Various observations on angles proceeding in geometric progression*

Leonhard Euler

1. Since a great many well known properties found about angles, or arcs, and the sines, cosines, tangents, cotangents, secants and cosecants of them have been derived from the consideration of arcs increasing in arithmetic progression, those properties which one may deduce from the consideration of arcs proceeding in geometric progression seem no less noteworthy, since for the most part the truth of these seem much more concealed; therefore I am determined here to unfold many properties of this kind.

2. The starting point that presents itself to us for these types of speculations is the very well known formula[1]

$$\sin. 2\varphi = 2 \sin. \varphi \cdot \cos. \varphi.$$

If s denotes any arc or angle, then

$$\sin. s = 2 \sin. \tfrac{1}{2}s \cdot \cos. \tfrac{1}{2}s,$$

and likewise

$$\sin. \tfrac{1}{2}s = 2 \sin. \tfrac{1}{4}s \cdot \cos. \tfrac{1}{4}s,$$

and substituting this value into the former yields

$$\sin. s = 4 \sin. \tfrac{1}{4}s \cdot \cos. \tfrac{1}{2}s \cdot \cos. \tfrac{1}{4}s.$$

Next, because also

$$\sin. \tfrac{1}{4}s = 2 \sin. \tfrac{1}{8}s \cdot \cos. \tfrac{1}{8}s,$$

with this value substituted

$$\sin. s = 8 \sin. \tfrac{1}{8}s \cdot \cos. \tfrac{1}{2}s \cdot \cos. \tfrac{1}{4}s \cdot \cos. \tfrac{1}{8}s.$$

*Presented to the St. Petersburg Academy on November 15, 1773. Originally published as *Variae observationes circa angulos in progressione geometrica progredientes*, Opuscula analytica **1** (1783), 345–352. E561 in the Eneström index. Translated from the Latin by Jordan Bell, Department of Mathematics, University of Toronto, Toronto, Canada. Email: jordan.bell@gmail.com

[1] Translator: cf. Thomas Heath, *A history of Greek mathematics*, vol. II, pp. 278–280, Oxford, 1921.

Proceeding in the same way it is

$$\sin. s = 16 \sin. \tfrac{1}{16}s \cdot \cos. \tfrac{1}{2}s \cdot \cos. \tfrac{1}{4}s \cdot \cos. \tfrac{1}{8}s \cdot \cos. \tfrac{1}{16}s,$$

and if one continues in this way to infinity, with i denoting an infinite number or rather an infinite power of 2 we will have

$$\sin. s = i \sin. \tfrac{s}{i} \cdot \cos. \tfrac{1}{2}s \cdot \cos. \tfrac{1}{4}s \cdot \cos. \tfrac{1}{8}s \cdot \cos. \tfrac{1}{16}s \cdot \text{etc.},$$

where, because the arc $\frac{s}{i}$ is infinitely small, $\sin. \frac{s}{i} = \frac{s}{i}$, and so $i \sin. \frac{s}{i} = s$, from which we obtain the outstanding property that

$$\sin. s = s \cos. \tfrac{1}{2}s \cdot \cos. \tfrac{1}{4}s \cdot \cos. \tfrac{1}{8}s \cdot \cos. \tfrac{1}{16}s \cdot \cos. \tfrac{1}{32}s \cdot \text{etc. to infinity}$$

3. Therefore the arc s itself can be very prettily defined by its sine and the cosines of arcs continually diminished in double ratio, as

$$s = \frac{\sin. s}{\cos. \frac{1}{2}s \cdot \cos. \frac{1}{4}s \cdot \cos. \frac{1}{8}s \cdot \cos. \frac{1}{16}s \cdot \cos. \frac{1}{32}s \cdot \text{etc.}};$$

and because $\frac{1}{\cos. \varphi} = \sec. \varphi$, it will be as a complete expression

$$s = \sin. s \cdot \sec. \tfrac{1}{2}s \cdot \sec. \tfrac{1}{4}s \cdot \sec. \tfrac{1}{8}s \cdot \sec. \tfrac{1}{16}s \cdot \sec. \tfrac{1}{32}s \cdot \sec. \tfrac{1}{64}s \cdot \text{etc.},$$

which expression can be represented quite properly geometrically, as I have shown before elsewhere.[2]

4. Because the arc s is expressed here as a product, by taking logarithms we will have

$$ls = l \sin. s + l \sec. \tfrac{1}{2}s + l \sec. \tfrac{1}{4}s + l \sec. \tfrac{1}{8}s + l \sec. \tfrac{1}{16}s + l \sec. \tfrac{1}{32}s + \text{etc.},$$

whence if we let $s = \frac{\pi}{2} = 90^\circ$ it will be

$$l\tfrac{\pi}{2} = 0 + l \sec. 45^\circ + l \sec. 22^\circ 30' + l \sec. 11^\circ 15' + l \sec. 5^\circ 37\tfrac{1}{2}' + \text{etc.},$$

[2]Translator: See Euler's 1738 *De variis modis circuli quadraturam numeris proxime exprimendi*, E74; Opera omnia I.14, p. 257.

whence, having done a calculation, it will be[3]

$$\begin{aligned}
l\,\text{sec.}\,45^\circ &= 0,1505150\\
l\,\text{sec.}\,22^\circ 30' &= 0,0343847\\
l\,\text{sec.}\,11^\circ 15' &= 0,0084261\\
l\,\text{sec.}\,5^\circ 37\tfrac{1}{2}' &= 0,0020963\\
l\,\text{sec.}\,2^\circ 48\tfrac{3}{4}' &= 0,0005234\\
l\,\text{sec.}\,1^\circ 24\tfrac{3}{8}' &= 0,0001308\\
l\,\text{sec.}\,0^\circ 42\tfrac{3}{16}' &= 0,0000327\\
l\,\text{sec.}\,0^\circ 21\tfrac{3}{32}' &= 0,0000082\\
\text{other terms} &= 0,0000027\\
l\tfrac{\pi}{2} &= 0,1961199\\
l2 &= 0,3010300\\
l\pi &= 0,4971499
\end{aligned}$$

and hence one finds rather closely that $\pi = 3,1415928$.

5. To deduce here new relations, we will differentiate the last logarithmic equation, and since

$$d.l\,\text{sec.}\,\varphi = \frac{d\varphi\,\text{sin.}\,\varphi}{\text{cos.}\,\varphi} = d\varphi\,\text{tag.}\,\varphi,$$

the following equation arises by dividing by ds

$$\frac{1}{s} = \text{cot.}\,s + \tfrac{1}{2}\,\text{tag.}\,\tfrac{1}{2}s + \tfrac{1}{4}\,\text{tag.}\,\tfrac{1}{4}s + \tfrac{1}{8}\,\text{tag.}\,\tfrac{1}{8}s + \tfrac{1}{16}\,\text{tag.}\,\tfrac{1}{16}s + \text{etc.},$$

which series converges very quickly, as will clearly be seen in the following example. Let us take $s = 90^\circ = \frac{\pi}{2}$, whence it will be

$$\tfrac{2}{\pi} = \tfrac{1}{2}\,\text{tag.}\,45^\circ + \tfrac{1}{4}\tan 22^\circ 30' + \tfrac{1}{8}\,\text{tag.}\,11^\circ 15' + \tfrac{1}{16}\tan 5^\circ 37\tfrac{1}{2}' + \text{etc.},$$

[3]Translator: Here l denotes the logarithm with base 10. To determine the approximate sum of the remaining terms, namely $\log\sec\frac{\pi}{2^{k+1}}$ from $k = 9$ to $k = \infty$, one can use the Euler-Maclaurin summation formula applied to the function $f(x) = \log\sec\frac{\pi}{2^{x+1}}$. However, the integral of $f(x)$ cannot be expressed in a closed form. I don't know if there are easier ways to approximate this sum.

and the values taken from tables give

$$\begin{aligned}
\tfrac{1}{2}\,\text{tag.}\,45^\circ &= 0,5000000\\
\tfrac{1}{4}\,\text{tag.}\,22^\circ 30' &= 0,1035534\\
\tfrac{1}{8}\,\text{tag.}\,11^\circ 15' &= 0,0248640\\
\tfrac{1}{16}\,\text{tag.}\,5^\circ 37\tfrac{1}{2}' &= 0,0061557\\
\tfrac{1}{32}\,\text{tag.}\,2^\circ 48\tfrac{3}{4}' &= 0,0015352\\
\tfrac{1}{64}\,\text{tag.}\,1^\circ 24\tfrac{3}{8}' &= 0,0003836\\
\text{for the remaining} &= 0,0001279\\
\tfrac{2}{\pi} &= 0,6366198,
\end{aligned}$$

hence[4]

$$\pi = \tfrac{2}{0,6366198} = \tfrac{1}{0,3183099}.$$

6. If we differentiate the last equation for a second time we will come to a much more convergent series; for since it is[5]

$$d.\cot.\varphi = \frac{-d\varphi}{\sin.\varphi^2} \qquad \text{and} \qquad d.\,\text{tag.}\,\varphi = \frac{d\varphi}{\cos.\varphi^2} = d\varphi\sec.\varphi^2,$$

we will get

$$-\tfrac{1}{ss} = \tfrac{-1}{\sin.s^2} + \tfrac{1}{4}\sec.\tfrac{1}{2}s^2 + \tfrac{1}{16}\sec.\tfrac{1}{4}s^2 + \tfrac{1}{64}\sec.\tfrac{1}{8}s^2 + \text{etc.}$$

or

$$\tfrac{1}{4}\sec.\tfrac{1}{2}s^2 + \tfrac{1}{16}\sec.\tfrac{1}{4}s^2 + \tfrac{1}{64}\sec.\tfrac{1}{8}s^2 + \tfrac{1}{256}\sec.\tfrac{1}{16}s^2 + \text{etc.} = \tfrac{1}{\sin.s^2} - \tfrac{1}{ss}.$$

7. Let us apply this reasoning to arcs which decrease in a triple ratio; to this end let us consider the formula

$$\sin.3\varphi = 4\sin.\varphi\cos.\varphi^2 - \sin.\varphi = \sin.\varphi(3 - 4\sin.\varphi^2),$$

which gives

$$\sin.3\varphi = 3\sin.\varphi\left(1 - \tfrac{4}{3}\sin.\varphi^2\right);$$

so if s denotes any arc, it will be

$$\sin.s = 3\sin.\tfrac{1}{3}s\left(1 - \tfrac{4}{3}\sin.\tfrac{s^2}{3}\right);$$

and in a similar way

$$\sin.\tfrac{1}{3}s = 3\sin.\tfrac{s}{9}\left(1 - \tfrac{4}{3}\sin.\tfrac{s^2}{9}\right),$$

[4]Translator: $\frac{1}{0,3183099} = 3.14159251\ldots$ while $\pi = 3.14159265\ldots$.
[5]Translator: $\sin.\varphi^2 = (\sin.\varphi)^2$.

so that now

$$\sin. s = 9\sin.\tfrac{1}{9}s\left(1-\tfrac{4}{3}\sin.\tfrac{s^2}{3}\right)\left(1-\tfrac{4}{3}\sin.\tfrac{s^2}{9}\right).$$

If such substitutions are continued to infinity, as before one will be led at last to this expression

$$\sin. s = s\left(1-\tfrac{4}{3}\sin.\tfrac{s^2}{3}\right)\left(1-\tfrac{4}{3}\sin.\tfrac{s^2}{9}\right)\left(1-\tfrac{4}{3}\sin.\tfrac{s^2}{27}\right)\cdot\text{etc.}$$

8. This factors are too complicated, and one may resolve them into simpler ones in the following way; namely, because $\sin.\varphi^2 = \frac{1}{2}-\frac{1}{2}\cos.2\varphi$, the general form $1-\frac{4}{3}\sin.\varphi^2$ is reduced to

$$\frac{1}{3}+\frac{2}{3}\cos.2\varphi,$$

which one can write as

$$\frac{2\cos.60^\circ+2\cos.2\varphi}{3}.$$

Now, since

$$\cos.a+\cos.b = 2\cos.\frac{a+b}{2}\cos.\frac{a-b}{2},$$

it will be

$$\cos.60^\circ+\cos.2\varphi = 2\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi),$$

which formula multiplied by $\frac{2}{3}$ yields

$$1-\frac{4}{3}\sin.\varphi^2 = \frac{4}{3}\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi).$$

Whereby if this reduction is applied to all the factors found above, we will have the following infinite product

$$\begin{aligned}\frac{\sin. s}{s} = {} & \tfrac{4}{3}\cos.(30^\circ+\tfrac{s}{3})\cos.(30^\circ-\tfrac{s}{3})\\ & \tfrac{4}{3}\cos.(30^\circ+\tfrac{s}{9})\cos.(30^\circ-\tfrac{s}{9})\\ & \tfrac{4}{3}\cos.(30^\circ+\tfrac{s}{27})\cos.(30^\circ-\tfrac{s}{27})\\ & \text{etc.,}\end{aligned}$$

which is expressed by secants

$$\begin{aligned}\frac{s}{\sin. s} = {} & \tfrac{3}{4}\sec.(30^\circ+\tfrac{s}{3})\sec.(30^\circ-\tfrac{s}{3})\\ & \tfrac{3}{4}\sec.(30^\circ+\tfrac{s}{9})\sec.(30^\circ-\tfrac{s}{9})\\ & \tfrac{3}{4}\sec.(30^\circ+\tfrac{s}{27})\sec.(30^\circ-\tfrac{s}{27})\\ & \text{etc.,}\end{aligned}$$

whose factors approach unity more closely the more the arc s is diminished.

9. If we now take logarithms and differentiate all the terms, the numerical factors $\frac{3}{4}$, since they are constants, will completely disappear from the calculation; and since as we saw before

$$d.l.\sec.\varphi = d\varphi\,\mathrm{tag}.\,\varphi,$$

the following equation will be obtained

$$\begin{array}{lclllll} \frac{1}{s} & = & \cot.s & +\frac{1}{3}\,\mathrm{tag}.(30^\circ+\frac{s}{3}) & +\frac{1}{9}\,\mathrm{tag}.(30^\circ+\frac{s}{9}) & +\frac{1}{27}\,\mathrm{tag}.(30^\circ+\frac{s}{27}) & +\mathrm{etc.} \\ & & & -\frac{1}{3}\,\mathrm{tag}.(30^\circ-\frac{s}{3}) & -\frac{1}{9}\,\mathrm{tag}.(30^\circ-\frac{s}{9}) & -\frac{1}{27}\,\mathrm{tag}.(30^\circ-\frac{s}{27}) & -\mathrm{etc.} \end{array}$$

It will be helpful to probe this by an example. Therefore let $s = \frac{\pi}{2}$, and it will be

$$\begin{array}{llllllll} \frac{2}{\pi} = & \frac{1}{3}\,\mathrm{tag}.\,60^\circ & +\frac{1}{9}\,\mathrm{tag}.\,40^\circ & +\frac{1}{27}\,\mathrm{tag}.(33^\circ 20') & +\frac{1}{81}\,\mathrm{tag}.(31^\circ 6\frac{2}{3}') & +\frac{1}{243}\,\mathrm{tag}.(30^\circ 22\frac{2}{9}') & +\mathrm{etc.} \\ & -\frac{1}{3}\,\mathrm{tag}.\,0^\circ & -\frac{1}{9}\,\mathrm{tag}.\,20^\circ & -\frac{1}{27}\,\mathrm{tag}.(26^\circ 40') & -\frac{1}{81}\,\mathrm{tag}.(28^\circ 53\frac{1}{3}') & -\frac{1}{243}\,\mathrm{tag}.(29^\circ 37\frac{7}{9}') & -\mathrm{etc.} \end{array}$$

10. This last series seems all the more noteworthy because scarcely anyone would have been able to demonstrate its truth unless the same method were used. This series is without doubt much of a much higher level of investigation than those to which we were led by the expansion of the previous case,[6] which was

$$\frac{1}{s} = \cot.s + \tfrac{1}{2}\,\mathrm{tag}.\,\tfrac{1}{2}s + \tfrac{1}{4}\,\mathrm{tag}.\,\tfrac{1}{4}s + \tfrac{1}{8}\,\mathrm{tag}.\,\tfrac{1}{8}s + \tfrac{1}{16}\,\mathrm{tag}.\,\tfrac{1}{16}s + \mathrm{etc.};$$

the truth of this follows from the well known formula

$$2\cot.2\varphi = \cot.\varphi - \mathrm{tag}.\,\varphi,$$

from which we have

$$\mathrm{tag}.\,\varphi = \cot.\varphi - 2\cot.2\varphi.$$

Then if the appropriate values are substituted in place of all the tangents, the series is arranged in the following way

$$\frac{1}{s} = \left\{ \begin{array}{llllll} \cot.s & +\frac{1}{2}\cot.\frac{1}{2}s & +\frac{1}{4}\cos.\frac{1}{4}s & +\frac{1}{8}\cos.\frac{1}{8}s & +\cdots & +\frac{1}{i}\cot.\frac{s}{i} \\ -\cot.s & -\frac{1}{2}\cot.\frac{1}{2}s & -\frac{1}{4}\cot.\frac{1}{4}s & -\frac{1}{8}\cot.\frac{1}{8}s & -\cdots, & \end{array} \right.$$

where one sees that all the terms eliminate each other up to the final term $\frac{1}{i}\cot.\frac{s}{i}$, which one can write in the form

$$\frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}}$$

[6]Translator: §5.

with i denoting an infinite number. Since the arc $\frac{s}{i}$ is now infinitely small, $\cos.\frac{s}{i}=1$ and indeed the sine of this arc will be equal to $\frac{s}{i}$, whence this final term will be $=\frac{1}{s}$, which is the same value that was found equal to the series.

11. And for the present case the direct demonstration which is to be given is exhibited from that formula by which the tangent of thrice an angle is expressed. For if one puts

$$\mathrm{tag.}\,\varphi = t,$$

since one has[7]

$$\mathrm{tag.}\,3\varphi = \frac{3t-t^3}{1-3tt},$$

it will be

$$\cot.3\varphi = \frac{1-3tt}{3t-t^3} \qquad \text{and} \qquad 3\cot.3\varphi = \frac{3-9tt}{t(3-tt)}.$$

Then one subtracts $\cot.\varphi = \frac{1}{t}$ and it will be

$$3\cot.3\varphi - \cot.\varphi = \frac{-8tt}{t(3-tt)} = \frac{-8t}{3-tt};$$

then in place of t let us substitute its value $\frac{\sin.\varphi}{\cos.\varphi}$ and we will have

$$3\cot.3\varphi - \cot.\varphi = \frac{-8\sin.\varphi\cos.\varphi}{3\cos.\varphi^2 - \sin.\varphi^2}.$$

Let us deal with the numerator and the denominator of the fraction in the following way: Since

$$\cos.\varphi^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos.2\varphi \qquad \text{and} \qquad \sin.\varphi^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos.2\varphi,$$

the denominator will take the form $1+2\cos.2\varphi$ which can therefore be written as

$$2\cos.60^\circ + 2\cos.2\varphi,$$

which further reduces, because

$$\cos.a + \cos.b = 2\cos.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2},$$

to

$$4\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi).$$

The numerator is clearly $-4\sin.2\varphi$, so that we now have

$$3\cot.3\varphi - \cot.\varphi = \frac{-\sin.2\varphi}{\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi)}.$$

Now, since in general

$$\sin.2\varphi = \sin.(a+\varphi)\cos.(a-\varphi) - \cos.(a+\varphi)\sin.(a-\varphi),$$

[7]Translator: This by applying the addition formula for tan twice. The addition formula for tan is $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

let us take $a = 30°$ and we will have the following equation

$$3\cot.3\varphi - \cot.\varphi = \frac{-\sin.(30° + \varphi)\cos.(30° - \varphi) + \cos.(30° + \varphi)\sin.(30° - \varphi)}{\cos.(30° + \varphi)\cos.(30° - \varphi)}$$
$$= -\operatorname{tag.}(30° + \varphi) + \operatorname{tag.}(30° - \varphi),$$

by which we arrive at this noteworthy equation

$$\cot.3\varphi = \tfrac{1}{3}\cot.\varphi - \tfrac{1}{3}\operatorname{tag.}(30° + \varphi) + \tfrac{1}{3}\operatorname{tag.}(30° - \varphi).$$

12. Now, by writing s in place of 3φ for our case we get at once

$$\cot.s = \tfrac{1}{3}\cot.\tfrac{s}{3} - \tfrac{1}{3}\operatorname{tag.}(30° + \tfrac{s}{3}) + \tfrac{1}{3}\operatorname{tag.}(30° - \tfrac{s}{3}).$$

Indeed in a similar way it will further be

$$\tfrac{1}{3}\cot.\tfrac{s}{3} = \tfrac{1}{9}\cot.\tfrac{s}{9} - \tfrac{1}{9}\operatorname{tag.}(30° + \tfrac{s}{9}) + \tfrac{1}{9}\operatorname{tag.}(30° - \tfrac{s}{9}).$$

Further in the same way,

$$\tfrac{1}{9}\cot.\tfrac{s}{9} = \tfrac{1}{27}\cot.\tfrac{s}{27} - \tfrac{1}{27}\operatorname{tag.}(30° + \tfrac{s}{27}) + \tfrac{1}{27}\operatorname{tag.}(30° - \tfrac{s}{27}),$$

and if we proceed this way to infinity, we will come finally to a cotangent of the form

$$\tfrac{1}{i}\cot.\tfrac{s}{i} = \frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}};$$

hence our equation will take the form[8]

$$\begin{array}{llllll} \cot.s & = -\tfrac{1}{3}\operatorname{tag.}(30° + \tfrac{s}{3}) & -\tfrac{1}{9}\operatorname{tag.}(30° + \tfrac{s}{9}) & -\tfrac{1}{27}\operatorname{tag.}(30° + \tfrac{s}{27}) & -\cdots & -\tfrac{1}{s} \\ & +\tfrac{1}{3}\operatorname{tag.}(30° - \tfrac{s}{3}) & +\tfrac{1}{9}\operatorname{tag.}(30° - \tfrac{s}{9}) & +\tfrac{1}{27}\operatorname{tag.}(30° - \tfrac{s}{27}) & +\cdots, & \end{array}$$

from which we deduce our very equation itself that was to be demonstrated

$$\begin{array}{lllllll} \tfrac{1}{s} & = & \cot.s & +\tfrac{1}{3}\operatorname{tag.}(30° + \tfrac{s}{3}) & +\tfrac{1}{9}\operatorname{tag.}(30° + \tfrac{s}{9}) & +\tfrac{1}{27}\operatorname{tag.}(30° + \tfrac{s}{27}) & +\text{etc.} \\ & & & -\tfrac{1}{3}\operatorname{tag.}(30° - \tfrac{s}{3}) & -\tfrac{1}{9}\operatorname{tag.}(30° - \tfrac{s}{9}) & -\tfrac{1}{27}\operatorname{tag.}(30° - \tfrac{s}{27}) & -\text{etc.} \end{array}$$

13. Furthermore, one may similarly exhibit for higher ratios series of this type in which the arc s is continually diminished. For since

$$\sin.4\varphi = 8\sin.\varphi\cos.(45° + \varphi)\cos.(45° - \varphi)\cos.\varphi,$$

it will be in a quadruple ratio

$$\begin{array}{llllll} \tfrac{1}{s} & = \cot.s & +\tfrac{1}{4}\operatorname{tag.}\tfrac{s}{4} & +\tfrac{1}{16}\operatorname{tag.}\tfrac{s}{16} & +\tfrac{1}{64}\operatorname{tag.}\tfrac{s}{64} & +\text{etc.} \\ & & +\tfrac{1}{4}\operatorname{tag.}(45° + \tfrac{s}{4}) & +\tfrac{1}{16}\operatorname{tag.}(45° + \tfrac{s}{16}) & +\tfrac{1}{64}\operatorname{tag.}(45° + \tfrac{s}{64}) & +\text{etc.} \\ & & -\tfrac{1}{4}\operatorname{tag.}(45° - \tfrac{s}{4}) & -\tfrac{1}{16}\operatorname{tag.}(45° - \tfrac{s}{16}) & -\tfrac{1}{64}\operatorname{tag.}(45° - \tfrac{s}{64}) & -\text{etc.} \end{array}$$

[8]Translator: $\frac{1}{i}\cot.\frac{s}{i} = \frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}} = \frac{1+O(i^{-2})}{i\cdot(\frac{s}{i}+O(i^{-3}))} = \frac{1+O(i^{-2})}{s+O(i^{-2})} = \frac{1}{s} + O(i^{-2})$ as $i \to \infty$.

Next since[9]

$$\sin. 5\varphi = 16 \sin. \varphi \cos.(18^\circ + \varphi) \cos.(18^\circ - \varphi) \cos.(54^\circ + \varphi) \cos.(54^\circ - \varphi)$$

we will find in quintuple ratio

$$\begin{array}{lllll} \frac{1}{s} & = \cot. s & +\frac{1}{5} \operatorname{tag.}(18^\circ + \frac{s}{5}) & +\frac{1}{25} \operatorname{tag.}(18^\circ + \frac{s}{25}) & +\text{etc.} \\ & & -\frac{1}{5} \operatorname{tag.}(18^\circ - \frac{s}{5}) & -\frac{1}{25} \operatorname{tag.}(18^\circ - \frac{s}{25}) & -\text{etc.} \\ & & +\frac{1}{5} \operatorname{tag.}(54^\circ + \frac{s}{5}) & +\frac{1}{25} \operatorname{tag.}(54^\circ + \frac{s}{25}) & +\text{etc.} \\ & & -\frac{1}{5} \operatorname{tag.}(54^\circ - \frac{s}{5}) & -\frac{1}{25} \operatorname{tag.}(54^\circ - \frac{s}{25}) & -\text{etc.} \end{array}$$

One can proceed further in exactly the same way, but truly the resulting series would be too muddled than would deserve our attention.

[9] Translator: See Euler's 1774 *Quomodo sinus et cosinus angulorum multiplorum per producta exprimi queant*, E562; Opera omnia I.15, p. 509.