t_\delta.
\end{aligned}$$

Combining this with (3.10) we prove the desired gradient estimate for $t \leq t_\delta$. By the gradient estimate for $t = t_\delta$ and the semigroup property, when $t > t_\delta$ we have

$$\|DP_tf(x)\|_{V_\theta^*} = \|DP_{t_\delta}(P_{t-t_\delta}f)(x)\|_{V_\theta^*} \le \delta\big\{P_{t_\delta}\big((P_{t-t_\delta}f)\log P_{t-t_\delta}f\big) - (P_tf)\log P_tf\big\}(x) + \frac{2}{\delta}\Big\{\frac{K_1}{t_\delta} + \frac{2K_2\mathrm{e}}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}P_tf(x).$$

This implies the desired gradient estimate for $t > t_\delta$ since due to the Jensen inequality

$$P_{t_\delta}\big((P_{t-t_\delta}f)\log P_{t-t_\delta}f\big) \le P_tf\log f.$$

(2) Repeating the proof of Corollary 1.3 (3) using the inequality in Theorem 1.4 (1) instead of Corollary 1.2 (2) for $\delta = \frac{\alpha-1}{\beta(s)\|x-y\|_{V_\theta}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\log P_tf^{\beta(s)}\big)^{\alpha/\beta(s)} \ge -\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha-1}\Big\{\frac{K_1}{t\wedge t_\delta} + \frac{2K_2\mathrm{e}}{\lambda_0^{1-\theta}}\big(\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t\big)\Big\}.$$

This completes the proof by integrating over $[0,1]$ w.r.t. ds and noting that

$$t_\delta = \frac{\delta^2\lambda_0^{3-\theta}}{4\|Q\|^2\mathrm{e}K_2} \ge \frac{(\alpha-1)^2\lambda_0^{3-\theta}}{4\alpha^2\|Q\|^2K_2\mathrm{e}\|x-y\|_{V_\theta}^2}$$

since

$$\delta = \frac{\alpha-1}{\beta(s)\|x-y\|_{V_\theta}} \ge \frac{\alpha-1}{\alpha\|x-y\|_{V_\theta}}.$$

□

4 Appendix

We aim to verify the existence and uniqueness of the solution to (1.1) by using the main result of [8].

Theorem 4.1. *Assume* **(A1)** *and* **(A2)**. *For any $X_0 \in H$ the equation (1.1) has a unique solution X_t, which is a continuous Markov process on H such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|X_t\|_H^p + \int_0^T \|X_t\|_V^2\mathrm{d}t\Big) < 0$$

holds for any $p > 1$ and $\mathbb{P}$-a.s.

$$X_t = X_0 - \int_0^t (LX_s + B(X_s))\mathrm{d}s + QW_t, \quad t \ge 0$$

holds on H.

Proof. Let V^* be the dual space of V w.r.t. H. Then for any $v \in V$,

$$A(v) := -(Lv + B(v)) \in V^*.$$

It suffices to verify assumptions (H1)-(H4) in [8, Theorem 1.1] for the functional A. The hemicontinuity assumption H1) follows immediately form the bilinear property of B. Next, by **(A2)** and the bilinear property of B, we have

$$\begin{aligned} {}_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V &= -\|v_1 - v_2\|_V^2 + \|B(v_2 - v_1, v_1), v_1 - v_2\rangle \\ &\le -\|v_1 - v_2\|_V^2 + C\|v_1 - v_2\|_H^2 \|v_1\|_V^2. \end{aligned}$$

So, the assumption (H2) in [8] holds for $\rho(v) := c\|v\|_V^2$. Moreover, by **(A1)** we have

$${}_{V^*}\langle A(v), v\rangle_V \le -\|v\|_V^2.$$

Thus, the coercivity assumption (H3) in [8] holds for $\theta = 1, \alpha = 2, K = 0$ and f =constant. Finally, **(A2)** implies that

$$\|A(v)\|_{V^*}^2 \le 2\|v\|_V^2 + 2\|L^{-1/2}B(v)\|_H^2 \le 2\|v\|_V^2 + \frac{2c}{\lambda_0}\|v\|_H^2\|v\|_V^2.$$

Therefore, the growth condition (H4) in [8] holds for some constant $f, K > 0$ and $\alpha = \beta = 2$. □

References

[1] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below,* Bull. Sci. Math. 130(2006), 223–233.

[2] M. Arnaudon, A. Thalmaier and F.-Y. Wang, *Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds,* Stoch. Proc. Appl. 119(2009), 3653–3670.

[3] G. Da Prato, A. Debussche, *Ergodicity for the 3D stochastic Navier-Stokes equations,* J. Math. Pures Appl. 82(2003), 877–947.

[4] G. Da Prato, M. Röckner and F.-Y. Wang, *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups,* J. Funct. Anal. 257(2009), 992–1017.

[5] F. Flandoli, M. Romito, *Markov selections for the 3D stochastic Navier-Stokes equations,* Probab. Theory Relat. Fields 140(2008), 407–458.

[6] M. Hairer, J. C. Mattingly, *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing,* Ann. Math. 164(2006), 993–1032.

[7] W. Liu, Doctor-Thesis, Bielefeld University, 2009.

[8] W. Liu, M. Röckner, *SPDE in Hilbert space with locally monotone coefficients,* arXiv:1005.0632v1, 2010.

[9] W. Liu and F.-Y. Wang, *Harnack inequality and strong Feller property for stochastic fast diffusion equations,* J. Math. Anal. Appl. 342(2008), 651–662.

[10] S.-X. Ouyang, Doctor-Thesis, Bielefeld University, 2009.

[11] S.-X. Ouyang, M. Röckner and F.-Y. Wang, *Harnack inequalities and applications for Ornstein-Uhlenbeck semigroups with jump,* arXiv:0908.2889

[12] M. Romito, L. Xu, *HErgodicity of the 3D stochastic Navier-Stokes equations driven by mildly degenerate noise,* 2009 Preprint.

[13] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis (2nd Ed),* CBMS-NSF Regional Conference Series in Appl. Math. V66, SIAM, Philadelphia, 1995.

[14] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds,* Probab. Theory Relat. Fields 109(1997), 417–424.

[15] F.-Y. Wang, *Harnack inequality and applications for stochastic generalized porous media equations,* Ann. Probab. 35(2007), 1333–1350.