A CONJUGATION-FREE GEOMETRIC PRESENTATION OF FUNDAMENTAL GROUPS OF ARRANGEMENTS II: EXPANSION AND SOME PROPERTIES

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Abstract. A *conjugation-free geometric presentation* of a fundamental group is a presentation with the natural topological generators $x_1, \dots, x_n$ and the cyclic relations:

$$x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2}$$

with no conjugations on the generators.

We have already proved in [13] that if the graph of the arrangement is a disjoint union of cycles, then its fundamental group has a conjugation-free geometric presentation. In this paper, we extend this property to arrangements whose graphs are a disjoint union of cycle-tree graphs.

Moreover, we study some properties of this type of presentations for a fundamental group of a line arrangement's complement. We show that these presentations satisfy a completeness property in the sense of Dehornoy, if the corresponding graph of the arrangement is triangle-free. The completeness property is a powerful property which leads to many nice properties concerning the presentation (as the left-cancellativity of the associated monoid and yields some simple criterion for the solvability of the word problem in the group).

1. Introduction

The fundamental group of the complement of plane curves is a very important topological invariant, which can be also computed for line arrangements. We count here some applications of this invariant.

Chisini [6], Kulikov [20, 21] and Kulikov-Teicher [22] have used the fundamental group of complements of branch curves of generic projections in order to distinguish between connected components of the moduli space of smooth projective surfaces, see also [15].

Moreover, the Zariski-Lefschetz hyperplane section theorem (see [24]) stated that:

$$\pi_1(\mathbb{CP}^N \setminus S) \cong \pi_1(H \setminus (H \cap S)),$$

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where S is an hypersurface and H is a generic 2-plane. Since $H \cap S$ is a plane curve, the fundamental groups of complements of curves can be used also for computing the fundamental groups of complements of hypersurfaces in $\mathbb{CP}^N$.

A different need for fundamental groups' computations is for obtaining more examples of Zariski pairs [31, 32]. A pair of plane curves is called *a Zariski pair* if they have the same combinatorics (to be exact: there is a degree-preserving bijection between the set of irreducible components of the two curves C_1, C_2, and there exist regular neighbourhoods of the curves $T(C_1), T(C_2)$ such that the pairs $(T(C_1), C_1), (T(C_2), C_2)$ are homeomorphic and the homeomorphism respects the bijection above [3]), but their complements in $\mathbb{P}^2$ are not homeomorphic. For a survey, see [5].

It is also interesting to explore new finite non-abelian groups which serve as fundamental groups of complements of plane curves in general, see for example [1, 2, 12, 31].

An *affine line arrangement* in $\mathbb{C}^2$ is a union of copies of $\mathbb{C}^1$ in $\mathbb{C}^2$. Such an arrangement is called *real* if the defining equations of all its lines can be written with real coefficients, and *complex* otherwise. Note that the intersection of a real arrangement with the natural copy of $\mathbb{R}^2$ in $\mathbb{C}^2$ is an arrangement of lines in the real plane, called the *real part* of the arrangement.

Similarly, a *projective line arrangement* in $\mathbb{CP}^2$ is a union of copies of $\mathbb{CP}^1$ in $\mathbb{CP}^2$. Note that the realization of the MacLane configuration [23] is an example of a complex arrangement, see also [4, 28].

For real and complex line arrangements $\mathcal{L}$, Fan [14] defined a graph $G(\mathcal{L})$ which is associated to its multiple points (i.e. points where more than two lines are intersected). We give here its version for real arrangements (the general version is more delicate to explain): Given a real line arrangement $\mathcal{L}$, the graph $G(\mathcal{L})$ of multiple points lies on the real part of $\mathcal{L}$. It consists of the multiple points of $\mathcal{L}$, with the segments between the multiple points on lines which have at least two multiple points. Note that if the arrangement consists of three multiple points on the same line, then $G(\mathcal{L})$ has three vertices on the same line (see Figure 1(a)). If two such lines happen to intersect in a simple point (i.e. a point where exactly two lines are intersected), it is ignored (i.e. the lines do not meet in the graph). See another example in Figure 1(b) (note that Fan's definition gives a graph different from the graph defined in [18, 29]).

In [13] we introduce the notion of a *conjugation-free geometric presentation* of the fundamental group of an arrangement:

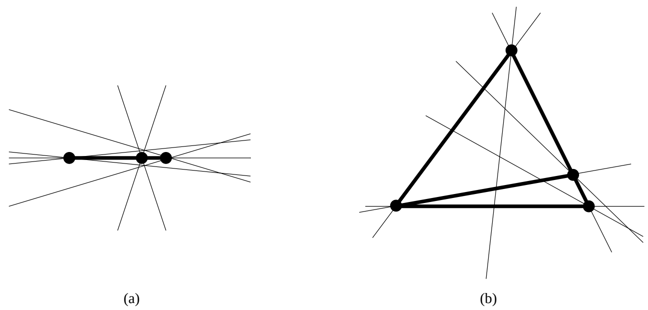

FIGURE 1. Examples for $G(\mathcal{L})$

Definition 1.1. *Let G be a fundamental group of the affine or projective complements of some line arrangement with n lines. We say that G has* a conjugation-free geometric presentation *if G has a presentation with the following properties:*

- *In the affine case, the generators $\{x_1, \dots, x_n\}$ are the meridians of lines at some far side of the arrangement, and therefore the number of generators is equal to n.*
- *In the projective case, the generators are the meridians of lines at some far side of the arrangement except for one, and therefore the number of generators is equal to $n-1$.*
- *In both cases, the relations are of the following type:*

$$x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2},$$

  where $\{i_1, i_2, \dots, i_k\} \subseteq \{1, \dots, m\}$ is an increasing subsequence of indices, where $m = n$ in the affine case and $m = n-1$ in the projective case. Note that for $k = 2$ we get the usual commutator.

Note that in usual geometric presentations of the fundamental group, most of the relations have conjugations.

The importance of this family of arrangements is that the fundamental group can be read directly from the arrangement or equivalently from its incidence lattice (where the *incidence lattice* of an arrangement is the partially-ordered set of non-empty intersections of the lines, ordered by inclusion, see [27]) without any computation. Hence, for this family of arrangements, the incidence lattice determines the fundamental group of the complement (this is based on Cordovil [7] too).

We start with the easy fact that there exist arrangements whose fundamental groups have no conjugation-free geometric presentation: The fundamental group of the Ceva arrangement (also known as the *braid* *arrangement*, appears in Figure 2) has no conjugation-free geometric presentation (see [13]).

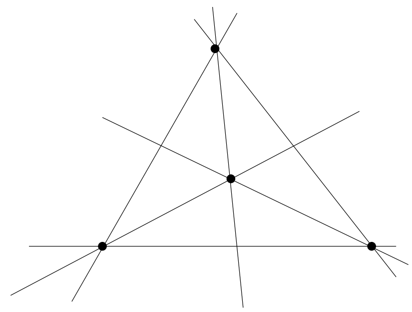

FIGURE 2. Ceva arrangement

Note also that if the fundamental groups of two arrangements $\mathcal{L}_1, \mathcal{L}_2$ have conjugation-free geometric presentations and the arrangements intersect transversally, then the fundamental group of $\mathcal{L}_1 \cup \mathcal{L}_2$ has a conjugation-free geometric presentation too. This is due to the important result of Oka and Sakamoto [26]:

Theorem 1.2. (Oka-Sakamoto) *Let C_1 and C_2 be algebraic plane curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 \cdot d_2$ points, where d_i $(i = 1, 2)$ are the respective degrees of C_1 and C_2. Then:*

$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2)$$

The main result of [13] is:

Proposition 1.3. *The fundamental groups of following family of arrangements have a conjugation-free geometric presentation: a real arrangement $\mathcal{L}$, where $G(\mathcal{L})$ is a disjoint union of cycles of any length, and the multiplicities of the multiple points are arbitrary.*

In this paper, we continue the investigation of the family of arrangements whose fundamental groups have conjugation-free geometric presentations in two directions. First, we extend this property to real arrangements whose graphs are a disjoint union of cycle-tree graphs, where an example for a cycle-tree graph is presented in Figure 3 (see Definition 2.4 below).

In the second direction, we study some properties of this type of presentations for a fundamental group of a line arrangement's complement. We prove:

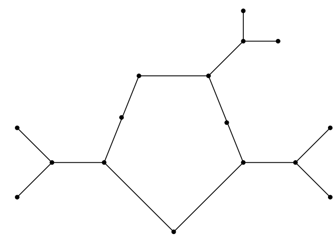

FIGURE 3. An example of a cycle-tree graph

Proposition 1.4. *Let $\mathcal{L}$ be a real arrangement whose fundamental group has a conjugation-free geometric presentation and its graph $G(\mathcal{L})$ is triangle-free (i.e. contains no cycles of length 3). Then, the presentation of the corresponding monoid is complete (and complemented).*

The completeness property is a powerful property which leads to many nice properties concerning the presentation (as the left-cancellativity of the associated monoid and yields some simple criterion for the solvability of the word problem in the group and for Garside groups).

The paper is organized as follows. In Section 2, we prove that arrangements whose graphs are a disjoint union of cycle-tree graphs have a conjugation-free geometric presentation of the fundamental group of the complement. In Section 3, we prove that conjugation-free geometric presentations are complemented presentations. Section 4 deals with complete presentations, and includes the proof of Proposition 1.4.

2. Adding a line through a single point preserves the conjugation-free geometric presentation

We start with the following obvious observation, which is based on the Oka-Sakamoto decomposition theorem (see Theorem 1.2 above):

Observation 2.1. *Let $\mathcal{L}$ be an arrangement whose fundamental group has a conjugation-free geometric presentation. Let L be a line which intersects $\mathcal{L}$ transversally. Then: $\mathcal{L} \cup L$ is also an arrangement whose fundamental group has a conjugation-free geometric presentation.*

In this section, we prove the following proposition, which is the next step:

Proposition 2.2. *Let $\mathcal{L}$ be a real arrangement whose affine fundamental group has a conjugation-free geometric presentation. Let L be a line not in $\mathcal{L}$, which passes through one intersection point P of $\mathcal{L}$. Then:*

$\mathcal{L} \cup L$ is also an arrangement whose affine fundamental group has a conjugation-free geometric presentation.

Proof. We can assume that the point P is the leftmost and lowest point of the arrangement $\mathcal{L}$ and all the intersection points of the line L (except for P) are to the left of all the intersection points of the arrangement $\mathcal{L}$ (except for P). We can also assume that the highest line in $\mathcal{L}$ (with respect to the global numeration of the lines) passes through P. See Figure 4 for an illustration, where the arrangement $\mathcal{L}$ is in the dashed rectangle.

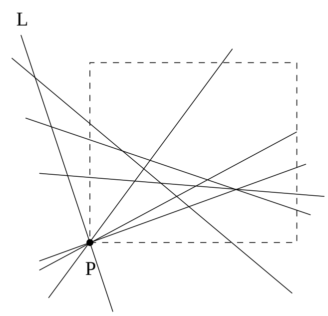

FIGURE 4. An illustration of the real part of $\mathcal{L} \cup L$

The above assumption is due to the following reasons: First, one can rotate a line that participates in only one multiple point as long as it does not unite with a different line (by Results 4.8 and 4.13 of [17]). Second, moving a line that participates in only one multiple point over a different line (see Figure 5) is permitted in the case of a triangle due to a result of Fan [14] that the family of configurations with 6 lines and three triple points is connected by a finite sequence of smooth equisingular deformations. Moreover, by Theorem 4.11 of [17], one can assume that the point P is the leftmost point of the arrangement $\mathcal{L}$.

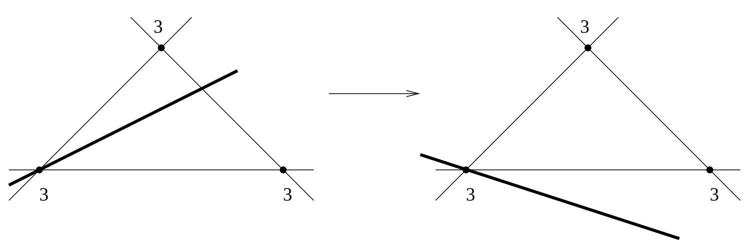

FIGURE 5. Moving a line that participates in only one multiple point over a different line

Let n be the number of lines in $\mathcal{L}$ and let m be the multiplicity of P in $\mathcal{L}$. So, the list of Lefschetz pairs of the arrangement $\mathcal{L}$ is

$$([a_1, b_1], [a_2, b_2], \ldots, [a_{q-1}, b_{q-1}], [1, m]),$$

where the Lefschetz pair $[1, m]$ corresponds to the point P (for the theory used here for computing the fundamental group of the complements of arrangements, see [13, 16, 19, 25]). Since we have that $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation, then we know that all the conjugations in the relations induced by the van Kampen theorem [19] (see also [13]) can be simplified.

Now, let us deal with the arrangement $\mathcal{L} \cup L$. By our assumptions, its list of Lefschetz pairs is (we write in small brackets the name of the corresponding point):

$$([a_1+1, b_1+1]_{(p_1)}, [a_2+1, b_2+1]_{(p_2)}, \ldots, [a_{q-1}+1, b_{q-1}+1]_{(p_{q-1})},$$
$$[1, m{+}1]_{(p_q)}, [m{+}1, m{+}2]_{(p_{q+1})}, [m{+}2, m{+}3]_{(p_{q+2})}, \ldots, [n, n{+}1]_{(p_{q+(n-m)})}).$$

We start with the relations induced from intersection points on the line L. We first choose a set of $n{+}1$ generators of the fundamental group of its complement corresponding to its lines, namely $\{x_1, \ldots, x_{n+1}\}$. By the Moishezon-Teicher algorithm [16, 25] (see also [13]), we now compute the skeletons corresponding to the points on the line L (i.e. the points p_j, where $q \leq j \leq q+n-m$). Note that:

$$\Delta\langle a_1+1, b_1+1\rangle \Delta\langle a_2+1, b_2+1\rangle \cdots \Delta\langle a_{q-1}+1, b_{q-1}+1\rangle = \Delta\langle 2, n+1\rangle \Delta^{-1}\langle 2, m+1\rangle,$$

since given an arrangement, the multiplication of all the halftwists based on its Lefschetz pairs is equivalent to a unique halftwist of all the lines. By this observation, we get the skeletons in Figure 6.

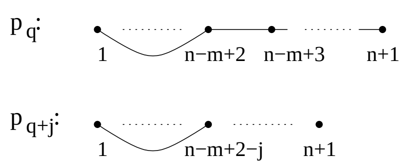

FIGURE 6. The skeletons of the points p_{q+j} where $0 \leq j \leq n-m$

Hence, we get the following relations:

For the point p_q:

$$\begin{aligned} x_{n+1}x_n \cdots x_{n-m+2}x_1 &= x_1 x_{n+1} x_n \cdots x_{n-m+2} = \\ &= \cdots = x_n \cdots x_{n-m+2} x_1 x_{n+1} \end{aligned}$$

For the points p_j, where $q+1 \leq j \leq q+n-m$:

$$x_{n-m+2-j}x_1 = x_1 x_{n-m+2-j}$$

These relations are obviously without conjugations.

Now, we move to the relations induced from points appearing in the original arrangement $\mathcal{L}$. The only change in the level of the Lefschetz pairs is an addition of one index in all the pairs, due to the line L. Therefore, the induced braid monodromy and the relations will be changed by adding 1 to every index, i.e. if we have a relation which involves the generators $x_{i_1}, \dots, x_{i_k}$, then after adding the line, we have the same relation but with generators $x_{i_1+1}, \dots, x_{i_k+1}$, respectively.

Now, we know that the fundamental group of $\mathcal{L}$ has a conjugation-free geometric presentation, hence we have that by a simplification process, one can reach a presentation without conjugations. If we imitate the simplification process of the presentation of the fundamental group of $\mathcal{L}$ for the presentation of the fundamental group of $\mathcal{L} \cup L$, the cases in which we need to use the relations induced from the point P are the relations that have been simplified by using the relations induced from P before adding the line. As above, the original relations induced from P are:

$$\begin{array}{rrcl} R_p: & x_n x_{n-1} \cdots x_{n-m+1} & = & x_{n-1} \cdots x_{n-m+1} x_n = \\ & & = & \cdots = x_{n-m+1} x_n \cdots x_{n-m+2}, \end{array}$$

while the new ones are:

$$\begin{array}{rrcl} \tilde{R}_p: & x_{n+1} x_n \cdots x_{n-m+2} x_1 & = & x_1 x_{n+1} x_n \cdots x_{n-m+2} = \\ & & = & \cdots = x_n \cdots x_{n-m+2} x_1 x_{n+1}. \end{array}$$

We can divide the relations induced from $\mathcal{L}$ before adding the line L into two subsets:

(1) Relations that during the simplification process contain the subword $x_{n-m+2}^{-1} \cdots x_{n-1}^{-1} x_n x_{n-1} \cdots x_{n-m+2}$.
(2) Relations that do not contain the above subword during its simplification process.

For the second subset, the simplification process will be identical before adding the line L and after it, since all the other relations induced by $\mathcal{L}$ have not been changed by adding the line L (except for adding 1 to the indices).

For the first subset, let us denote the relation by R. Except for applying the relations induced from P, the rest of simplification process is identical to the one before adding the line (again, except for adding 1 to the indices). The only change is in the step of applying R_p. In this step, before adding the line L, the generator x_1 has not been involved in R_p, but after adding the line L, it appears in $\tilde{R}_p$. Hence, for applying $\tilde{R}_p$, we have to conjugate relation R by x_1, and using the commutative relations which x_1 is involved in, we can diffuse x_1 into the relation R, so we can use the relation $\tilde{R}_p$ instead of R_p.

Hence, we can simplify all the conjugations in all the relations, so we have a conjugation-free geometric presentation, as needed. □

Remark 2.3. *Note that adding a line which closes a cycle in $\mathcal{L}$ might not preserve the conjugation-free geometric presentation property. For example, adding a line to an arrangement of 5 lines which creates the Ceva arrangement (see Figure 2) is not an action which preserves the conjugation-free geometric presentation property.*

Hence, we can extend the family of arrangements whose fundamental groups have a conjugation-free geometric presentation. We start with the following definition:

Definition 2.4. *A* cycle-tree graph *is a graph which consists of a cycle, where each vertex of the cycle can be a root of a tree, see Figure 7. It is possible that there exist some vertices also in the middle of an edge of the cycle or the trees.*

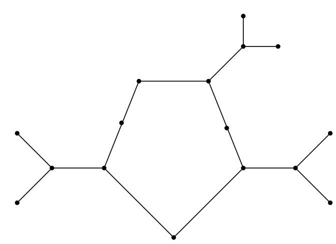

FIGURE 7. An example of a cycle-tree graph

Corollary 2.5. *Let $\mathcal{L}$ be a real line arrangement whose graph is a disjoint union of cycle-tree graphs. Then the fundamental group of $\mathcal{L}$ has a conjugation-free geometric presentation.*

Proof. We start by proving that a real arrangement whose graph is a cycle-tree graph has a fundamental group which has a conjugation-free geometric presentation. We already have from [13] that a real arrangement whose graph is a cycle, has a fundamental group which has a conjugation-free geometric presentation. By Proposition 2.2, adding a line which is either transversal to an arrangement or passes through one intersection point, preserves the property that the fundamental group has a conjugation-free geometric presentation. One can easily construct an arrangement whose graph is a cycle-tree graph from an arrangement whose graph is a cycle by inductively adding a line which is either transversal to the arrangement or passes through one of its intersection points. Hence, we get that an arrangement whose graph is a cycle-tree graph has a fundamental group which has a conjugation-free geometric presentation.

In the next step, using the theorem of Oka and Sakamoto [26] (see Theorem 1.2 above), we can generalize the result from the case of one cycle-tree graph to the case of a disjoint union of cycle-tree graphs. $\square$

3. Complemented presentations

A *semigroup presentation* $(\mathcal{S}, \mathcal{R})$ consists of a nonempty set $\mathcal{S}$ and a family of pairs of nonempty words $\mathcal{R}$ in the alphabet $\mathcal{S}$. The corresponding monoid $(\mathcal{S}, \mathcal{R})$ is $\langle \mathcal{S} | \mathcal{R} \rangle^+ \cong (\mathcal{S}^* / \equiv^+_{\mathcal{R}})$.

Dehornoy [8] has defined the notion of a *complemented presentation* of a semigroup:

Definition 3.1. *A semigroup presentation* $(\mathcal{S}, \mathcal{R})$ *is called* complemented *if, for each* $s \in \mathcal{S}$*, there is no relation* $s \ldots = s \ldots$ *in* $\mathcal{R}$ *and, for* $s, s' \in \mathcal{S}$*, there is at most one relation* $s \ldots = s' \ldots$ *in* $\mathcal{R}$*.*

Our type of presentations satisfies this property:

Lemma 3.2. *A conjugation-free geometric presentation is a complemented presentation.*

Proof. Any pair of lines intersect exactly once, hence their corresponding generators appear as prefixes in exactly one relation. Since there are no conjugations, this is their unique appearance as a pair of prefixes. $\square$

Remark 3.3.

1. *This property does not hold for presentations of fundamental groups in general (due to the conjugations in the relations).*
2. *This property does not hold in the homogeneous minimal presentations introduced by Yoshinaga* [30]*.*

4. Complete presentations

In this section, we will study which cases of conjugation-free geometric presentations are also *complete* in the sense of Dehornoy [10]. The completeness property is a very important and powerful property. In Section 4.1, we supply some background on this property. In Section 4.2, we count some important consequences and applications arising from the completeness property. In Section 4.3, we present our results in this direction.

4.1. **Background on complete presentations.** We follow the survey of Dehornoy [11]. We start by defining the notion of a *word reversing*:

Definition 4.1. *For a semigroup presentation* $(\mathcal{S},\mathcal{R})$ *and* $w,w' \in (\mathcal{S}\cup\mathcal{S}^{-1})^*$, w reverses to w' in one step, *denoted by* $w \curvearrowright^1_{\mathcal{R}} w'$, *if there exist a relation* $sv' = s'v$ *of* $\mathcal{R}$ *and* u,u' *satisfying:*

$$w = us^{-1}s'u' \text{ and } w' = uv'v^{-1}u'.$$

We say that w reverses to w' in k steps, *denoted by* $w \curvearrowright^k_R w'$, *if there exist words* $w_0,\ldots,w_k$ *satisfying* $w_0 = w, w_k = w'$ *and* $w_i \curvearrowright^1_R w_{i+1}$ *for each* i. *The sequence* $(w_0,\ldots,w_k)$ *is called* an R-reversing sequence from w to w'.

We write $w \curvearrowright w'$, *if* $w \curvearrowright^k_{\mathcal{R}} w'$ *holds for some* $k \in \mathbb{N}$.

Definition 4.2. *A semigroup presentation* $(\mathcal{S},\mathcal{R})$ *is called* complete *if, for all words* $w,w' \in \mathcal{S}^*$:

$$w \equiv^+_{\mathcal{R}} w' \quad\Rightarrow\quad w^{-1}w' \curvearrowright_{\mathcal{R}} \varepsilon.$$

where ε *is the empty word.*

In the next definition, we define the *cube condition*, which is a useful tool for verifing the completeness property:

Definition 4.3. *Let* $(\mathcal{S},\mathcal{R})$ *be a semigroup presentation, and* $u,u',u'' \in \mathcal{S}^*$. *We say that* $(\mathcal{S},\mathcal{R})$ *satisfies the* cube condition for (u,u',u'') *if:*

$$u^{-1}u''u''^{-1}u' \curvearrowright_{\mathcal{R}} v'v^{-1} \qquad\Rightarrow\qquad (uv')^{-1}(vu') \curvearrowright_{\mathcal{R}} \varepsilon.$$

For $X \subseteq \mathcal{S}^*$, *we say that* $(\mathcal{S},\mathcal{R})$ *satisfies* the cube condition on X *if it satisfies the cube condition for every triple* (u,u',u'') *where* $u,u',u'' \in X$.

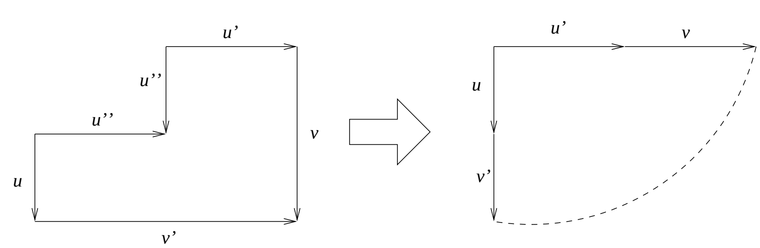

FIGURE 8. An illustration of the cube condition

Definition 4.4. *A semigroup presentation* $(\mathcal{S},\mathcal{R})$ *is said to be* homogeneous *if there exists an* $\equiv^+_{\mathcal{R}}$*-invariant mapping* $\lambda : \mathcal{S}^* \to \mathbb{N}$ *satisfying, for* $s \in \mathcal{S}$ *and* $w \in \mathcal{S}^*$,

$$\lambda(sw) > \lambda(w).$$

A typical case of an homogeneous presentation is where all relations in $\mathcal{R}$ preserve the length of words, i.e. they have the form $v' = v$ where v' and v have the same length.

Dehornoy [10] has proved the following result:

Proposition 4.5. *Assume that* $(\mathcal{S},\mathcal{R})$ *is a homogeneous semigroup presentation. Then:* $(\mathcal{S},\mathcal{R})$ *is complete if and only if it satisfies the cube condition on* $\mathcal{S}$.

The next definition is needed for introducing an operation used in an equivalent condition for the cube condition:

Definition 4.6. *For a complemented semigroup presentation* $(\mathcal{S},\mathcal{R})$ *and* $w, w' \in \mathcal{S}^*$*, the* $\mathcal{R}$-complement of w' in w, denoted $w\backslash w'$*, ("*w *under* w'*"), is the unique word* $v' \in \mathcal{S}^*$ *such that* $w^{-1}w'$ *reverses to* $v'v^{-1}$ *for some* $v \in \mathcal{S}^*$*, if such a word exists.*

Dehornoy [10] has proved that the cube condition is equivalent to some expression involving the complement operation:

Proposition 4.7. *Assume that* $(\mathcal{S},\mathcal{R})$ *is a complemented semigroup presentation. Then, for all words* $u, u', u'' \in \mathcal{S}^*$*, the following are equivalent:*

(1) $(\mathcal{S},\mathcal{R})$ *satisfies the cube condition on* $\{u, u', u''\}$.
(2) *either* $(u\backslash u')\backslash(u\backslash u'')$ *and* $(u'\backslash u)\backslash(u'\backslash u'')$ *are* $\mathcal{R}$*-equivalent or they are not defined, and the same holds for all permutations of* u, u', u''.

4.2. **Consequences of complete presentations.**

In this section, we survey some important consequences and applications arising from the completeness property.

Proposition 4.8 ([10], Proposition 6.1)**.** *Every monoid that admits a complete complemented presentation is left-cancellative (i.e.* $xy = xz \Rightarrow y = z$*).*

Proposition 4.9 ([10], Proposition 6.10)**.** *Assume that* $(\mathcal{S},\mathcal{R})$ *is a complete semigroup presentation. If* $(\mathcal{S},\mathcal{R})$ *is complemented, then the monoid* $\langle \mathcal{S}|\mathcal{R}\rangle^+$ *admits least common multiples.*

Proposition 4.10 ([10], Proposition 7.7)**.** *Assume that* $(\mathcal{S},\mathcal{R})$ *is a complete semigroup presentation and there exists* $\widehat{\mathcal{S}} \subseteq \mathcal{S}^*$ *that includes* $\mathcal{S}$ *and satisfies the following conditions:*

(1) *For all* $u, u' \in \widehat{\mathcal{S}}$, *there exist* $v, v' \in \widehat{\mathcal{S}}$ *such that* $u^{-1}u' \curvearrowright_{\mathcal{R}} v'v^{-1}$.
(2) *For all* $u, u' \in \widehat{\mathcal{S}}$ *and for all* $v, v' \in \mathcal{S}^*$, *we have:*

$$u^{-1}u' \curvearrowright_{\mathcal{R}} v'v^{-1} \;\Rightarrow\; v, v' \in \widehat{\mathcal{S}}.$$

Then, every $\mathcal{R}$*-reversing sequence leads in finitely many steps to a positive–negative word. If* $\widehat{\mathcal{S}}$ *is finite, then the word problem of the presented monoid* $\langle \mathcal{S}|\mathcal{R}\rangle^+$ *is solvable in exponential time, and in quadratic time if* $(\mathcal{S}, \mathcal{R})$ *is complemented.*

If, in addition, the monoid $\langle \mathcal{S}|\mathcal{R}\rangle^+$ *is right-cancellative, the word problem of the presented group* $\langle \mathcal{S}|\mathcal{R}\rangle$ *is solvable in exponential time, and in quadratic time if* $(\mathcal{S}, \mathcal{R})$ *is complemented.*

Definition 4.11. [9] *A monoid* M *is called* Garside, *and its group of fractions is called a* Garside group, *if it satisfies the following conditions:*

(1) M *is cancellative.*
(2) M *contains no invertible elements except* ε.
(3) *Any two elements of* M *admit a left and right least common multiples and greatest common divisors.*
(4) *There exists an element* $\Delta \in M$, *called the* Garside element, *such that the sets of left and right divisors of* Δ *coincide, generate* M, *and are finite in number.*

Dehornoy [9] proved the following proposition with respect to the Garside element:

Proposition 4.12. *Let* $(\mathcal{S}, \mathcal{R})$ *be a complemented presentation of a monoid* M. *Then the Garside element, if exists, is the longest element in the smallest set of words that includes* $\mathcal{S}$ *and is closed under the complement and right-lcm operations.*

4.3. **Completeness of conjugation-free geometric presentations.** In this section, we prove that a conjugation-free geometric presentation is complete if its corresponding graph is triangle-free:

Proposition 4.13. *Let* $\mathcal{L}$ *be a real arrangement whose fundamental group has a conjugation-free geometric presentation and its graph* $G(\mathcal{L})$ *is triangle-free (i.e. contains no triangles). Then, the presentation of the corresponding monoid is complete (and complemented).*

Proof. It is obvious that the conjugation-free geometric presentations are homogeneous (since all the words in the same relation are of the same length). Hence, we prove this proposition by verifying the equivalent version of the cube condition (for any triple $(u, u', u'') \in (\mathcal{S}^*)^3$, the words $(u\backslash u')\backslash(u\backslash u'')$ and $(u'\backslash u)\backslash(u'\backslash u'')$ are $\mathcal{R}$-equivalent) case-by-case.

Case 1: The three generators correspond to three lines ℓ_i, ℓ_j, ℓ_k intersecting in three simple points, see Figure 9.

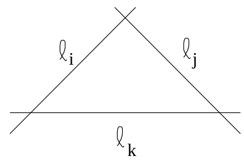

FIGURE 9. Case 1

In this case, the relations induced by the three simple points are: $[x_i, x_j] = [x_i, x_k] = [x_j, x_k] = e$, where x_i, x_j, x_k are the generators of the lines ℓ_i, ℓ_j, ℓ_k respectively. So, we have:

$$(x_i\backslash x_j)\backslash(x_i\backslash x_k) = x_k = (x_j\backslash x_i)\backslash(x_j\backslash x_k),$$

which are indeed $\mathcal{R}$-equivalent.

By symmetry, this holds to any permutation of x_i, x_j, x_k as needed.

Case 2: The three generators correspond to three lines ℓ_i, ℓ_j, ℓ_k passing through the same multiple point, see Figure 10. For this case, we have two subcases: in the first case, the corresponding lines appear consecutively in the intersection point. In the second case, the corresponding lines appear separately in the intersection point.

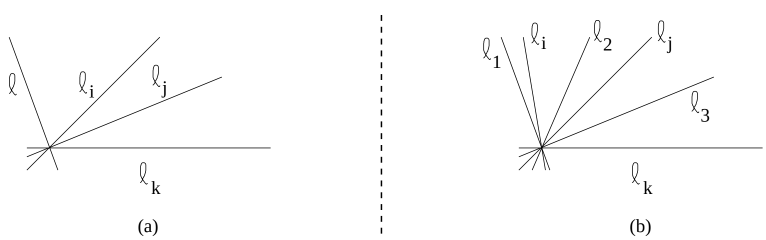

FIGURE 10. Case 2

Case 2a: *The lines appear consecutively in the intersection point:* Without loss of generality, we can assume that the multiple point has multiplicity 4, see Figure 10(a). Hence, the relations induced by this multiple point are:

$$xx_kx_jx_i = x_kx_jx_ix = x_jx_ixx_k = x_ixx_kx_j,$$

where x_i, x_j, x_k, x are the generators of the lines $\ell_i, \ell_j, \ell_k, \ell$ respectively. Hence, we have:

$$(x_i \backslash x_j) \backslash (x_i \backslash x_k) = e = (x_j \backslash x_i) \backslash (x_j \backslash x_k),$$

which are indeed $\mathcal{R}$-equivalent.
Any other permutation of x_i, x_j, x_k yields e at both sides of the condition, so the condition is satisfied for any permutation.

Case 2b: *The lines do not appear consecutively in the intersection point:* Without loss of generality, we can assume that the multiple point has multiplicity 6, see Figure 10(b). Hence, the relations induced by this multiple point are:

$$\begin{aligned} zx_kyx_jxx_i &= x_kyx_jxx_iz = yx_jxx_izx_k = x_jxx_izx_ky = \\ &= xx_izx_kyx_j = x_izx_kyx_jx, \end{aligned}$$

where x_i, x_j, x_k, x, y, z are the generators of the lines ℓ_i,ℓ_j, ℓ_k,ℓ_1,ℓ_2,ℓ_3 respectively. Hence, we have:

$$(x_i \backslash x_j) \backslash (x_i \backslash x_k) = e = (x_j \backslash x_i) \backslash (x_j \backslash x_k),$$

which are indeed $\mathcal{R}$-equivalent.
Any other permutation of x_i, x_j, x_k yields e at both sides of the condition, so the condition is satisfied for any permutation.

Case 3: The three generators correspond to three lines ℓ_i, ℓ_j, ℓ_k, where two of the lines ℓ_i, ℓ_j pass through the same multiple point, and the third line ℓ_k intersects them transversally, see Figure 11. In this case also, we have two subcases: in the first case, the two lines involved in the multiple point appear consecutively in the intersection point. In the second case, these lines appear separately in the multiple point.

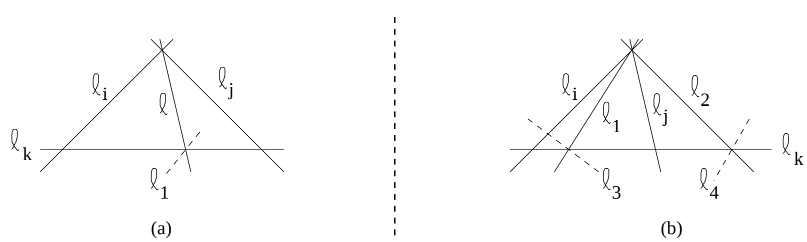

FIGURE 11. Case 3

Case 3a: *The two lines appear consecutively in the multiple point:* Without loss of generality, we can assume that the multiple point has multiplicity 3, see Figure 11(a). Hence, the relations induced by this multiple point are:

$$xx_jx_i = x_jx_ix = x_ixx_j.$$

Moreover, we have: $[x_i, x_k] = [x_j, x_k] = e$, where x_i, x_j, x_k, x are the generators of the lines $\ell_i, \ell_j, \ell_k, \ell$ respectively. Hence, we have:

$$(x_i \backslash x_j)\backslash(x_i \backslash x_k) = x_k = (x_j \backslash x_i)\backslash(x_j \backslash x_k),$$

and in case there exists another line ℓ_1 (whose generator is y) which passes through the intersection point of the lines ℓ_k and ℓ, we have:

$$(x_i \backslash x_j)\backslash(x_i \backslash x_k) = yx_k = (x_j \backslash x_i)\backslash(x_j \backslash x_k),$$

which are $\mathcal{R}$-equivalent too.
Any other permutation of x_i, x_j, x_k satisfies the condition as well.

Case 3b: *The two lines do not appear consecutively in the multiple point:* Without loss of generality, we can assume that the multiple point has multiplicity 4, see Figure 11(b). Hence, the relations induced by this multiple point are:

$$yx_jxx_i = x_jxx_iy = xx_iyx_j = x_iyx_jx.$$

Moreover, we have: $[x_i, x_k] = [x_j, x_k] = e$, where x_i, x_j, x_k, x, y are the generators of the lines $\ell_i, \ell_j, \ell_k, \ell_1, \ell_2$ respectively. Now, we have three different situations:

(i) If ℓ_3 and ℓ_4 both do not exist, then we have:

$$(x_i \backslash x_j)\backslash(x_i \backslash x_k) = x_k = (x_j \backslash x_i)\backslash(x_j \backslash x_k).$$

(ii) If ℓ_3 exists (and its corresponding generator is z) but ℓ_4 does not exist, then we have:

$$(x_i \backslash x_j)\backslash(x_i \backslash x_k) = zx_k = (x_j \backslash x_i)\backslash(x_j \backslash x_k).$$

(iii) If ℓ_4 exists (and its corresponding generator is u) but ℓ_3 does not exist, then we have:

$$(x_i \backslash x_j)\backslash(x_i \backslash x_k) = ux_k = (x_j \backslash x_i)\backslash(x_j \backslash x_k).$$

All are $\mathcal{R}$-equivalent (note that it is impossible that both ℓ_3 and ℓ_4 exist, since in that case we get a triangle in the graph). Hence, in all cases, the condition is satisfied.

Any other permutation of x_i, x_j, x_k satisfies the condition as well.

Case 4: The three generators correspond to three lines ℓ_i, ℓ_j, ℓ_k, where the line ℓ_i passes through two multiple points, the line ℓ_j passes through the first multiple point, and the line ℓ_k passes through the other multiple point, see Figure 12. In this case too, we have two subcases: in the first case, the two lines ℓ_i and ℓ_j involved in the first multiple point appear consecutively in the intersection point. In the second case, these lines appear separately in the multiple point (actually there are two more cases which are related to the intersection point of the lines ℓ_i and ℓ_k, but they can be treated similar to the cases appeared here).

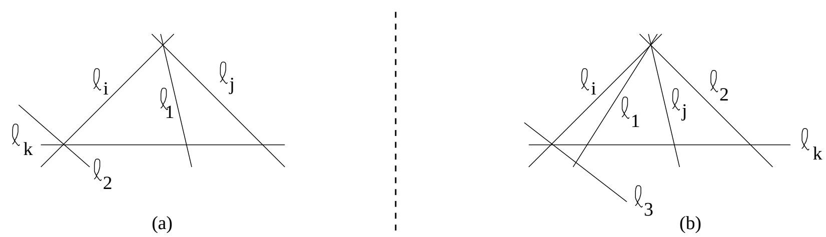

FIGURE 12. Case 4

Case 4a: *The lines ℓ_i and ℓ_j appear consecutively in the first multiple point:* Without loss of generality, we can assume that this multiple point has multiplicity 3, see Figure 12(a). Hence, the relations induced by this multiple point are:

$$xx_jx_i = x_jx_ix = x_ixx_j.$$

Moreover, we have: $yx_kx_i = x_kx_iy = x_iyx_k$ and $[x_j, x_k] = e$, where x_i, x_j, x_k, x, y are the generators of the lines $\ell_i, \ell_j, \ell_k, \ell_1, \ell_2$ respectively. Hence, we have:

$$(x_i\backslash x_j)\backslash(x_i\backslash x_k) = yx_k = (x_j\backslash x_i)\backslash(x_j\backslash x_k),$$

which is $\mathcal{R}$-equivalent too.
Any other permutation of x_i, x_j, x_k satisfies the condition as well.

Case 4b: *The lines ℓ_i and ℓ_j do not appear consecutively in the first multiple point:* Without loss of generality, we can assume that the multiple point has multiplicity 4, see Figure 12(b). Hence, the relations induced by this multiple point are:

$$yx_jxx_i = x_jxx_iy = xx_iyx_j = x_iyx_jx.$$

Moreover, we have: $zx_kx_i = x_kx_iz = x_izx_k$ and $[x_j, x_k] = e$, where x_i, x_j, x_k, x, y, z are the generators of the lines $\ell_i, \ell_j, \ell_k, \ell_1, \ell_2, \ell_3$ respectively. Hence, we have:

$$(x_i\backslash x_j)\backslash(x_i\backslash x_k) = zx_k = (x_j\backslash x_i)\backslash(x_j\backslash x_k),$$

which is $\mathcal{R}$-equivalent too.
Any other permutation of x_i, x_j, x_k satisfies the condition as well.

Hence, we have verified the equivalent version of the cube condition ($(u\backslash u')\backslash(u\backslash u'')$ and $(u'\backslash u)\backslash(u'\backslash u'')$ are $\mathcal{R}$-equivalent) for any triple of generators u, u', u'' in any case that the graph is triangle-free, so we are done. □

Hence, we have the following corollary:

Corollary 4.14. *Let $\mathcal{L}$ be a real arrangement whose fundamental group has a conjugation-free geometric presentation and its graph $G(\mathcal{L})$ is triangle-free. Then, the corresponding monoid is cancellative and has least common multiples.*

Remark 4.15. *The condition that the graph is triangle-free is essential, since if we take a line arrangement whose graph contains triangles, and its fundamental group has a conjugation-free geometric presentation, we can find a triple of generators for which the cube condition is not satisfied anymore.*

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References

[1] E. Artal-Bartolo, *Fundamental group of class of rational cuspidal curves*, Manuscripta Math. **93**, 273–281 (1997).
[2] E. Artal-Bartolo, *A curve of degree five with non-abelian fundamental group*, Topology Appl. **83**, 13–29 (1997).
[3] E. Artal-Bartolo and J. Carmona-Ruber, *Zariski pairs, fundamental groups and Alexander polynomials*, J. Math. Soc. Japan **50**(3), 521–543 (1998).
[4] E. Artal-Bartolo, J. Carmona-Ruber, J.I. Cogolludo and M.A. Marco, *Invariants of combinatorial line arrangements and Rybnikov's example*, in: Singularity theory and its applications, 1–34, Adv. Stud. Pure Math. **43**, Math. Soc. Japan, Tokyo, 2006.

[5] E. Artal-Bartolo, J.I. Cogolludo and H. Tokunaga, *A survey on Zariski pairs*, in: *Algebraic geometry in east Asia, Hanoi 2005*, Adv. Stud. Pure Math. **50**, Math. Soc. Japan, Tokyo, 1–100 (2008).

[6] O. Chisini, *Sulla identità birazionale delle funzioni algebriche di due variabili dotate di una medesima curva di diramazione*, Rend. Ist. Lombardo **77**, 339–356 (1944).

[7] R. Cordovil, *The fundamental group of the complement of the complexification of a real arrangement of hyperplanes*, Adv. Appl. Math. **21**, 481–498 (1998).

[8] P. Dehornoy, *Groups with a complemented presentation*, J. Pure Appl. Alg. **116**, 115–137 (1997).

[9] P. Dehornoy, *Groupes de Garside*, Ann. Scient. Ec. Norm. Sup. **35**, 267–306 (2002).

[10] P. Dehornoy, *Complete positive group presentations*, J. Alg. **268**, 156–197 (2003).

[11] P. Dehornoy, *The subword reversing method*, preprint (2009) [Available online: http://arxiv.org/abs/0912.4272].

[12] A.I. Degtyarev, *Quintics in* $\mathbb{CP}^2$ *with nonabelian fundamental group*, Algebra i Analiz **11**(5), 130–151 (1999) [Russian]; English translation: St. Petersburg Math. J. **11**(5), 809–826 (2000).

[13] M. Eliyahu, D. Garber and M. Teicher, *A conjugation-free geometric presentation of fundamental groups of arrangements*, Manuscripta Math. **133**(1–2), 247–271 (2010).

[14] K.M. Fan, *Direct product of free groups as the fundamental group of the complement of a union of lines*, Michigan Math. J. **44**(2), 283–291 (1997).

[15] M. Friedman and M. Teicher, *On non fundamental group equivalent surfaces*, Alg. Geom. Topo. **8**, 397–433 (2008).

[16] D. Garber and M. Teicher, *The fundamental group's structure of the complement of some configurations of real line arrangements*, Complex Analysis and Algebraic Geometry, edited by T. Peternell and F.-O. Schreyer, de Gruyter, 173-223 (2000).

[17] D. Garber, M. Teicher and U. Vishne, *Classes of wiring diagrams and their invariants*, J. Knot Theory Ramifications **11**(8), 1165–1191 (2002).

[18] T. Jiang and S.S.-T. Yau, *Diffeomorphic types of the complements of arrangements of hyperplanes*, Compositio Math. **92**(2), 133–155 (1994).

[19] E.R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55**, 255–260 (1933).

[20] V.S. Kulikov, *On Chisini's conjecture*, Izv. Ross. Akad. Nauk Ser. Mat. **63**(6), 83–116 (1999) [Russian]; *English translation:* Izv. Math. **63**, 1139–1170 (1999).

[21] V.S. Kulikov, *On Chisini's conjecture II*, Izv. Ross. Akad. Nauk Ser. Mat. **72**(5), 63–76 (2008) [Russian]; *English translation:* Izv. Math. **72**(5), 901–913 (2008).

[22] V.S. Kulikov and M. Teicher, *Braid monodromy factorizations and diffeomorphism types*, Izv. Ross. Akad. Nauk Ser. Mat. **64**(2), 89–120 (2000) [Russian]; *English translation:* Izv. Math. **64**(2), 311–341 (2000).

[23] S. MacLane, *Some interpretations of abstract linear independence in terms of projective geometry*, Amer. J. Math. **58**, 236–241 (1936).

[24] J. Milnor, *Morse Theory*, Ann. Math. Stud. **51**, Princeton University Press, Princeton, NJ (1963).

[25] B. Moishezon and M. Teicher, *Braid group technique in complex geometry I: Line arrangements in* $\mathbb{CP}^2$, Contemp. Math. **78**, 425–555 (1988).

[26] M. Oka and K. Sakamoto, *Product theorem of the fundamental group of a reducible curve*, J. Math. Soc. Japan **30**(4), 599–602 (1978).

[27] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften **300**, Springer-Verlag, Berlin, 1992.

[28] G. Rybnikov, *On the fundamental group of the complement of a complex hyperplane arrangement*, DIMACS Tech. Report **94-13**, 33–50 (1994).

[29] S. Wang and S.S.-T. Yau, *Rigidity of differentiable structure for new class of line arrangements*, Comm. Anal. Geom. **13**(5), 1057–1075 (2005).

[30] M. Yoshinaga, *Hyperplane arrangements and Lefschetz's hyperplane section theorem*, Kodai Math. J. **30**, 157–194 (2007).

[31] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51**, 305–328 (1929).

[32] O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. **58**, 607–619 (1936).

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