NON-TRIVIAL ELEMENTS IN THE ABEL-JACOBI KERNELS OF HIGHER DIMENSIONAL VARIETIES

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ABSTRACT. The purpose of this paper is to construct new non-trivial elements in the kernels of the Abel-Jacobi maps from the Chow groups of codimension p algebraic cycles to the p-th intermediate Jacobians on higher-dimensional varieties. Our method consists of the specialization of correspondences with non-trivial Hodge-theoretical invariants at the generic point on one variable. To provide explicit examples of rationally non-trivial cycles in the Albanese kernels we use correspondences on threefolds whose inverse Lefschetz operator is algebraic and p-fold self-intersections of the Chern class of the Poincaré line bundle for higher dimensional varieties.

1. INTRODUCTION

A standard approach in the study of algebraic cycles presumes the mapping of Chow groups in to the Weil cohomology groups via so-called cycle-class maps. The kernels of such mappings can not be always easily understood, so that we send then homologically trivial cycle classes into the intermediate Jacobians via the Abel-Jacobi mappings. The kernels of the Abel-Jacobi maps are even harder to understand and, actually, they are quite mysterious. For example, if X is a $K3$ surface, its Albanese kernel is unknown, in spite of the fact that such surfaces are intensively studied from many other points of view. If X is a surface of general type with $p_g = 0$ then Bloch's conjecture, which is a codimension 2 part of the general Bloch-Beilinson philosophy, predicts that the kernel of the Abel-Jacobi map is zero, but we are too far from the understanding of this conjecture. Working over the algebraic closure $\bar{\mathbb{Q}}$ of rational numbers we have the Bloch-Beilinson's conjecture saying that there are no non-trivial cycle classes in the Albanese kernel of any smooth projective variety over $\bar{\mathbb{Q}}$, so that first non-trivial examples of elements in the Abel-Jacobi kernels are expected to be rational over an extension of $\mathbb{Q}$ whose transcendence degree is at least one. The first example of such a cycle class was constructed in [14] by Schoen. He also refers to unpublished Nori's results along the same line. Schoen's algebraic cycle is an external product of two cycles on the product of two curves. The general case in dimension 2 has been considered in [5] where the authors proved an existence of a non-trivial class in the Albanese kernel for an arbitrary surface over $\bar{\mathbb{Q}}$ which is defined over a field of transcendence degree 1. In higher dimension the problem was considered in [13] and [9].

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In general, we are not afraid to say that the lack of concrete examples of non-trivial algebraic cycle classes in the Abel-Jacobi kernels is one of the main obstacles to further progress in our understanding of algebraic cycles and intersection theory, over $\mathbb{C}$ and specifically over $\bar{\mathbb{Q}}$.

Our method of constructing of non-trivial algebraic cycle classes in the Abel-Jacobi kernels is not relevant to external product of cycles on two varieties, but rather rely upon the interplay between algebraic cycles defined over a field of non-zero transcendence degree, and correspondences, i.e. algebraic cycles on products of varieties, see [2] and [3]. Correspondences act on Chow groups and cohomology, and as such they have more powerful cohomological and Hodge-theoretic invariants which allow detect non-triviality of the original cycle classes. At the same time, correspondences control the motivic information "at large". Searching correspondences which act non-trivially on suitable pieces in the Hodge-decomposition, and specializing them at the generic point on one variable, we construct desirable non-trivial classes in the Abel-Jacobi kernels.

Our first main result consists of the following general principle (Theorem 11 below):

> *Let S and X be two smooth projective varieties over an algebraically closed subfield in $\mathbb{C}$, and let α be a correspondence from S to X. Suppose that $\alpha_{\mathbb{C}}$ acts non-trivially on the appropriate Hodge pieces of the cohomology groups. Then the specialization of α at the generic point of S is a non-trivial cycle class on the variety $X_{\mathbb{C}}$.*

Notice that this principle appears already in the paper by Green and Griffiths, [6], which is the departing point in our considerations. In op.cit. the authors construct a nice version of the Bloch-Beilinson filtration assuming the generalized Hodge conjecture and Bloch-Beilinson's conjecture, while we concentrate mostly on the task of constructing of concrete non-trivial algebraic cycle classes in the Abel-Jacobi kernels not assuming any conjecture.

To provide explicit examples of new rationally non-trivial cycles we use the following result, which is arises from the above principle (Theorem 12):

> *Let S, X and α be as above, and assume that $\alpha_{\mathbb{C}}$ acts non-trivially on suitable Hodge pieces of the cohomology groups. Then the difference between specializations of α, modified by the Albanese projector, at the generic and a closed points on S is always a non-trivial element in the Abel-Jacobi kernel for the variety $X_{\mathbb{C}}$.*

In particular, it allows to construct non-trivial elements in the Abel-Jacobi kernels for $K3$-surfaces and threefolds whose inverse Lefschetz operator is algebraic.

Our next construction is based on specialization at the generic point of self-intersections of the Chern class of the Poincaré line bundle. Namely, if α is a p-fold self-intersection of the Chern class of the Poincaré line bundle β on a smooth projective variety over k then the following result (see Theorem 15 in the text) gives a wide range of concrete non-trivial algebraic cycle classes in the Abel-Jacobi maps in arbitrary codimension p:

> *Let $\alpha = \beta^p$ be as above. Then the specialization of α at the generic point is in the Abel-Jacobi kernel. Moreover, if there exist p holomorphic one-forms on $X_{\mathbb{C}}$, such that*
>
> $$\omega_1 \wedge \ldots \wedge \omega_p \neq 0 \ ,$$
>
> *then the specialization of α at the generic point of S is a non-trivial element in the Abel-Jacobi kernel for $X_{\mathbb{C}}$.*

The paper is organized as follows. First we recall the language of Chow motives and prove some easy motivic lemmas in Section 2. Section 3 is devoted to the Abel-Jacobi kernels. Some important lemma (Lemma 6) is proved there. In Section 4 we prove Theorem 11 and Theorem 12. In Section 5 we consider the case of $K3$ surfaces and then threefolds with known Lefschetz conjecture. In the last Section 6 we explain how to use Theorem 12 in order to produce new concrete non-trivial cycle classes in the Abel-Jacobi kernels of arbitrary codimension $p \geq 2$ for a large class of higher dimensional varieties.

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2. Motivic factorization

Below we will intensively use the language of Chow motives. As this language is not yet absolutely standard, for the convenience of the reader, we first recall some basic definitions and fix notation on them. Then we will prove a few motivic lemmas and deduce one proposition on motivic factorization, which will be used in what follows.

Let k be an algebraically closed field. For any algebraic scheme over k let $CH^n(X)$ be the Chow group of codimension n algebraic cycles on X with coefficients in $\mathbb{Q}$. Thus, $CH^n(X)$ is, actually, a $\mathbb{Q}$-vector space, rather than a group. The category of Chow motives $\mathscr{CM}$ over k will be given in contravariant notation. That is, if X and Y are two smooth projective varieties over k, and $X = \cup_j X_j$ is the decomposition of X into its connected components, then the group of correspondences of degree m from X to Y is defined by the formula

$$CH^m(X, Y) = \oplus_j CH^{e_j+m}(X_j \times Y) \ ,$$

where e_j is the dimension of the component X_j. For example, given a regular morphism $f : X \to Y$ over k, the class of the transpose Γ_f^t of its graph Γ_f is in $CH^0(X, Y)$. For any two correspondences $f \in CH^n(X, Y)$ and $g \in CH^m(Y, Z)$ their composition $g \circ f$ is defined by the standard formula

$$g \circ f = p_{13*}(p_{12}^*(f) \cdot p_{23}^*(g)) \ ,$$

in which the central dot denotes the intersection of cycle classes in Fulton's sense, [4], and the projections are obvious. Notice that the composition $g \circ f$ belongs to the group $CH^{m+n}(Y,Z)$, so that the group $CH^0(Y,Z)$ is an associative algebra with respect to the above composition, and the class of the diagonal on a smooth projective variety X over k is the unit in this algebra. Objects in $\mathscr{CM}$ are triples (X,p,n) where p is a projector on X, i.e. an idempotent in the algebra $CH^0(X,X)$, and n is an integer. For two motives $M=(X,p,m)$ and $N=(Y,q,n)$, the Hom-group of morphisms from M to N is defined by the formula

$$\mathrm{Hom}(M,N) = q \circ CH^{n-m}(X,Y) \circ p\ .$$

Given a smooth projective variety X over k its motive $M(X)$ is defined by the diagonal Δ_X weighted by zero, $M(X)=(X,\Delta_X,0)$. For any morphism $f : X \to Y$ over k the transpose Γ_f^t represents a cycle class in $CH^0(X,X)$ denoted by $M(f)$. Then we have a contravariant functor $M : \mathscr{SP} \longrightarrow \mathscr{CM}$ from the category $\mathscr{SP}$ of smooth projective varieties over k to the category of Chow motives $\mathscr{CM}$, also over k.

The category $\mathscr{CM}$ is tensor,

$$(X,p,m)\otimes(Y,q,n) = (X\times Y, p\otimes q, m+n)\ ,$$

the motive

$$\mathbb{1} = M(\mathrm{Spec}(k)) = (\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)}, 0)$$

is the unit of this tensor product, the triple

$$\mathbb{L} = (\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)}, -1)$$

is called the Lefschetz motive, and its geometrical meaning is provided by the isomorphism $M(\mathbb{P}^1) = \mathbb{1}\oplus\mathbb{L}$. For any integer m we will write $\mathbb{L}^m$ for the m-fold tensor power $\mathbb{L}^{\otimes m}$. The duality in $\mathscr{CM}$ is defined by the formula

$$(X,p,m)^\vee = (X, p^{\mathrm{t}}, d-m)\ ,$$

where X is a smooth projective variety of pure dimension d over S. The category $\mathscr{CM}$ is rigid with respect to the above defined structures.

A degree m correspondence β from a smooth projective variety X to a smooth projective variety Y over k acts on the corresponding Chow groups,

$$\beta_* : CH^i(X) \longrightarrow CH^{i+m}(Y)\ ,$$

by the formula

$$\beta_*(\alpha) = (p_2)_*(p_1^*(\alpha)\cdot\beta)\ ,$$

for any $\alpha\in CH^i(X)$. Given a motive $M=(X,p,m)$ in $\mathscr{CM}$ its Chow groups are defined as

$$CH^i(M) = \mathrm{im}(p_* : CH^{i+m}(X) \to CH^{i+m}(X))\ .$$

Fixing a prime $l \neq \mathrm{char}(k)$ let

$$H^i(X)(j) = H^i_{\acute{e}t}(X, \mathbb{Q}_l(j))$$

be the l-adic étale cohomology group for a smooth projective variety X over k. Correspondences act also on cohomology, and cohomology groups of a Chow motive $M = (X, p, m)$ are defined by the formula

$$H^i(M)(j) = p_* H^{i+2m}(X)(j+m) \ .$$

Lemma 1. *Let $f : X \to Y$ be a surjective proper morphism of varieties over k. Then for any $p \geq 0$, the push-forward homomorphism*

$$f_* : CH_p(X) \longrightarrow CH_p(Y)$$

is surjective.

Proof. Use the fact that any proper surjective morphism of algebraic varieties admits a multisection. □

Lemma 2. *Let X, Y and Z be three smooth projective varieties over k, X is equi-dimensional, and let $f : Y \to Z$ be a regular morphism over k. Let α be an element in $CH^m(Y, X)$, i.e. a correspondence of degree m from Y to X. Consider the element $\gamma = (f \times \mathrm{id}_X)_*(\alpha)$ sitting in the group $CH^m(Z, X)$. Then the following diagram commutes:*

$$\begin{tikzcd} M(Z) \arrow[r, "M(f)"] \arrow[dr, "\gamma"'] & M(Y) \arrow[d, "\alpha"] \\ & M(X) \otimes \mathbb{L}^{-m} \end{tikzcd}$$

Proof. Let $g : Y \to Z \times Y$ be a morphism defined by the morphisms f and id_X, so that $[\Gamma_f^t] = g_*[Y]$. Let p_{12}, p_{23} and p_{13} are projections of the product $Z \times Y \times X$ onto the corresponding factors. Then we see that

$$p_{12}^*[\Gamma_f^t] = [\Gamma_f^t] \times X = g_*[Y] \times X = (g \times \mathrm{id}_X)_*[Y \times X] \ .$$

Therefore,

$$\begin{aligned} \alpha \circ [\Gamma_f^t] &= p_{13*}(p_{12}^*[\Gamma_f^t] \circ p_{23}^*(\alpha)) \\ &= p_{13*}((g \times \mathrm{id}_X)_*[Y \times X] \cdot p_{23}^*(\alpha)) \end{aligned}$$

By the projection formula:

$$\begin{aligned} \alpha \circ [\Gamma_f^t] &= p_{13*}((g \times \mathrm{id}_X)_*[Y \times X] \cdot p_{23}^*(\alpha)) \\ &= p_{13*}((g \times \mathrm{id}_X)_*((g \times \mathrm{id}_X)^* p_{23}^*(\alpha) \cdot [Y \times X])) \\ &= p_{13*}((g \times \mathrm{id}_X)_*(g \times \mathrm{id}_X)^* p_{23}^*(\alpha)) \\ &= p_{13*}((g \times \mathrm{id}_X)_*(p_{23} \circ (g \times \mathrm{id}_X))^*(\alpha)) \end{aligned}$$

But

$$p_{23} \circ (g \times \mathrm{id}_X) = \mathrm{id}_{Y \times X} \ ,$$

so that

$$\begin{aligned} \alpha \circ [\Gamma_f^t] &= p_{13*}((g \times \mathrm{id}_X)_*(p_{23} \circ (g \times \mathrm{id}_X))^*(\alpha)) \\ &= p_{13*}(g \times \mathrm{id}_X)_*(\alpha) \\ &= (p_{13} \circ (g \times \mathrm{id}_X))_*(\alpha) \end{aligned}$$

Since

$$p_{13} \circ (g \times \mathrm{id}_X) = g \times \mathrm{id}_X \ ,$$

we obtain:

$$\begin{aligned} \alpha \circ [\Gamma_f^t] &= (p_{13} \circ (g \times \mathrm{id}_X))_*(\alpha) \\ &= (g \times \mathrm{id}_X)_*(\alpha) = \gamma \ . \end{aligned}$$

□

Let X and Y be two equi-dimensional varieties over k. Following [1], we will say that a correspondence $\alpha \in CH^m(X,Y)$ is *balanced on the left* (respectively, *on the right*) if there exists an equi-dimensional Zariski closed subscheme $Z \subset X$ with $\dim(Z) < \dim(X)$ and an algebraic cycle Γ on $X \times Y$, such that $[\Gamma] = \alpha$ in $CH^m(X,Y)$ and the support of Γ is contained in $Z \times X$ (respectively, in $X \times Z$). Such Z will be called the *pans* of balancing. We say that a correspondence $\alpha \in CH^m(X,Y)$ is *balanced* if $\alpha = \alpha_1 + \alpha_2$, where α_1 is balanced on the left, and α_2 is balanced on the right.

Lemma 3. *Let X and S be equidimensional smooth projective varieties over k, and let $\alpha \in CH^m(S,X)$. Then α is balanced on the left if and only if there exists an equidimensional smooth projective variety Z over k with $\dim(Z) < \dim(S)$, such that α factors through $M(Z)$, that is α is a composition*

$$M(S) \longrightarrow M(Z) \longrightarrow M(X) \otimes \mathbb{L}^{-m} \ .$$

Proof. Suppose α is balanced on the left by the pan Z. Let $s : \tilde{Z} \to Z$ be the resolution of singularities on Z. Since $s \times \mathrm{id}_X$ is a surjective morphism, the induced homomorphism

$$(s \times \mathrm{id}_X)_* : CH_d(\tilde{Z} \times X) \longrightarrow CH_d(Z \times X)$$

is surjective too, where d is the dimension of X. Let $\tilde{\gamma}$ be a cycle class in $CH_d(\tilde{Z} \times X)$, such that $(s \times \mathrm{id}_X)_* \tilde{\gamma} = \gamma$. It exists by Lemma 1. Let $f = i \circ s$. Then

$$(f \times \mathrm{id}_X)_* \tilde{\gamma} = (i \times \mathrm{id}_X)_* (s \times \mathrm{id}_X)_* \tilde{\gamma} = (i \times \mathrm{id}_X)_* \gamma = \pi.$$

By Lemma 2, the following diagram is commutative:

$$\begin{tikzcd} M(S) \arrow[r, "M(f)"] \arrow[dr, "\alpha"'] & M(\tilde{Z}) \arrow[d, "\tilde{\gamma}"] \\ & M(X) \otimes \mathbb{L}^{-m} \end{tikzcd}$$

Let us prove the converse. Suppose α factors through the Chow motive $M(Z)$,

$$M(S) \xrightarrow{\beta} M(Z) \xrightarrow{\gamma} M(X) \otimes \mathbb{L}^{-m} \ ,$$

with Z as in the statement of the lemma. Let

$$W = \sum_i n_i W_i$$

be a cycle on $S \times Z$, such that $[W] = \beta$. Then $\dim(W_i) = \dim(Z)$ and, therefore, $\dim(p_S(W_i)) < \dim(S)$, where $p_S : S \times Z \to S$ is the natural projection. This shows that there exists a subvariety $T \subset S$ with $\dim(T) < \dim(S)$, such that the support of W is contained in $T \times Z \subset S \times Z$. Let $f : \widetilde{T} \to S$ be the composition of a resolution of singularities $\widetilde{T} \to T$ and the closed embedding $T \to S$.

Notice that the element

$$\beta = [W] \in CH^0(S, Z) = CH_{\dim(Z)}(S \times Z)$$

is the push-forward of an element in $CH_{\dim(Z)}(T \times Z)$ with respect to the closed embedding $T \times Z \to S \times Z$. In addition, the natural morphism $\widetilde{T} \times Z \to T \times Z$ is surjective and proper. Hence, by Lemma 1, there exists an element

$$\tilde{\beta} \in CH^0(\widetilde{T}, Z) = CH_{\dim(Z)}(\widetilde{T} \times Z) \ ,$$

such that $(f \times \mathrm{id}_Z)_*(\tilde{\beta}) = \beta$. By Lemma 2, we have that

$$\beta = \tilde{\beta} \circ M(f) \ .$$

Consequently,

$$\alpha = \gamma \circ \beta = (\gamma \circ \tilde{\beta}) \circ M(f) \ .$$

Again by Lemma 2, one has

$$(\gamma \circ \tilde{\beta}) \circ M(f) = (f \times \mathrm{id}_X)_*(\gamma \circ \tilde{\beta}) \ ,$$

where $\gamma \circ \tilde{\beta} \in CH^m(\widetilde{T}, X)$. Since all elements in the image of the push-forward map

$$(f \times \mathrm{id}_X)_* : CH^m(\widetilde{T}, X) \longrightarrow CH^m(S, X)$$

have representatives with support in $T \times X$, we obtain that $\alpha = (f \times \mathrm{id}_X)_*(\gamma \circ \tilde{\beta})$ is balanced on the left. □

Corollary 4. *Let X and S be equidimensional smooth projective varieties over k, and let $\alpha \in CH^m(S, X)$. Then α is balanced on the right if and only if there exists an equidimensional smooth projective variety Z over k with $n := \dim(X) - \dim(Z) > 0$ such that α factors through $M(Z) \otimes \mathbb{L}^{n-m}$, that is, α is a composition*

$$M(S) \longrightarrow M(Z) \otimes \mathbb{L}^{n-m} \longrightarrow M(X) \otimes \mathbb{L}^{-m} \ .$$

Proof. The corollary follows from Lemma 3 by taking transpositions of the involed correspondences. □

Remark 5. Certainly, one can now easily deduce a general result saying that α is balanced if and only if there exists equidimensional smooth projective varieties Z and T over k, such that α factors,

$$M(S) \longrightarrow M(T) \oplus (M(Z) \otimes \mathbb{L}^{n-m}) \longrightarrow M(X) \otimes \mathbb{L}^{-m} \ ,$$

and the difference $n = \dim(X) - \dim(Z)$ is greater than 0. But we will not use it in the paper.

3. Abel - Jacobi kernels

Here we recall some basic facts about the Abel-Jacobi maps and then prove some Lemma about specialization of correspondences lying in the Abel-Jacobi kernels.

Let X be an irreducible smooth projective variety over $\mathbb{C}$. The Betti cohomology groups $H^i(X,\mathbb{Q})$ carry a pure Hodge structure. It means that there is a decreasing Hodge filtration F^p on the $\mathbb{C}$-vector space

$$H^i(X,\mathbb{C}) \cong H^i(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} ,$$

compatible in a sense with the complex conjugate filtration $\bar{F}^p$. Let $H^{p,q}(X)$ be the adjoint quotient $(F^p/F^{p+1})H^{p+q}(X,\mathbb{C})$.

Consider two irreducible smooth projective complex algebraic varieties X and S and a correspondence $\alpha \in CH^m(S,X)$. The correspondence α defines a map

$$\alpha_* : H^i(S,\mathbb{Q}) \longrightarrow H^{i+2m}(X,\mathbb{Q}) .$$

and shifts the Hodge filtration by the formula

$$\alpha_*(F^p H^i(S,\mathbb{C})) \subset F^{p+m} H^{i+2m}(X,\mathbb{C}) .$$

Let us describe this in some more detail. Let

$$p_X : S \times X \to X \quad \text{and} \quad p_S : S \times X \to S$$

denote the natural projections. For any element $h \in F^p H^i(S,\mathbb{C})$, we have that

$$p_S^*(h) \in F^p H^i(S \times X,\mathbb{C}) ,$$

because pull-backs of elements in cohomology groups preserve the Hodge filtration, see [15, Section 7.3.2]. Let d be the dimension of the variety S. Since α is a correspondence of degree m from S to X, we obtain that

$$cl(\alpha) \in F^{d+m} H^{2d+2m}(S \times X) ,$$

see Proposition 11.20 in loc. cit. Therefore,

$$cl(\alpha) \cdot p_S^*(h) \in F^{d+m+p} H^{2d+2m+i}(S \times X,\mathbb{C}) .$$

It follows that

$$(p_X)_*(cl(\alpha) \cdot p_S^*(h)) \in F^{m+p} H^{2m+i}(X,\mathbb{C}) ,$$

as push-forwards of elements in cohomology groups shifts the Hodge filtration by the dimension of fibers, see Section 7.3.2 in Voisin's book [15].

Thus, the action α_* induces maps also on adjoint quotients:

$$\alpha_* : H^{p,q}(S) \longrightarrow H^{p+m,q+m}(X) .$$

For any $p \geq 0$ let $CH^p_{\mathbb{Z}}(X)_0$ be the kernel of the cycle class homomorphism

$$cl : CH^p_{\mathbb{Z}}(X) \longrightarrow H^{2p}(X,\mathbb{Z}) .$$

Extending coefficients from $\mathbb{Z}$ to $\mathbb{Q}$ we obtain a rational cycle class map

$$cl : CH^p(X) \longrightarrow H^{2p}(X, \mathbb{Q}) ,$$

whose kernel will be denoted by $CH^p(X)_0$.

A p-th intermediate Jacobian is a compact complex torus defined by the formula

$$J^{2p-1}(X) = H^{2p-1}(X, \mathbb{C})/(\mathrm{Im}(H^{2p-1}(X, \mathbb{Z})) + F^p H^{2p-1}(X, \mathbb{C})) ,$$

where $\mathrm{Im}(H^{2p-1}(X, \mathbb{Z}))$ is the image of the natural map

$$H^{2p-1}(X, \mathbb{Z}) \longrightarrow H^{2p-1}(X, \mathbb{C}) ,$$

see [15, Section 12.1.1]. There is a special group homomorphism

$$AJ : CH^p_{\mathbb{Z}}(X)_0 \longrightarrow J^{2p-1}(X) ,$$

called the Abel–Jacobi map, whose precise description can be found in Section 12.1.2 of [15]. We will also work with the group

$$J^{2p-1}(X)_{\mathbb{Q}} = J^{2p-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which can be also described as a quotient

$$J^{2p-1}(X)_{\mathbb{Q}} = H^{2p-1}(X, \mathbb{C})/((H^{2p-1}(X, \mathbb{Q}) + F^p H^{2p-1}(X, \mathbb{C})) .$$

Extending coefficients of the integral Abel-Jacobi map from $\mathbb{Z}$ to $\mathbb{Q}$, we obtain the rational Abel-Jacobi map

$$AJ : CH^p(X)_0 \longrightarrow J^{2p-1}(X)_{\mathbb{Q}} ,$$

whose kernel we denote by $T^p(X)$. Thus, an element $\alpha \in CH^p_{\mathbb{Z}}(X)$ is in $T^p(X)$ if and only if $cl(N\alpha) = 0$ for an integer N and $AJ(N\alpha)$ is in the torsion of the group $J^{2p-1}(X)$.

Notice that the action of correspondences on Chow groups is compatible with their actions on Hodge structures via the cycle class map and the Abel-Jacobi map.

Denote by $J^{2p-1}(X)_{\mathrm{alg}}$ the largest complex subtorus of $J^{2p-1}(X)$ whose tangent space is contained in $H^{p-1,p}(X)$, see [15, Section 12.2.2]. It follows $J^{2p-1}(X)_{\mathrm{alg}}$ is an abelian variety over $\mathbb{C}$, loc.cit. Denote by $CH^p(X)_{\mathrm{alg}}$ (respectively, $CH^p_{\mathbb{Z}}(X)_{\mathrm{alg}}$) the group of algebraically trivial cycles with rational (respectively, integral) coefficients modulo rational equivalence. Then $CH^p(X)_{\mathrm{alg}} \subset CH^p(X)_0$, i.e.

$$cl(CH^p(X)_{\mathrm{alg}}) = 0 .$$

It is not hard to show that

$$AJ(CH^p_{\mathbb{Z}}(X)_{\mathrm{alg}}) \subset J^{2p-1}(X)_{\mathrm{alg}}$$

and

$$AJ(CH^p(X)_{\mathrm{alg}}) \subset (J^{2p-1}(X)_{\mathrm{alg}})_{\mathbb{Q}} ,$$

loc.cit.

Let k be a field of characteristic zero, and let X be an irreducible smooth projective variety over k. An embedding $k \hookrightarrow \mathbb{C}$ defines the cohomology groups $H^i(X_{\mathbb{C}}, \mathbb{Z})$, and different embeddings give isomorphic complex algebraic varieties $X_{\mathbb{C}}$. This shows that for any element $\alpha \in CH^p(X)$ the conditions $cl(\alpha_{\mathbb{C}}) = 0$ and $AJ(\alpha_{\mathbb{C}}) = 0$ do not depend on the choice of the embedding. Thus, it is correct to write that $cl(\alpha) = 0$ or $AJ(\alpha) = 0$, and so the groups $CH^p(X)_0$ and $T^p(X)$ are well-defined.

From now on we assume that k is an algebraically closed field of characteristic zero.

Let X be an irreducible smooth projective variety over k, and let S be an irreducible variety over k. Given an element $\alpha \in CH^p(S \times X)$ and a schematic point $x \in S$, denote by $\alpha_x \in CH^p(X_{k(x)})$ the pull-back of the cycle class α with respect to the natural morphism

$$\mathrm{Spec}(k(x)) \times X \longrightarrow S \times X ,$$

where $k(x)$ is the residue field at x. Let η be the generic point of the scheme S, and let also

$$\Phi : CH^p(S \times X) \longrightarrow CH^p(X_{k(S)}), \quad \alpha \mapsto \alpha_{\eta}$$

be the pulling-back homomorphism induced by the morphism $\eta \times X \to S \times X$.

Let α be a cycle class in the group $CH^p(S \times X)$. By $\alpha_{\mathbb{C}} \in CH^p((S \times X)_{\mathbb{C}})$ we denote the extension of scalars for the class α from k to $\mathbb{C}$. That is, $\alpha_{\mathbb{C}}$ is the pull-back of α with respect to the natural flat morphism of schemes

$$(S \times X)_{\mathbb{C}} \to S \times X .$$

Notice that

$$(S \times X)_{\mathbb{C}} = S_{\mathbb{C}} \times X_{\mathbb{C}} ,$$

so that $\alpha_{\mathbb{C}}$ is a correspondence from $S_{\mathbb{C}}$ to $X_{\mathbb{C}}$.

Lemma 6. *For any element $\alpha \in CH^p(S \times X)$, we have that:*

(i) $cl(\Phi(\alpha)) = 0$ if and only if $cl((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$.

(ii) Assuming one of the equivalent conditions in (i), $AJ(\Phi(\alpha)) = 0$ if and only if $AJ((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$.

Proof. First let us prove (i). Any embedding $k(S) \hookrightarrow \mathbb{C}$ over k gives a closed point $\eta_{\mathbb{C}} \in S_{\mathbb{C}}$. Suppose that $cl((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$. In particular, we have

$$cl((\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}}) = 0 .$$

The extension of scalars satisfies

$$(\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}} = (\alpha_{\eta})_{\mathbb{C}} .$$

By definition, we have

$$(\alpha_{\eta})_{\mathbb{C}} = \Phi(\alpha)_{\mathbb{C}} .$$

This shows that $cl(\Phi(\alpha)_{\mathbb{C}}) = 0$, because $cl((\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}}) = 0$.

Now suppose that $cl(\Phi(\alpha)_{\mathbb{C}}) = 0$. Since

$$\Phi(\alpha)_{\mathbb{C}} = (\alpha_{\eta})_{\mathbb{C}} = (\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}} ,$$

we have

$$cl((\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}}) = 0 .$$

Further, for any closed point $x \in S_{\mathbb{C}}$ the elements

$$(\alpha_{\mathbb{C}})_x \in CH^p(X_{\mathbb{C}})$$

and

$$(\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}} \in CH^p(X_{\mathbb{C}})$$

are algebraically equivalent to each other. Therefore,

$$cl((\alpha_{\mathbb{C}})_x) = cl((\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}}) = 0 ,$$

which finishes the proof of (i).

Now let us prove (ii). Suppose that $cl((\alpha_{\mathbb{C}})_x) = 0$ and $AJ((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$. In particular, one has

$$AJ((\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}}) = 0 .$$

As above,

$$\Phi(\alpha)_{\mathbb{C}} = (\alpha_{\eta})_{\mathbb{C}} = (\alpha_{\mathbb{C}})_{\eta_{\mathbb{C}}} .$$

Therefore, $AJ(\Phi(\alpha)_{\mathbb{C}}) = 0$.

Thus, it remains only to show the implication "from the left to the right" in (ii). Namely, suppose that

$$AJ(\Phi(\alpha)) = AJ(\alpha_{\eta}) = 0 .$$

There exists an element $\beta \in CH^p_{\mathbb{Z}}(S \times X)$ such that $\beta = N\alpha$ for some integer N, and $AJ(\beta_{\eta}) = 0$. A similar argument as in the proof of (i) shows that for any point $x \in S_{\mathbb{C}}$ we have that

$$AJ((\beta_{\mathbb{C}})_x) \in J^{2p-1}(X_{\mathbb{C}})_{\mathrm{alg}} .$$

This gives a set-theoretic map

$$f : S_{\mathbb{C}} \longrightarrow J^{2p-1}(X_{\mathbb{C}})_{\mathrm{alg}}$$

sending any point $x \in S_{\mathbb{C}}$ to the Anel-Jacobi class $AJ((\beta_{\mathbb{C}})_x)$. This map f is holomorphic, see [15, Section 7.2.1].

Since f is a holomorphic map between projective complex algebraic varieties, f is an algebraic morphism between them, see [11, 4.14]. It follows that there exists a finitely generated field subextension

$$k \subset K \subset \mathbb{C} ,$$

an abelian variety A over K, and a morphism of varieties over K

$$g : S_K \longrightarrow A ,$$

such that $A_{\mathbb{C}} = J^{2p-1}(X)_{\mathrm{alg}}$ and $g_{\mathbb{C}} = f$.

Choose an embedding $K(S_K) \hookrightarrow \mathbb{C}$ over K. Automatically, it fixes embeddings

$$k(S) \hookrightarrow K(S_K) \hookrightarrow \mathbb{C} .$$

Let

$$\xi : \mathrm{Spec}(K(S_K)) \longrightarrow S_K$$

be the generic point of S_K. Consider the point

$$g \circ \xi : \mathrm{Spec}(K(S_K)) \longrightarrow A \ .$$

The extension of scalars satisfies

$$(g \circ \xi)_{\mathbb{C}} = g_{\mathbb{C}}(\xi_{\mathbb{C}}) \ .$$

As $g_{\mathbb{C}} = f$, we have that

$$g_{\mathbb{C}}(\xi_{\mathbb{C}}) = f(\xi_{\mathbb{C}}) \ .$$

Further, since $\xi = \eta_{K(S_K)}$, we obtain that $\xi_{\mathbb{C}} = \eta_{\mathbb{C}}$ and

$$f(\xi_{\mathbb{C}}) = f(\eta_{\mathbb{C}}) \ .$$

By the definition of f,

$$f(\eta_{\mathbb{C}}) = AJ((\beta_{\,\mathbb{C}})_{\eta_{\mathbb{C}}}) \ .$$

Finally, the extension of scalars gives

$$AJ((\beta_{\,\mathbb{C}})_{\eta_{\mathbb{C}}}) = AJ((\beta_{\,\eta})_{\mathbb{C}}) \ .$$

We conclude that

$$(g \circ \xi)_{\mathbb{C}} = AJ((\beta_{\,\eta})_{\mathbb{C}}) \ .$$

Since $AJ(\beta_{\,\eta}) = 0$, we have

$$g \circ \xi = 0 \ .$$

Thus the morphism of varieties over K

$$g : S_K \to A$$

sends the generic point ξ of S_K to the closed K-rational point of A which is the zero of the abelian variety A. Therefore, g sends S_K to the zero point of A, that is, $g(S_K) = 0$.

It gives, of course, that

$$f(S_{\mathbb{C}}) = 0 \ ,$$

because $f = g_{\mathbb{C}}$. By the definition of f, this means that

$$AJ((\beta_{\,\mathbb{C}})_x) = 0$$

for any point $x \in S_{\mathbb{C}}$. Since $\beta = N\alpha$, we see that

$$AJ((N\alpha_{\,\mathbb{C}})_x) = 0$$

for any point $x \in S_{\mathbb{C}}$. Therefore,

$$N \cdot AJ((\alpha_{\,\mathbb{C}})_x) = 0$$

for any point $x \in S_{\mathbb{C}}$.

Since the rational Abel-Jacobi map

$$AJ : CH^p(X_{\mathbb{C}}) \longrightarrow J^{2p-1}(X_{\mathbb{C}})_{\mathbb{Q}}$$

takes its values in the $\mathbb{Q}$-vector space $J^{2p-1}(X_{\mathbb{C}})_{\mathbb{Q}}$, we obtain that

$$AJ((\alpha_{\,\mathbb{C}})_x) = 0$$

for any point $x \in S_{\mathbb{C}}$. This finishes the proof of the lemma. □

Remark 7. A similar argument as in the proof of Lemma 6 shows that this lemma is also true for Chow groups with coefficients in $\mathbb{Z}$. Moreover, for an element $\alpha \in CH^p_{\mathbb{Z}}(S \times X)$ there is the following equivalence: $AJ((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$ if and only if $AJ(\alpha_x) = 0$ for any (k-rational) closed point $x \in S$.

Remark 8. One can also try to prove an analogue of Lemma 6 for any, i.e. not necessarily trivial, family of smooth projective varieties $\mathscr{X} \to S$, and for an element α in $CH^p(\mathscr{X})$. The implications "from the right to the left" in both (i) and (ii) remain to be true. The implication "from the left to the right" in (i) is true as well. The reason for that is that the map $x \mapsto cl(\alpha_x)$, $x \in S_{\mathbb{C}}$, is a section over $S_{\mathbb{C}}$ of the local system with fibers $H^{2p}((\mathscr{X}_{\mathbb{C}})_x, \mathbb{Q})$. Besides, if

$$AJ((\alpha_{\mathbb{C}})_x) \in (J^{2p-1}((\mathscr{X}_{\mathbb{C}})_x)_{\mathrm{alg}})_{\mathbb{Q}}$$

for any point $x \in S_{\mathbb{C}}$, then the implication "from the left to the right" in (ii) is true by a similar argument as in the proof of (ii).

4. Main result

The following two propositions can be deduced with the aid of Lemma 6. Implicitly they appear in [6] without a proof (see the first statement in Case 2 on p. 488 in loc.cit.).

Proposition 9. *Let X and S be two irreducible smooth projective varieties over k, let $d = \dim(S)$, and let α be an element in the group*

$$CH^p(S \times X) = CH^{p-d}(S, X) \ .$$

Then the following conditions are equivalent:

(i) $cl(\Phi(\alpha)) = 0$;
(ii) the image of the homomorphism $(\alpha_{\mathbb{C}})_* : CH^d(S_{\mathbb{C}}) \to CH^p(X_{\mathbb{C}})$ is contained in the group $CH^p(X_{\mathbb{C}})_0$;
(iii) the map $\alpha_* : H^{2d}(S_{\mathbb{C}}, \mathbb{Q}) \to H^{2p}(X_{\mathbb{C}}, \mathbb{Q})$ is trivial.

Proof.

$(i) \Leftrightarrow (ii)$

Suppose (i). Then, by Lemma 6 (i), for any closed point $x \in S_{\mathbb{C}}$ we have that $cl((\alpha_{\mathbb{C}})_x) = 0$. Since, moreover,

$$(\alpha_{\mathbb{C}})_*[x] = (\alpha_{\mathbb{C}})_x \ ,$$

we get

$$cl((\alpha_{\mathbb{C}})_*[x]) = 0 \ .$$

By the definition of the group $CH^p(X_{\mathbb{C}})_0$, we obtain that the element $(\alpha_{\mathbb{C}})_*[x]$ is in the group $CH^p(X_{\mathbb{C}})_0$. As the latter group is generated by the classes $[x]$, we get (ii).

Now suppose (ii). Then

$$cl((\alpha_{\mathbb{C}})_*[x]) = 0$$

for any closed point $x \in S_{\mathbb{C}}$. As above, $(\alpha_{\mathbb{C}})_*[x] = (\alpha_{\mathbb{C}})_x$. Then we see that $cl((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$. Applying Lemma 6 (i), we obtain (i).

$(ii) \Leftrightarrow (iii)$

Suppose (ii). Take any closed point $x \in S_{\mathbb{C}}$. As the action of the correspondence α on Chow groups commutes with the action of α on the cohomology groups via the cycle class map, we see that

$$\alpha_*(cl([x])) = cl(\alpha_*[x]) \ .$$

The image $\alpha_*(CH^d(S_{\mathbb{C}}))$ is contained in the group $CH^p(X_{\mathbb{C}})_0$, whence

$$\alpha_*(cl([x])) = 0 \ .$$

Since the element $cl([x])$ generates the one-dimensional $\mathbb{Q}$-vector space $H^{2d}(S_{\mathbb{C}}, \mathbb{Q})$, we get (iii).

Suppose (iii) and take any element β in $CH^d(S_{\mathbb{C}})$. As

$$cl(\alpha_*(\beta)) = \alpha_*(cl(\beta))$$

and the map

$$\alpha_* : H^{2d}(S_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^{2p}(X_{\mathbb{C}}, \mathbb{Q})$$

is zero, one has

$$cl(\alpha_*(\beta)) = 0$$

for any element $\beta \in CH^d(S_{\mathbb{C}})$. This gives (ii). $\square$

Proposition 10. *Let again X and S be two irreducible smooth projective varieties over k, $d = \dim(S)$, and let α be an element in the group $CH^p(S \times X) = CH^{p-d}(S, X)$. Assume one of the three equivalent conditions in Proposition 9. Then the following two conditions are equivalent:*

(i) $AJ(\Phi(\alpha)) = 0$;
(ii) the image of the homomorphism $(\alpha_{\mathbb{C}})_* : CH^d(S_{\mathbb{C}}) \to CH^p(X_{\mathbb{C}})$ is in $T^p(X_{\mathbb{C}})$;
(iii) the map $\alpha_* : H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}) \to H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q})$ is zero, and there is a closed point $x \in S_{\mathbb{C}}$, such that $AJ((\alpha_{\mathbb{C}})_x) = 0$.

Proof.

$(i) \Leftrightarrow (ii)$

Suppose (i). Then, by Lemma 6(ii), for any closed point $x \in S_{\mathbb{C}}$ we have that $AJ((\alpha_{\mathbb{C}})_x) = 0$. Since

$$(\alpha_{\mathbb{C}})_*[x] = (\alpha_{\mathbb{C}})_x$$

and $AJ((\alpha_{\mathbb{C}})_x) = 0$, we obtain

$$AJ((\alpha_{\mathbb{C}})_*[x]) = 0 \ .$$

By the definition of the group $T^p(X_{\mathbb{C}})$, the element $(\alpha_{\mathbb{C}})_*[x]$ sits in $T^p(X_{\mathbb{C}})$. Since the group $CH^d(S_{\mathbb{C}})$ is generated by the classes $[x]$, we arrive to (ii).

Now suppose (ii). Then

$$AJ((\alpha_{\mathbb{C}})_*[x]) = 0$$

for any closed point $x \in S_{\mathbb{C}}$. Since $(\alpha_{\mathbb{C}})_*[x] = (\alpha_{\mathbb{C}})_x$, we see that $AJ((\alpha_{\mathbb{C}})_x) = 0$ for any closed point $x \in S_{\mathbb{C}}$. Applying Lemma 6 (ii), we get (i).

$(ii) \Leftrightarrow (iii)$

Assuming (ii), for any closed point $x \in S_{\mathbb{C}}$, we have

$$AJ((\alpha_{\mathbb{C}})_x) = AJ((\alpha_{\mathbb{C}})_*[x]) = 0 \ .$$

So, to deduce (iii) we need only to show that the map

$$\alpha_* : H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q})$$

is zero. Let

$$\gamma \in CH^d(S_{\mathbb{C}})_0$$

be a zero-cycle class of degree zero. We have that

$$AJ((\alpha_{\mathbb{C}})_*(\gamma)) = (\alpha_{\mathbb{C}})_*(AJ(\gamma)) \ .$$

As the element $(\alpha_{\mathbb{C}})_*(\gamma)$ sits in $T^p(X_{\mathbb{C}})$, the element $AJ((\alpha_{\mathbb{C}})_*(\gamma))$ vanishes, so that

$$(\alpha_{\mathbb{C}})_*(AJ(\gamma)) = 0 \ .$$

Since the Albanese map

$$AJ : CH^d(S_{\mathbb{C}})_0 \to J^{2d-1}(S_{\mathbb{C}})_{\mathbb{Q}} = \mathrm{Alb}(S_{\mathbb{C}})_{\mathbb{Q}}$$

is surjective and $(\alpha_{\mathbb{C}})_*(AJ(\gamma)) = 0$, we have that the map

$$(\alpha_{\mathbb{C}})_* : J^{2d-1}(S_{\mathbb{C}})_{\mathbb{Q}} \to J^{2p-1}(X_{\mathbb{C}})_{\mathbb{Q}}$$

is zero.

Let $\beta \in CH^p_{\mathbb{Z}}(S \times X)$ be such that $\beta = N\alpha$ for some non-zero integer N. The map

$$(\beta_{\mathbb{C}})_* : J^{2e-1}(S_{\mathbb{C}}) \to J^{2p-1}(X_{\mathbb{C}})$$

satisfies

$$(\beta_{\mathbb{C}})_* \otimes_{\mathbb{Z}} \mathbb{Q} = N(\alpha_{\mathbb{C}})_* \ .$$

Therefore, we obtain that $(\beta_{\mathbb{C}})_* \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. This means that the image of $(\beta_{\mathbb{C}})_*$ is contained in the torsion of the group $J^{2p-1}(X_{\mathbb{C}})$. Since $(\beta_{\mathbb{C}})_*$ is a continuous map between complex compact tori, the image of $(\beta_{\mathbb{C}})_*$ is compact. So, the subgroup $(\beta_{\mathbb{C}})_*(J^{2d-1}(S_{\mathbb{C}}))$ in $J^{2p-1}(X_{\mathbb{C}})$ is both compact and contained in the torsion. This implies that the group $(\beta_{\mathbb{C}})_*(J^{2d-1}(S_{\mathbb{C}}))$ is finite, because $J^{2p-1}(X_{\mathbb{C}})$ is a compact complex tori. Let M be the order of the finite group $(\beta_{\mathbb{C}})_*(J^{2d-1}(S_{\mathbb{C}}))$. The map

$$M \cdot (\beta_{\mathbb{C}}) : J^{2d-1}(S_{\mathbb{C}}) \to J^{2p-1}(X_{\mathbb{C}})$$

vanishes. Therefore the induced map on first homology groups with rational coefficients

$$H_1(J^{2d-1}(S_{\mathbb{C}}), \mathbb{Q}) \to H_1(J^{2p-1}(X_{\mathbb{C}}), \mathbb{Q})$$

also vanishes. Notice that by the construction of the intermediate Jacobians, we have identifications

$$H_1(J^{2d-1}(S_{\mathbb{C}}), \mathbb{Q}) = H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}),$$

$$H_1(J^{2p-1}(X_{\mathbb{C}}), \mathbb{Q}) = H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}).$$

Moreover, the action on first homology groups induced by $M \cdot (\beta_{\mathbb{C}})_*$ agrees with the map

$$M \cdot (\beta_{\mathbb{C}})_* : H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}) \to H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}) \ .$$

This implies the vanishing of the map

$$M \cdot (\beta_{\mathbb{C}})_* : H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}) \to H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}) \ .$$

Since $M \cdot (\beta_{\mathbb{C}})_* = \frac{M}{N}(\alpha_{\mathbb{C}})_*$, the map

$$(\alpha_{\mathbb{C}})_* : H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}) \to H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q})$$

also vanishes. This gives (iii).

Now suppose (iii). By the construction of the groups $J^{2d-1}(S_{\mathbb{C}})_{\mathbb{Q}}$ and $J^{2p-1}(X_{\mathbb{C}})_{\mathbb{Q}}$, the vanishing of the map $(\alpha_{\mathbb{C}})_* : H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q}) \to H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q})$ implies the vanishing of the map

$$(\alpha_{\mathbb{C}})_* : J^{2d-1}(S_{\mathbb{C}})_{\mathbb{Q}} \to J^{2p-1}(X_{\mathbb{C}})_{\mathbb{Q}} \ .$$

Therefore, for any zero-cycle of degree zero $\beta \in CH^d(S_{\mathbb{C}})_0$, we have

$$(\alpha_{\mathbb{C}})_*(AJ(\beta)) = 0 \ .$$

Since $AJ((\alpha_{\mathbb{C}})_*\beta) = (\alpha_{\mathbb{C}})_*(AJ(\beta))$, we obtain the vanishing of $AJ((\alpha_{\mathbb{C}})_*\beta)$. Therefore, the image of the map

$$(\alpha_{\mathbb{C}})_* : CH^d(S_{\mathbb{C}})_0 \to CH^p(X_{\mathbb{C}})$$

is contained in $T^p(X_{\mathbb{C}})$. Further, take the closed point $x \in S_{\mathbb{C}}$ as in (iii). Since $AJ((\alpha_{\mathbb{C}})_x) = 0$ and $(\alpha_{\mathbb{C}})_*[x] = (\alpha_{\mathbb{C}})_x$, we have

$$(\alpha_{\mathbb{C}})_*[x] \in T^p(X_{\mathbb{C}}) \ .$$

Since the element $[x]$ and the subgroup $CH^d(S_{\mathbb{C}})_0$ in $CH^d(S_{\mathbb{C}})$ generate together the group $CH^d(S_{\mathbb{C}})$, we obtain (ii). □

Now let us give a sufficient condition for $\Phi(\alpha)$ to be non-zero. It is equivalent to non-vanishing of some Hodge-theoretic invariants constructed in [6]. More precisely, the condition in Theorem 11 below is equivalent to the vanishing of invariants denoted by $[\mathscr{Z}]_m$ and defined on p.483 in loc.cit.

Theorem 11. *Let X and S be smooth projective irreducible varieties over k, let d be the dimension of S, and let α be an element in the group*

$$CH^p(S \times X) = CH^{p-d}(S, X) \ .$$

Suppose that there exists i, such that the map

$$\alpha_* : H^{i,d}(S_{\mathbb{C}}) \longrightarrow H^{i+p-d,p}(X_{\mathbb{C}})$$

is non-zero. Then the cycle class $\Phi(\alpha)$ is non-zero in $CH^p(X_{k(S)})$.

Proof. Suppose $\Phi(\alpha) = 0$. It is equivalent to say that α is balanced on the left, see [8]. By Lemma 3, there exists an equidimensional smooth projective variety Z over k with $\dim(Z) < d$, such that the morphism of motives

$$\alpha_* : M(S) \longrightarrow M(X) \otimes \mathbb{L}^{\otimes(d-p)}$$

is equal to a composition

$$M(S) \to M(Z) \to M(X) \otimes \mathbb{L}^{\otimes(d-p)} .$$

Therefore, the non-zero map

$$\alpha_* : H^{i,e}(S_{\mathbb{C}}) \longrightarrow H^{i+p-e,p}(X_{\mathbb{C}})$$

factors through the group $H^{i,e}(Z_{\mathbb{C}})$. As $\dim(Z) < d$, the latter group is trivial, which is in contradiction with the non-triviality of the map α_*. □

Let S be a smooth projective irreducible variety S over k of dimension d. Recall that, by [12], there exist the Picard and the Albanese projectors of S generating, respectively, the Picard motive $M^1(S)$ and its dual motive $M^{2d-1}(S)$ of the variety S. Denote by $\pi_{2d-1} \in CH^0(S,S)$ the projector that corresponds to the Chow motive $M^{2d-1}(S)$. Also, denote by $\pi_{2d} \in CH^0(S,S)$ the projector that corresponds to a direct summand $\mathbb{L}^{\otimes d}$ in $M(S)$. We put

$$\tau_S := [\Delta_S] - \pi_{2d-1} - \pi_{2d} ,$$

where $[\Delta_S]$ is the class of the diagonal in the group $CH^0(S,S)$. The correspondence τ_S acts as zero on the groups $H^{2d}(S_{\mathbb{C}}, \mathbb{Q})$ and $H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q})$, and it acts identically on the groups $H^i(S_{\mathbb{C}}, \mathbb{Q})$ with $i < 2d-1$.

Our main result in in this paper is as follows:

Theorem 12. *Let X and S be two irreducible smooth projective varieties over k, $d = \dim(S)$, and let α be an element in*

$$CH^p(S \times X) = CH^{p-d}(S,X) .$$

Suppose that there exists $i \leq d-2$, such that the induced map

$$\alpha_* : H^{i,d}(S_{\mathbb{C}}) \longrightarrow H^{i+p-d,p}(X_{\mathbb{C}})$$

is non-zero. Then, for any closed point $x \in S$, the cycle class

$$\Psi_x(\alpha) = \Phi(\alpha \circ \tau_S) - (\alpha \circ \tau_S)_x$$

is a non-zero element in the Abel–Jacobi kernel $T^p(X_{k(S)})$.

Proof. By the construction of τ_S, the correspondence $\alpha \circ \tau_S$ acts as zero on the group $H^{2d}(S_{\mathbb{C}}, \mathbb{Q})$. Thus, it satisfies the condition in Proposition 9 (iii). Therefore, by that corollary, we obtain that

$$(\alpha \circ \tau_S)_*(CH^d(S_{\mathbb{C}})) \subset CH^p(X_{\mathbb{C}})_0 .$$

In particular,

$$cl((\alpha \circ \tau_S)_*[x]) = 0 ,$$

and therefore

$$cl((\alpha \circ \tau_S)_x) = 0 .$$

The correspondence $((\alpha \circ \tau_S)_x \times [S])$ acts as zero on the all groups $H^i(S_{\mathbb{C}}, \mathbb{Q})$ with $i \neq 2d$. Moreover, for any element

$$\beta \in H^{2d}(S_{\mathbb{C}}, \mathbb{Q})$$

we have that

$$((\alpha \circ \tau_S)_x \times [S])_*(\beta) = cl((\alpha \circ \tau_S)_x) \cdot \deg(\beta) = 0 \ ,$$

where

$$\deg : H^{2d}(S_{\mathbb{C}}, \mathbb{Q}) \longrightarrow \mathbb{Q}$$

is the natural isomorphism. Thus, the correspondence $(\alpha \circ \tau_S)_x \times [S]$ acts as zero on all the cohomology groups $H^i(S_{\mathbb{C}}, \mathbb{Q})$.

Next, the correspondence $\alpha \circ \tau_S$ also acts as zero on the group $H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q})$. Consequently, the correspondence

$$\gamma := \alpha \circ \tau_S - (\alpha \circ \tau_S)_x \times [S]$$

acts as zero on the groups $H^{2d}(S_{\mathbb{C}}, \mathbb{Q})$ and $H^{2d-1}(S_{\mathbb{C}}, \mathbb{Q})$.

We have that

$$\gamma_*[x] = 0 \ .$$

Therefore, the correspondence γ satisfies the conditions of Proposition 9 (iii) and Proposition 10 (iii).

Also, by definition, we have that

$$\Phi(\gamma) = \Psi_x(\alpha) \ .$$

Therefore, by Proposition 9 and Proposition 10 we obtain that

$$cl(\Psi_x(\alpha)) = 0, \quad \text{and} \quad AJ(\Psi_x(\alpha)) = 0 \ ,$$

that is, $\Psi_x(\alpha)$ is in $T^p(X_{k(S)})$.

Moreover, the action of the correspondence γ on $H^i(S_{\mathbb{C}}, \mathbb{Q})$ with $i \leq 2d-2$ coincides with the action of the correspondence α. Since α acts non-trivially on $H^i(S_{\mathbb{C}}, \mathbb{Q})$ for some $i \leq 2d-2$, we see that γ acts non-trivially on $H^i(S_{\mathbb{C}}, \mathbb{Q})$. By Theorem 11, $\Psi_x(\alpha)$ is non-trivial. This completes the proof of the theorem. □

Remark 13. Under the notation of Theorem 12, consider the class

$$cl(\alpha) = \sum_{r+s=2p} cl(\alpha)_{r,s} \in H^{2p}((S \times X)_{\mathbb{C}}, \mathbb{C}) \cong \bigoplus_{r+s=2p} H^s(S_{\mathbb{C}}, \mathbb{C}) \otimes_{\mathbb{C}} H^r(X_{\mathbb{C}}, \mathbb{C}) \ .$$

The Hodge theory implies that the element α satisfies the condition of Theorem 12 if and only if the image of the class $cl(\alpha)_{2p+i-d,d-i}$ under the projection

$$H^{d-i}(S_{\mathbb{C}}, \mathbb{C}) \otimes_{\mathbb{C}} H^{2p+i-d}(X_{\mathbb{C}}, \mathbb{C}) \to H^{d-i,0}(S_{\mathbb{C}}) \otimes_{\mathbb{C}} H^{i+p-d,p}(X_{\mathbb{C}})$$

does not vanish.

5. Some examples

In this section we construct algebraic cycle classes in the Abel-Jacobi kernels for $K3$ surfaces and then for threefolds and higher dimensional varieties with known Lefschetz conjecture.

Let first X be a smooth projective surface over k, let $S = X$ and let $m = 0$. Then a correspondence $\alpha \in CH^2(X \times X) = CH^0(X, X)$ satisfies the condition of Theorem 12 if and only if the map

$$\alpha_* : H^{0,2}(X) \longrightarrow H^{0,2}(X)$$

is non-zero. Suppose, for example, X is a $K3$ surface with an algebraic symplectomorphism $s : X \to X$ on it. Without loss of generality one can assume that k is big enough that s is defined over k. Take as α the graph of s.

Now let X be a smooth projective irreducible variety over k of dimension d, such that for some $p \geq 2$ the group $H^{p,0}(X_{\mathbb{C}})$ is non-trivial and the inverse of the Lefschetz operator

$$L : H^p(X_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^{2d-p}(X_{\mathbb{C}}, \mathbb{Q})$$

is represented by an algebraic correspondence

$$\alpha \in CH^p(X \times X) = CH^{p-d}(X, X) \ .$$

Let Y be an intersection of general $d - p$ hyperplane sections of X, and let

$$i : Y \hookrightarrow X$$

be the corresponding closed embedding. Then one has a morphism of Chow motives over the field k,

$$\beta = \alpha \circ M(i)^{\mathrm{t}} : M(Y) \longrightarrow M(X) \ ,$$

where $M(i)^{\mathrm{t}}$ is the transposition of the graph of the embedding i. Notice that β is an element in the group $CH^p(Y \times X) = CH^0(Y, X)$.

Under the above notation, the elements α and β satisfy the condition of Theorem 12 because, by the Lefschetz theorem, the maps

$$(\alpha_{\mathbb{C}})_* : H^{d-p,d}(X_{\mathbb{C}}) \longrightarrow H^{0,p}(X_{\mathbb{C}}) \ ,$$

$$(\beta_{\mathbb{C}})_* : H^{0,p}(S_{\mathbb{C}}) \longrightarrow H^{0,p}(X_{\mathbb{C}})$$

are isomorphisms.

Thus, taking two points $x \in X$ and $y \in Y$, we obtain two non-zero elements

$$\Psi_x(\alpha) \in T^p(X_{k(X)})$$

and

$$\Psi_y(\beta) \in T^p(X_{k(Y)}) \ .$$

Let us consider in some more detail the case when $p = 2$ and $d = 3$. The existence of α as above is equivalent to say that the Lefschetz conjecture holds for X, which implies the existence of a Künneth decomposition for the homological motive of X. Assuming finite-dimensionality of Chow motive $M(X)$ or, equivalently, that the homological realization functor is conservative, we obtain a Chow-Künneth decomposition for the motive of X, see [7]:

$$M(X) \cong \oplus_{i=0}^{6} M^i(X) \ .$$

As Y is a surface, it also has a Chow-Künneth decomposition for its motive $M(Y)$. The morphism

$$\beta : M(Y) \longrightarrow M(X)$$

induces the morphisms

$$\beta : M^i(Y) \to M^i(X), \quad i \geq 0 \ ,$$

and $M^2(X)$ is a direct summand of the middle motive $M^2(Y)$. According to the conjectural formulas for the Bloch-Beilinson filtration, for any field extension $k \subset K$, we have that

$$T^2(X_K) = CH^2(M^2(X)_K)$$

(possibly, this also follows from finite-dimensionality).

Thus, the big conjectures applied to X imply the following statement: the map

$$(\beta_K)_* : T^2(Y_K) \longrightarrow T^2(X_K)$$

is surjective.

One may wish to ask the following question: is it possible to prove the above statement unconditionally?

Notice that, by construction, the element $\Psi_y(\beta)$ belongs to the image of the map

$$(\beta_{k(Y)})_* : T^2(Y_{k(Y)}) \longrightarrow T^2(X_{k(Y)}) \ .$$

A more particular question: is it possible to prove unconditionally that the element $\Psi_x(\alpha)$ belongs to the image of the map

$$(\beta_{k(X)})_* : T^2(Y_{k(X)}) \longrightarrow T^2(X_{k(X)})\,?$$

The above questions can be tested on a wide range of examples of threefolds that satisfy the Lefschetz conjecture.

6. Main example

Let X be an irreducible smooth projective variety over k and let A be the Picard variety $\mathrm{Pic}^0(X)$ of the variety X. Fix a closed point $x_0 \in X$ and consider the associated Poincaré line bundle $\mathscr{P}$ over $A \times X$. Recall that $\mathscr{P}$ is the unique line bundle on $A \times X$, such that for any point $a \in A$ the isomorphism class of the line bundle on X

$$\mathscr{P}|_{a\times X}$$

is equal to a, and

$$\mathscr{P}|_{A\times x_0} \cong \mathscr{O}_A \ .$$

Let

$$\beta \in CH^1(A \times X)$$

be the first Chern class of $\mathscr{P}$, and let

$$\alpha = \beta^p$$

be the p-fold self-intersection of the class β in $CH^p(A \times X)$ with $p \geq 2$.

The following fact is classical:

Lemma 14. *Let g be the dimension of the Picard variety A. The correspondence*

$$\beta \in CH^1(A \times X) = CH^{1-g}(A, X)$$

induces an map

$$(\beta_{\mathbb{C}})_* : H^{2g-1}(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^1(X_{\mathbb{C}}, \mathbb{Q}) \ ,$$

and this map is an isomorphism.

Proof. Since $H^*((-)_{\mathbb{C}}, \mathbb{Q})$ is a Weil cohomology theory, the lemma can be deduced by the same method as in [10]. $\square$

Theorem 15. *Under the above notation, the element $\Phi(\alpha)$ belongs to $T^p(X_{k(A)})$. Moreover, if there exist p differential forms*

$$\omega_1, \ldots, \omega_p \in H^0(X_{\mathbb{C}}, \Omega^1_{X_{\mathbb{C}}}) = H^{1,0}(X_{\mathbb{C}}) \ ,$$

such that

$$\omega_1 \wedge \ldots \wedge \omega_p \neq 0 \ ,$$

then

$$\Phi(\alpha) \neq 0$$

in $T^p(X_{k(A)})$.

Proof. By the construction of the cycle class β, for any closed point $a \in A_{\mathbb{C}}$ the element $(\beta_{\mathbb{C}})_a$ is equal, in the group $CH^1(X_{\mathbb{C}})$, to the first Chern class of the line bundle $\mathscr{P}|_{X\times a}$. Hence, the class $cl((\beta_{\mathbb{C}})_a)$ is equal to the first Chern class of $\mathscr{P}|_{X\times a}$ with value in the second Betti cohomology group $H^2(X_{\mathbb{C}}, \mathbb{Q})$. By the definition of the Poincaré line bundle, for any $a \in \mathrm{Pic}^0(X_{\mathbb{C}})$ the isomorphism class of $\mathscr{P}|_{X\times a}$ is exactly a. Remind also that elements in $\mathrm{Pic}^0(X_{\mathbb{C}})$ are isomorphism classes of line bundles with a trivial first Chern class with value in the second Betti cohomology group $H^2(X_{\mathbb{C}}, \mathbb{Q})$. Therefore, we have that

$$cl((\beta_{\mathbb{C}})_a) = 0$$

for any closed point $a \in A_{\mathbb{C}}$.

Then, by Lemma 6(i), we have that

$$cl(\Phi(\beta)) = 0 \ ,$$

i.e.

$$\Phi(\beta) \in CH^1(X_{k(A)})_0 \ .$$

Since

$$cl(\Phi(\alpha)_{\mathbb{C}}) = cl(\Phi(\beta)_{\mathbb{C}})^p \ ,$$

we have that

$$cl(\Phi(\alpha)) = 0 \ ,$$

so that one can apply the Abel-Jacobi map AJ to the cycle class $\Phi(\alpha)_{\mathbb{C}}$.

Applying AJ we obtain:

$$AJ(\Phi(\alpha)_{\mathbb{C}}) = AJ(\Phi(\beta)^{p-1}_{\mathbb{C}} \cdot \Phi(\beta)_{\mathbb{C}}) \ .$$

Then

$$AJ(\Phi(\alpha)_{\mathbb{C}}) = cl(\Phi(\beta)^{p-1}_{\mathbb{C}}) \cdot AJ(\Phi(\beta)_{\mathbb{C}})$$

by Proposition 9.23 in [15].

Since $p \geq 2$, and $cl(\Phi(\beta)) = 0$, cycle class map vanishes at $\Phi(\beta)^{p-1}_{\mathbb{C}}$, i.e.

$$cl(\Phi(\beta)^{p-1}_{\mathbb{C}}) = 0 \ .$$

Therefore,

$$AJ(\Phi(\alpha)_{\mathbb{C}}) = 0 \ .$$

Thus,

$$\Phi(\alpha) \in T^p(X) .$$

The rest of the proof of Theorem 15 is devoted to showing that α satisfies the condition of Theorem 11, which will allow us to prove that $\Phi(\alpha)$ is non-zero.

With this aim we first describe the action of β on the cohomology groups of $A_{\mathbb{C}}$. Then we use the interplay between elements in tensor products of vector spaces and linear operators via duality (in our case this will be the Poincaré duality on cohomology). As a consequence we will find the shape of the Künneth components of the class $cl(\beta_{\mathbb{C}})$ in the cohomology groups of the product $A \times X$. Taking products in cohomology gives Künneth components for the cycle class $\alpha = \beta^p$. This will allow to describes explicitly the action of α on $H^{2g-p}(A_{\mathbb{C}}, \mathbb{Q})$. Then, using the fact that A is an abelian variety, we will construct an explicit element in $H^{g,g-p}(A_{\mathbb{C}})$ with a non-trivial image under the map $(\alpha_{\mathbb{C}})_*$. Finally, we will use Theorem 11 in order to show that $\Phi(\alpha)$ is non-zero.

So, let us implement this program.

For each index i the correspondence β, being an element in $CH^1(A \times X) = CH^{1-g}(A, X)$, gives maps on cohomology groups

$$(\beta_{\mathbb{C}})_{*,i} : H^i(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^{i+2-2g}(X_{\mathbb{C}}, \mathbb{Q}) .$$

We want to describe the maps $(\beta_{\mathbb{C}})_{*,i}$.

Suppose that $(\beta_{\mathbb{C}})_{*,i} \neq 0$. Since the source and the target of $(\beta_{\mathbb{C}})_{*,i}$ need to be non-zero and $\dim(A) = g$, we have that $i \leq 2g$ and $i + 2 - 2g \geq 0$, that is, $2g - 2 \leq i \leq 2g$.

We have shown already that $cl((\beta_{\mathbb{C}})_a) = 0$ for any point a in $A_{\mathbb{C}}$. Hence, by Proposition 9, the map

$$(\beta_{\mathbb{C}})_{*,2g} : H^{2g}(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^2(X_{\mathbb{C}}, \mathbb{Q})$$

vanishes.

By Lemma 14, the map

$$(\beta_{\mathbb{C}})_{*,2g-1} : H^{2g-1}(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^1(X_{\mathbb{C}}, \mathbb{Q})$$

is an isomorphism.

Consider the map

$$(\beta_{\mathbb{C}})_{*,2g-2} : H^{2g-2}(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^0(X_{\mathbb{C}}, \mathbb{Q}) .$$

We claim that this map is zero. Let us give a detailed proof of this fact.

The group in the target is canonically isomorphic to $\mathbb{Q}$. Let

$$i_x : x \longrightarrow X_{\mathbb{C}}$$

be a closed point on $X_{\mathbb{C}}$. Then the pull-back

$$i_x^* : H^0(X_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^0(x, \mathbb{Q})$$

is an isomorphism. Thus, as soon as $i_x^*(\beta_{\mathbb{C}})_*(\gamma) = 0$ for any element $\gamma \in H^{2g-2}(A_{\mathbb{C}}, \mathbb{Q})$ we get that $(\beta_{\mathbb{C}})_*(\gamma) = 0$ for any element $\gamma \in H^{2g-2}(A_{\mathbb{C}}, \mathbb{Q})$, that is, $(\beta_{\mathbb{C}})_{*,2g-2} = 0$.

We are going to prove that $i_x^*(\beta_{\mathbb{C}})_*(\gamma) = 0$ for any element $\gamma \in H^{2g-2}(A_{\mathbb{C}}, \mathbb{Q})$. By the definition of the action of correspondences, we have that

$$i_x^*(\beta_{\mathbb{C}})_*(\gamma) = i_x^*(p_X)_*(p_A^*\gamma \cdot cl(\beta_{\mathbb{C}})) \ .$$

Consider the Cartesian square

$$\begin{tikzcd}
A \times x \arrow[r, "\mathrm{id}_A \times i_x"] \arrow[d, "p_x"'] & A \times X \arrow[d, "p_X"] \\
x \arrow[r, "i_x"] & X
\end{tikzcd}$$

where $p_x : A \times x \to x$ is the natural projection onto the point. Since pull-backs and push-forwards for Chow groups commute in Cartesian squares, we see that

$$i_x^*(p_X)_*(p_A^*\gamma \cdot cl(\beta_{\mathbb{C}})) = (p_x)_*(\mathrm{id}_A \times i_x)^*(p_A^*\gamma \cdot cl(\beta_{\mathbb{C}})) \ .$$

As pull-backs commute with products in cohomology, we deduce:

$$(p_x)_*(\mathrm{id}_A \times i_x)^*(p_A^*\gamma \cdot cl(\beta_{\mathbb{C}})) = (p_x)_*((\mathrm{id}_A \times i_x)^*(p_A^*\gamma) \cdot (\mathrm{id}_A \times i_x)^*(cl(\beta_{\mathbb{C}}))) \ .$$

Notice that

$$(\mathrm{id}_A \times i_x)^*(p_A^*\gamma) = \gamma$$

after the identification of $A \times x$ with A. Therefore,

$$(p_x)_*((\mathrm{id}_A \times i_x)^*(p_A^*\gamma) \cdot (\mathrm{id}_A \times i_x)^*(cl(\beta_{\mathbb{C}}))) = (p_x)_*(\gamma \cdot (\mathrm{id}_A \times i_x)^*(cl(\beta_{\mathbb{C}}))) \ .$$

Since the cycle class map commutes with pull-backs, we also have

$$(p_x)_*(\gamma \cdot (\mathrm{id}_A \times i_x)^*(cl(\beta_{\mathbb{C}}))) = (p_x)_*(\gamma \cdot cl((\mathrm{id}_A \times i_x)^*\beta_{\mathbb{C}})) \ .$$

Next, the element $(\mathrm{id}_A \times i_x)^*(\beta_{,\mathbb{C}})$ is equal to $(\beta^{\mathrm{t}}_{\mathbb{C}})_x$, where β^{t} is the transpose of the correspondence β. In particular, β^{t} is an element in in the Chow group $CH^{1-\dim(X)}(X, A)$.

Thus, we have shown that

$$i_x^*(\beta_{\mathbb{C}})_*(\gamma) = (p_x)_*(\gamma \cdot cl((\beta^{\mathrm{t}}_{\mathbb{C}})_x)) \ .$$

As above, $(\beta^{\mathrm{t}}_{\mathbb{C}})_{x_0}$ is the first Chern class of the line bundle $\mathscr{P}|_{A \times x_0}$, where $x_0 \in X$ is the closed point that has been fixed above for the choice of the Poincaré line bundle. By definition of $\mathscr{P}$, the line bundle $\mathscr{P}|_{A \times x_0}$ is trivial. Hence,

$$(\beta^{\mathrm{t}}_{\mathbb{C}})_{x_0} = 0$$

and, in particular,

$$cl((\beta^{\mathrm{t}}_{\mathbb{C}})_{x_0}) = 0 \ .$$

Notice that the cycles $(\beta^{\mathrm{t}}_{\mathbb{C}})_x$ and $(\beta^{\mathrm{t}}_{\mathbb{C}})_{x_0}$ are algebraically equivalent. Therefore,

$$cl((\beta^{\mathrm{t}}_{\mathbb{C}})_x) = 0 \ .$$

Since

$$i_x^*(\beta_{\mathbb{C}})_*(\gamma) = (p_x)_*(\gamma \cdot cl((\beta^{\mathrm{t}}_{\mathbb{C}})_x)) \ ,$$

we get the vanishing

$$i_x^*(\beta_{\mathbb{C}})_*(\gamma) = 0 ,$$

which implies the vanishing

$$(\beta_{\mathbb{C}})_{*,2g-2} = 0 ,$$

as explained above.

Thus, we obtained a description of the action of β on all the cohomology groups of $A_{\mathbb{C}}$. We see that the only non-zero action is

$$(\beta_{\mathbb{C}})_* : H^{2g-1}(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^1(X_{\mathbb{C}}, \mathbb{Q}) ,$$

and this map is an isomorphism.

Now consider the Künneth components of $cl(\beta_{\mathbb{C}})$ in the cohomology group $H^*(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X_{\mathbb{C}}, \mathbb{Q})$:

$$cl(\beta_{\mathbb{C}}) = \sum_{i+j=2} cl(\beta_{\mathbb{C}})_{i,j} \in \oplus_{i+j=2} H^i(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(X_{\mathbb{C}}, \mathbb{Q}) .$$

The Poincaré duality establishes isomorphisms for all i:

$$H^i(A_{\mathbb{C}}, \mathbb{Q}) \cong H^{2g-i}(A_{\mathbb{C}}, \mathbb{Q})^{\vee} ,$$

where by $V^{\vee}$ we denote the dual vector space to a finite-dimensional vector space V.

Therefore, the tensor product

$$H^i(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^j(X_{\mathbb{C}}, \mathbb{Q})$$

is canonically isomorphic to the space of $\mathbb{Q}$-linear operators

$$\mathrm{Hom}_{\mathbb{Q}}(H^{2g-i}(A_{\mathbb{C}}, \mathbb{Q}), H^j(X_{\mathbb{C}}, \mathbb{Q})) ,$$

so that one can consider elements $cl(\beta_{\mathbb{C}})_{i,j}$ in tensor products as operators.

With this identification, for any element

$$\gamma \in H^{2g-i}(A_{\mathbb{C}}, \mathbb{Q})$$

we have that

$$(\beta_{\mathbb{C}})_* \gamma = cl(\beta_{\mathbb{C}})_{i,j}(\gamma) .$$

In notation introduced above, we have that

$$(\beta_{\mathbb{C}})_{*,i} = cl(\beta_{\mathbb{C}})_{2g-i,2-2g+i} .$$

Using the known vanishing of $(\beta_{\mathbb{C}})_{*,i}$, we deduce that

$$cl(\beta_{\mathbb{C}}) = cl(\beta_{\mathbb{C}})_{1,1} \in H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_{\mathbb{C}}, \mathbb{Q}) .$$

Now let us look at the Künneth decomposition for the element $cl(\alpha_{\mathbb{C}})$. By the construction of α,

$$cl(\alpha_{\mathbb{C}}) = cl(\beta_{\mathbb{C}})^p .$$

And, as we have seen above,

$$cl(\beta_{\mathbb{C}}) \in H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_{\mathbb{C}}, \mathbb{Q}) .$$

Therefore, it makes sense to compute the Künneth components of p-th powers of a given element in $H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_{\mathbb{C}}, \mathbb{Q})$.

Consider the following p-th power map in cohomology of $A \times X$:

$$H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_{\mathbb{C}}, \mathbb{Q}) \to H^{2p}(A_{\mathbb{C}} \times X_{\mathbb{C}}, \mathbb{Q}) ,$$

$$\sum_i a_i \otimes x_i \mapsto \left(\sum_i a_i \otimes x_i \right)^p ,$$

where $a_i \in H^1(A_{\mathbb{C}}, \mathbb{Q})$ and $x_i \in H^1(X_{\mathbb{C}}, \mathbb{Q})$.

Notice that the product in cohomology groups of $A \times X$ factors into the product in the cohomology groups of A and the product in the cohomology groups of X. In other words, the algebra $H^*(A_{\mathbb{C}} \times X_{\mathbb{C}}, \mathbb{Q})$ is isomorphic to the tensor product of the algebras $H^*(A_{\mathbb{C}}, \mathbb{Q})$ and $H^*(X_{\mathbb{C}}, \mathbb{Q})$. Another thing is that the product in the first cohomology group is skew-symmetric.

Denote by $\diamond$ the multiplication in the tensor product

$$\wedge^* H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \wedge^* H^1(X_{\mathbb{C}}, \mathbb{Q})$$

of the Grassman algebras.

The above p-th power map is equal to the composition of the product map

$$\lambda : H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_{\mathbb{C}}, \mathbb{Q}) \to \wedge^p H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \wedge^p H^1(X_{\mathbb{C}}, \mathbb{Q}) ,$$

$$\sum_i a_i \otimes x_i \mapsto \left(\sum_i a_i \otimes x_i \right)^{\diamond p} ,$$

the map

$$\mu : \wedge^p H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \wedge^p H^1(X_{\mathbb{C}}, \mathbb{Q}) \to H^p(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^p(X_{\mathbb{C}}, \mathbb{Q})$$

given by the multiplication in cohomology groups of A and X, and the natural embedding

$$H^p(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^p(X_{\mathbb{C}}, \mathbb{Q}) \to H^{2p}(A_{\mathbb{C}} \times X_{\mathbb{C}}, \mathbb{Q}) .$$

We need to describe the maps λ and μ in terms of operators between cohomology groups. Since the wedge powers of dual vector spaces are still dual, there is a canonical isomorphism

$$\wedge^p H^1(A_{\mathbb{C}}, \mathbb{Q}) \cong \wedge^p H^{2g-1}(A_{\mathbb{C}}, \mathbb{Q})^\vee .$$

Therefore, the target of λ and the source of μ is the space of operators

$$\mathrm{Hom}_{\mathbb{Q}}(\wedge^p H^{2g-1}(A_{\mathbb{C}}, \mathbb{Q}), \wedge^p H^p(X_{\mathbb{C}}, \mathbb{Q})) .$$

By Poincaré duality, the target of μ is the space of operators

$$\mathrm{Hom}_{\mathbb{Q}}(H^{2g-p}(A_{\mathbb{C}}, \mathbb{Q}), H^p(X_{\mathbb{C}}, \mathbb{Q})) .$$

Consider the following map induced by the product in cohomology of X,

$$\rho : \wedge^p H^1(X_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^p(X_{\mathbb{C}}, \mathbb{Q}) ,$$

and the map

$$\delta : H^{2g-p}(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow \wedge^p H^{2g-1}(A_{\mathbb{C}}, \mathbb{Q})$$

which is dual to the map

$$\wedge^p H^1(A_{\mathbb{C}}, \mathbb{Q}) \longrightarrow H^p(A_{\mathbb{C}}, \mathbb{Q})$$

induced by the product in the cohomology groups of A.

It follows from linear algebra that the p-th power map $\mu\circ\lambda$ sends an operator

$$\psi : H^{2g-1}(A_{\mathbb{C}},\mathbb{Q}) \longrightarrow H^1(X_{\mathbb{C}},\mathbb{Q})$$

to an operator

$$\phi : H^{2g-p}(A_{\mathbb{C}},\mathbb{Q}) \longrightarrow H^p(X_{\mathbb{C}},\mathbb{Q})$$

defined as the composition

$$\phi = \rho\circ\wedge^p(\psi)\circ\delta\ .$$

Let ψ is the isomorphism given by the action $(\beta_{\mathbb{C}})_{*,2g-1}$. Then α acts on the cohomology groups of A by the formula

$$(\alpha_{\mathbb{C}})_{*,2g-p} = \rho\circ\wedge^p((\beta_{\mathbb{C}})_{*,2g-1})\circ\delta :\ H^{2g-p}(A_{\mathbb{C}},\mathbb{Q}) \to H^p(X_{\mathbb{C}},\mathbb{Q})\ .$$

Now we are ready to construct an element γ in $H^{g,g-p}(A_{\mathbb{C}})$ such that $\beta_*(\gamma)\neq 0$.

Recall that, by assumption, we have p forms $\omega_1,\dots,\omega_p$ on $X_{\mathbb{C}}$, such that

$$\omega_1\wedge\ldots\wedge\omega_p\neq 0\ .$$

For each index i consider the element

$$\xi_i = ((\beta_{\mathbb{C}})_{*,2g-1})^{-1}(\omega_i)$$

in the group $H^{2g-1}(A_{\mathbb{C}},\mathbb{Q})$.

Since A is an abelian variety, the multiplication map

$$\wedge^p H^1(A_{\mathbb{C}},\mathbb{Q}) \longrightarrow H^p(A_{\mathbb{C}},\mathbb{Q})$$

is an isomorphism. Hence, the dual map δ is also an isomorphism.

Take the element

$$\gamma = \delta^{-1}(\xi_1\wedge\ldots\wedge\xi_p)$$

in the cohomology group $H^{2g-p}(A_{\mathbb{C}},\mathbb{Q})$. By the above formula for $(\alpha_{\mathbb{C}})_{*,2g-p}$, we have that

$$\begin{array}{rcl}
(\alpha_{\mathbb{C}})_*(\gamma) & = & (\rho\circ\wedge^p((\beta_{\mathbb{C}})_{*,2g-1})\circ\delta)(\gamma)\\
& = & (\rho\circ\wedge^p((\beta_{\mathbb{C}})_{*,2g-1}))(\xi_1\wedge\ldots\wedge\xi_p)\\
& = & \rho(\omega_1\wedge\ldots\wedge\omega_p)\\
& = & \omega_1\wedge\ldots\wedge\omega_p\\
& \neq & 0
\end{array}$$

Moreover,

$$\omega_1\wedge\ldots\wedge\omega_p\in H^{p,0}(X_{\mathbb{C}})\ .$$

Since the correspondence α is of degree $p-g$, the map $(\alpha_{\mathbb{C}})_*$ shifts the indices in the Hodge filtration by $p-g$, so that

$$\gamma\in H^{g,g-p}(A_{\mathbb{C}})\ .$$

Then, by Theorem 11, the cycle class $\Phi(\alpha)$ is non-trivial. $\square$

Now we would like to take a closer look at the condition in Theorem 15.

Let ω be a holomorphic 1-form on X, i.e. a section of the cotangent bundle T^*_X on the complex manifold $X_{\mathbb{C}}$. For a closed point $x\in X_{\mathbb{C}}$ denote by $\omega|_x$ the value of the differential form ω at x. Thus, $\omega|_x$ is an element in the cotangent space $T^*_{X,x}$ to $X_{\mathbb{C}}$ at x. The condition $\omega_1 \wedge \ldots \wedge \omega_p \neq 0$ in Theorem 15 is equivalent to say that there exists a closed point $x \in X_{\mathbb{C}}$, such that

$$\omega_1|_x \wedge \ldots \wedge \omega_p|_x \neq 0 \ .$$

It follows from linear algebra that $\omega_1|_x \wedge \ldots \wedge \omega_p|_x \neq 0$ if and only if the vectors

$$\omega_1|_x, \ldots, \omega_p|_x$$

are linearly independent in the cotangent space $T^*_{X,x}$. The next lemma shows that latter thing is equivalent to the condition that the image of the Albanese mapping $X \to \mathrm{Alb}(X)$ is at least p-dimensional.

Lemma 16. *Let X be an irreducible smooth projective variety X over k. Then it has p holomorphic one-forms $\omega_1, \ldots, \omega_p$ in $H^0(X_{\mathbb{C}}, \Omega^1_{X_{\mathbb{C}}}) = H^{1,0}(X_{\mathbb{C}})$, such that*

$$\omega_1 \wedge \ldots \wedge \omega_p \neq 0 \ ,$$

if and only if the dimension of the image of the Albanese map $X \to \mathrm{Alb}(X)$ is greater or equal than p.

Proof. For short, let $B = \mathrm{Alb}(X)$, and let

$$f : X \longrightarrow B$$

be the Albanese mapping.

First assume that there are p forms $\omega_1, \ldots, \omega_p$ as above. Denote by Y the image of the Albanese map $f : X \to B$. Since X is irreducible, Y is irreducible too. Let U be a non-empty open subset in Y, such that U is a smooth variety over k, and put

$$V = f^{-1}(U) \ .$$

Notice that V is a non-empty open subset in X because the natural map $X \to Y$ is surjective. Let

$$i : V \longrightarrow X$$

be the corresponding open embedding. Denote by

$$g : V \longrightarrow U$$

the restriction of the natural map $X \to Y$ to V, and let

$$j : U \longrightarrow B$$

be the composition $U \to Y \to B$.

Let W be the set of closed points $x \in X$ such that $\omega_1|_x \wedge \ldots \wedge \omega_p|_x \neq 0$. Then W is an open subset in X and, moreover, W is non-empty because $\omega_1 \wedge \ldots \wedge \omega_p \neq 0$.

Since X is irreducible, the intersection of two non-empty open subsets $W \cap V$ is non-empty. Thus, there exists a closed point $x \in V$, such that $\omega_1|_x \wedge \ldots \wedge \omega_p|_x \neq 0$. Therefore, the p-form $i^*_{\mathbb{C}}(\omega_1 \wedge \ldots \wedge \omega_p)$ on V is non-zero too.

As all holomorphic 1-forms on $X_{\mathbb{C}}$ are obtained by pulling-back via $f^*_{\mathbb{C}}$ of holomorphic 1-forms on $B_{\mathbb{C}}$, there are p holomorphic 1-forms $\xi_1, \ldots, \xi_p$ on $B_{\mathbb{C}}$ such that

$$f^*_{\mathbb{C}} \xi_i = \omega_i$$

for all i.

Since $f \circ i = j \circ g$, we have that

$$g_{\mathbb{C}}^* j_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p) = i_{\mathbb{C}}^* f_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p) \ .$$

Then

$$f_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p) = \omega_1 \wedge \ldots \wedge \omega_p$$

and, as we have seen above, the form

$$i_{\mathbb{C}}^*(\omega_1 \wedge \ldots \wedge \omega_p)$$

does not vanish. Therefore,

$$i_{\mathbb{C}}^* f_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p) \neq 0 \ ,$$

whence

$$g_{\mathbb{C}}^* j_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p) \neq 0 \ .$$

This implies that

$$j_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p) \neq 0 \ .$$

Thus, there exists a non-zero p-form, namely $j_{\mathbb{C}}^*(\xi_1 \wedge \ldots \wedge \xi_p)$, on the smooth variety U. Hence $\dim(U) \geq p$. Since U is dense in Y, we have that $\dim(Y) = \dim(U)$. Thus, $\dim(Y) \geq p$.

Suppose now that $\dim(f(X)) = \dim(Y) \geq p$. Take a smooth closed point $x \in Y$. Choose any codimension p linear subspace

$$E \subset T_x B$$

that intersects with $T_x Y$ only by zero. Then there are p linearly independent vectors

$$l_1, \ldots, l_p$$

in the cotangent space $T_x^* B$, such that

$$l_i|_E = 0 \ ,$$

where elements in the cotangent space $T_x^* B$ are considered as linear functionals on the tangent space $T_x B$.

Since B is an abelian variety, the tangent bundle on B is trivial. Therefore, there are p forms

$$\xi_1, \ldots, \xi_p$$

in $H^0(B, \Omega_B^1)$, such that

$$(\xi_1)|_x = l_1, \ldots, (\xi_p)|_x = l_p \ .$$

We now have that

$$(\xi_1)|_x \wedge \ldots \wedge (\xi_p)|_x \neq 0 \ .$$

It follows from linear algebra that the skew covector

$$l_1 \wedge \ldots \wedge l_p$$

remains non-zero when being restricted to the subspace $T_x Y$ in $T_x B$. Therefore, the restriction of the form $\xi_1 \wedge \ldots \wedge \xi_p$ from B to Y is non-zero at the point x. Thus, the restriction of the whole form $\xi_1 \wedge \ldots \wedge \xi_p$ from B to Y is non-zero.

Let now

$$\omega_i = f^* \xi_i$$

for each index i. Since the morphism $X \to Y$ is surjective, the form

$$\omega_1 \wedge \ldots \wedge \omega_p$$

is non-zero too. □

Now we can make our construction to be absolutely concrete. Namely, take any abelian variety B over k. Then take any subvariety Y in B, such that $\dim(Y) \geq p$. Consider a smooth projective variety X that admits a surjective morphism onto Y. Then the dimension of the image of the Albanese map $f : X \to \mathrm{Alb}(X)$ is greater or equal than p. Thus, by Theorem 12, we get a concrete non-zero element in the Abel-Jacobi kernel $T^p(X)$ of the variety X.

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