REDUCTION OF SINGULARITIES OF THREE-DIMENSIONAL LINE FOLIATIONS

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Dedicated to Heisuke Hironaka on the occasion of his 80th birthday.

Contents

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0. Introduction

We give a birational reduction of singularities for one dimensional foliations in ambient spaces of dimension three. To do this, we first prove the existence of a Local Uniformization in the sense of Zariski [19]. The reduction of singularities is then obtained by a gluing procedure for Local Uniformization similar to Zariski's one in [20].

Let K be the field of rational functions of a projective algebraic variety M_0 of dimension n over an algebraically closed field k of characteristic zero. We prove the following theorem

Theorem 1 (Local Uniformization)**.** *Assume that $n = 3$. Consider a k-valuation ν of K and a foliation by lines $\mathcal{L} \subset \mathrm{Der}_k K$. There is a composition of a finite sequence of blow-ups with non singular centers $M \to M_0$ such that $\mathcal{L}$ is log-elementary at the center $Y \subset M$ of ν.*

A *foliation by lines* (or simply a *foliation*) is any 1-dimensional K-vector subspace $\mathcal{L} \subset \mathrm{Der}_k K$. Recall that space of k-derivations $\mathrm{Der}_k K$ is a n-dimensional K-vector space. The notion of "log-elementary" comes from results in [4]. Let us explain it. Take a regular point P in a projective model M. We know that $\mathrm{Der}_k \mathcal{O}_{M,P} \subset \mathrm{Der}_k K$ is a free $\mathcal{O}_{M,P}$-module of rank n generated by the partial derivatives $\partial/\partial x_i$, $i = 1, 2, \ldots, n$, for any regular system of parameters $x_1, x_2, \ldots, x_n$ of the local ring $\mathcal{O}_{M,P}$. Moreover

$$\mathcal{L}_{M,P} = \mathcal{L} \cap \mathrm{Der}_k \mathcal{O}_{M,P}$$

is a free rank one sub-module of $\mathrm{Der}_k \mathcal{O}_{M,P}$ that we call the *local foliation induced by* $\mathcal{L}$ at M, P. We say that $\mathcal{L}$ is *non-singular* at P if $\mathcal{L}_{M,P} \not\subset \mathcal{M}_{M,P} \mathrm{Der}_k \mathcal{O}_{M,P}$, where $\mathcal{M}_{M,P} \subset \mathcal{O}_{M,P}$ is the maximal ideal. We say that $\mathcal{L}$ is *log-elementary* at P if there is a regular system of parameters $z_1, z_2, \ldots, z_n$, an integer $0 \leq e \leq n$ and $\xi \in \mathcal{L}_{M,P}$ of the form

$$\xi = \sum_{i=1}^{e} a_i z_i \frac{\partial}{\partial z_i} + \sum_{i=e+1}^{n} a_i \frac{\partial}{\partial z_i}, \ (a_i \in \mathcal{O}_{M,P}, i = 1, 2, \ldots, n)$$

with $a_j \notin \mathcal{M}^2_{M,P}$ for at least one index j. If $Y \subset M$ is an irreducible subvariety, we say that $\mathcal{L}$ is *non-singular at Y*, respectively, *log-elementary at Y*, if it is so at a generic point of Y. Note in particular that M must be non-singular at a generic point of Y.

Theorem 1 may be globalized as a consequence of a patching procedure developed by O. Piltant [12], which is an axiomatic adaptation of the one given by Zariski in the case of varieties [19]. We obtain the following birational result of reduction of singularities of foliations in an ambient space of dimension three

Theorem 2. *Assume that $n=3$ and let $\mathcal{L}\subset Der_k K$ be a foliation. Consider a birational model M_0 of K. There is a birational morphism $M\to M_0$ such that $\mathcal{L}$ is log-elementary at all the points of M.*

The reduction of singularities of foliations in an ambient space of dimension two is proved in the classical Seidenberg's paper [14]. In dimension three or higher one would like to be able to obtain *elementary singularities*, that is singularities with a non-nilpotent linear part. This is not possible in a birational way as an example of F. Sanz and F. Sancho shows (see for instance the introduction of [11]). There is no general result in dimension $n\geq 4$, except for the case of absolutely isolated singularities [3]. In dimension three Panazzolo [11] gives a global but non-birational result over the real numbers, getting elementary singularities after doing ramifications and blow-ups. There is also a preprint of Panazzolo and McQuillan, where they announce and adaptation to the results in [11] to the language of stacks. In [5] there is a local result, along a trajectory of a real vector field, obtained also by the use of ramifications and blow-ups. Finally, in [4] there is a strategy to solve by means of blow-ups a "formal version" of the local uniformization problem, where formal non-algebraic centers of blow-up are allowed.

Let us give an outline of the proof of Theorem 1. We organize the proof by taking account of the ranks and dimension of the valuation and of the existence of "maximal contact" with a formal series.

In Part I, we consider the case of a real valuation $\nu:K\setminus\{0\}\to\mathbb{R}$ with residual field $\kappa_\nu=k$. In the classical situations of Zariski's Local Uniformization [19] this one is considered to be the most difficult case. Note that since $\kappa_\nu=k$ the center of ν at any projective model is a closed point. Our first result is

Theorem 3. *Assume that $n=3$ and ν is a real k-valuation of K with residual field $\kappa_\nu=k$. There is a finite composition of blow-ups with non-singular centers $M\to M_0$ such that M is non-singular at the center P of ν at M and and one of the following properties holds*

(1) *$\mathcal{L}$ is log-elementary at P.*
(2) *There is $\hat f\in\widehat{\mathcal{O}}_{M,P}$ having transversal maximal contact with ν.*

A formal series $\hat f\in\widehat{\mathcal{O}}_{M,P}$ has *transversal maximal contact with ν* if it is the Krull-limit of a sequence $f_i\in\mathcal{O}_{M,P}$ with strictly increasing values and moreover we have the following property of transversality: there is a part of a regular system of parameters $x_1,x_2,\ldots,x_r$ of $\mathcal{O}_{M,P}$ such that the values $\nu(x_1),\nu(x_2),\ldots,\nu(x_r)$ are $\mathbb{Z}$-independent, where r is the rational rank of ν, and $x_1,x_2,\ldots,x_r,\hat f$ is a part of a regular system of parameters of the complete local ring $\widehat{\mathcal{O}}_{M,P}$.

In order to prove Theorem 3, we work over the rational rank r of ν and we study the three following cases in an ordered way:

(1) $r=n$. Here we get $\mathcal{L}$ elementary for any ambient dimension n. This is a combinatorial case with few differences with respect to the classical situations of varieties.

(2) $r = n - 1$. The statement of Theorem 3 is valid for any n. We use Newton Polygon technics to give the proof. If $n = 2$ the result is slightly stronger: we get either maximal contact of a non-singular foliation. This will be useful in the next case.
(3) $r = 1, n = 3$. This is the hardest situation. We have important difficulties due to the fact that ν is not a discrete valuation.

We end Part I by giving a proof of

Theorem 4. *Assume that $n = 3$. Let ν be a real k-valuation of K with residual field $\kappa_\nu = k$ and suppose that $\hat{f} \in \widehat{\mathcal{O}}_{M,P}$ has transversal maximal contact with ν. There is a finite composition of blow-ups with non-singular centers $M \to M_0$ such that $\mathcal{L}$ is log-elementary at the center P of ν.*

Part II is devoted to the remaining cases. We obtain many of the results by an inductive use of the technics in Part I. In Part III we prove the validity of Piltant's patching axioms and hence we obtain the proof of Theorem 2.

Part 1. Zero dimensional arquimedean valuations

In all this part $\nu : K \setminus \{0\} \to \Gamma$ denotes a valuation such that $\Gamma \subset (\mathbb{R}, +)$ and $\kappa_\nu = k$. In other words, the (arquimedean) rank of ν is one and it is a zero-dimensional k-valuation of K. We denote by r the *rational rank* of ν, that is, the maximum number of $\mathbb{Z}$-linearly independent elements in the value group of ν. We know that $1 \leq r \leq n$ by Abhyankar's inequality. In particular, for the case $n = 3$ we have the possibilities $r = 3, r = 2$ and $r = 1$.

1. Parameterized regular local models

A *parameterized regular local model* $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, \mathbf{y}))$ for K, ν is a pair with $\mathcal{O} = \mathcal{O}_{M,P}$, where M is a projective model of K, the point $P \in M$ is the center of ν in M and the sequence

$$(z_1, z_2, \ldots, z_n) = \mathbf{z} = (\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, x_r, y_{r+1}, y_{r+2}, \ldots, y_n)$$

is a regular system of parameters of $\mathcal{O}$ such that $\nu(x_1), \nu(x_2), \ldots, \nu(x_r)$ are $\mathbb{Z}$-linearly independent values. We call $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ the *independent variables* and $\mathbf{y} = (y_{r+1}, y_{r+2}, \ldots, y_n)$ the *dependent variables*. The existence of parameterized regular local models is a consequence of Hironaka's reduction of singularities [10]. More precisely, we have

Proposition 1. *Given a projective model M_0 of K, there is a composition of a finite sequence of blow-ups with non-singular centers $M \to M_0$ such that the center P of ν at M provides a local ring $\mathcal{O} = \mathcal{O}_{M,P}$ for a parameterized regular local model $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, \mathbf{y}))$.*

Proof. By Hironaka's reduction of the singularities (see [10]) of M_0, we get a non-singular projective model M' of K jointly with a birational morphism $M' \to M_0$ that is the composition of a finite sequence of blow-ups with non-singular centers. Consider the local ring $\mathcal{O}_{M',P'}$ of M' at the center P' of ν and chose elements $f_1, f_2, \ldots, f_r \in \mathcal{O}_{M,P}$ such that $\nu(f_1), \nu(f_2), \ldots, \nu(f_r)$ are $\mathbb{Z}$-linearly independent. Another application of Hironaka's theorem gives a birational morphism $M \to M'$, that is also a composition of a finite sequence of blow-ups with non-singular centers, such that$f = \prod_{i=1}^{r} f_i$, is a monomial (times a unit) in a suitable regular system of parameters at any point of M and hence each of the f_i, $i = 1, 2, \dots, r$ is also a monomial (times a unit) in that regular system of parameters. In particular, if P is the center of ν at M there is a regular system of parameters $\mathbf{z} = (z_1, z_2, \dots, z_n)$ of $\mathcal{O}_{M,P}$ such that

$$f_i = U_i \mathbf{z}^{\mathbf{m}_i}, U_i \in \mathcal{O}_{M,P} \setminus \mathcal{M}_{M,P}, \text{ for } i = 1, 2, \dots, n,$$

where $\mathbf{m}_i = (m_{i,1}, m_{i,2}, \dots, m_{i,n}) \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{z}^{\mathbf{m}_i} = z_1^{m_{i,1}} z_2^{m_{i,2}} \cdots z_n^{m_{i,n}}$. In terms of values, we have $\nu(f_i) = \sum_{j=1}^n m_{ij}\nu(z_j)$. This implies that there are r variables among the z_j whose values are $\mathbb{Z}$-linearly independent. □

1.1. **Coordinate changes and blow-ups.** Take a parameterized regular local model $\mathcal{A} = (\mathcal{O}, \mathbf{z})$. We will do "atomic" transformations of $\mathcal{A}$ of two types: *coordinate changes in the dependent variables* and *coordinate blow-ups with codimension two centers*. Our "basic" transformations, called *Puiseux packages* will be certain sequences of coordinate changes and blow-ups.

Let us describe the two types of transformations. Each one produces a parameterized local model $\mathcal{A}' = (\mathcal{O}', \mathbf{z}')$.

Coordinate changes in the dependent variables. Consider j with $r + 1 \leq j \leq n$. A *j-coordinate change* is such that $z'_i = z_i$ for $i \neq j$ and y'_j is one of the following

a) $y'_j = y_j - c\mathbf{x}^{\mathbf{a}}$, $\nu(y'_j) \geq \nu(y_j)$, $c \in k$, $\mathbf{a} \in \mathbb{Z}^r_{\geq 0}$.
b) $y'_j = y_j + y_s$, for another $s \neq j$ with $r + 1 \leq s \leq n$.

If $r = n$ we do not do coordinate changes.

Coordinate blow-ups with codimension two centers. Take a pair i, j of distinct indices with $1 \leq i \leq r$ and $1 \leq j \leq n$. We say that $\mathcal{A}' = (\mathcal{O}, \mathbf{z}')$ is obtained from $\mathcal{A}$ by an (i, j)-*blow-up* if the following holds. First $z'_s = z_s$ for any $s \notin \{i, j\}$. In order two determine z'_i, z'_j we have three cases

(1) $\nu(x_i) < \nu(z_j)$. We put $x'_i = x_i$ and $z'_j = z_j/x_i$.
(2) $\nu(x_i) > \nu(z_j)$. We put $x'_i = x_i/z_j$ and $z'_j = z_j$.
(3) $\nu(x_i) = \nu(z_j)$. Note that in this case we necessarily have that $j \geq r + 1$ and hence $z_j = y_j$. Since $\kappa_\nu = k$, there is $c \in k$ with $\nu(y_j/x_i - c) > 0$. We put $x'_i = x_i$ and $y'_j = y_j/x_i - c$.

The first two cases above are called *combinatorial* and the third one corresponds to a blow-up *with translation*. If x_i, x_j are independent variables, we have always a combinatorial case, since $\nu(x_i) \neq \nu(x_j)$.

The local ring $\mathcal{O}'$ is the (algebraic) localization of $\mathcal{O}[\mathbf{z}']$ at the ideal $(\mathbf{z}')$.

In the case that $j \geq r + 1$ the above blow-up will also be referred as a *j-blow-up*.

Remark 1. Let M be a projective model for K such that $\mathcal{O} = \mathcal{O}_{M,P}$, where P is the center of ν at M. There is a closed irreducible algebraic subvariety $Y \subset M$ of codimension two defined by the equations $x_i = z_j = 0$ that is non singular at P. Let $\pi : M' \to M$ be the blow-up of M with center Y and let P' be the center of ν at M'. Then $\mathcal{O}' = \mathcal{O}_{M',P'}$.

1.2. **Puiseux packages of blow-ups.** Let $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, \mathbf{y}))$ be a parameterized regular local model. Consider a dependent variable y_j. Then $\nu(y_j)$ can be expressed uniquely as a $\mathbb{Q}$-linear combination of $\nu(x_1), \nu(x_2), \dots, \nu(x_r)$. More precisely, there are unique integer numbers $d > 0$ and $p_1, p_2, \dots, p_r$ such that

$$d\nu(y_j) = p_1\nu(x_1) + p_2\nu(x_2) + \cdots + p_r\nu(x_r)$$

and $gcd(d; p_1, p_2, \ldots, p_r) = 1$. In particular, the rational function

$$\Phi = y_j^d / \mathbf{x}^{\mathbf{p}}, \quad \mathbf{x}^{\mathbf{p}} = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r},$$

has value equal to zero. We call this function *the j-contact rational function* and d is the *j-ramification index for $\mathcal{A}$*. Note that there is a unique scalar $c \in k$ such that $\nu(\Phi - c) > 0$, since $\kappa_\nu = k$.

A coordinate (i, s)-blow-up is said to be *j-admissible* if either $1 \leq s \leq r$ with $p_i \neq 0 \neq p_s$ or $p_i \neq 0$ and $s = j$.

Remark 2. Assume that $\mathcal{A}'$ has been obtained from $\mathcal{A}$ by a j-admissible coordinate (i, s)-blow-up. There are two possibilities:

A) The blow-up is combinatorial. In this case Φ is also the j-contact rational function for $\mathcal{A}'$.
B) The blow-up has a translation. Then $\Phi = y_j / x_i$ and $s = j$. Moreover, we have $y'_j = \Phi - c$.

Definition 1. *A j-*Puiseux package *starting at $\mathcal{A}$ is a finite sequence*

$$\mathcal{A} = \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_N = \mathcal{A}'$$

where $\mathcal{A}_{t-1} \to \mathcal{A}_t$ is a combinatorial j-admissible blow-up for $t = 1, 2, \ldots, N-1$ and $\mathcal{A}_{N-1} \to \mathcal{A}_N$ is a j-admissible blow-up with translation. In this situation, we say that $\mathcal{A}'$ has been obtained from $\mathcal{A}$ by a j-Puiseux package.

Note that $y'_j = \Phi - c$, in view of the above Remark.

Proposition 2. *Given $\mathcal{A}$ and j, with $r < j \leq n$, there is at least one j-Puiseux package starting at $\mathcal{A}$.*

Proof. There are many known algorithms for doing this (see [10, 16, 15, 18, 2]). We include a proof for the sake of completeness. Let us write

$$\Phi = \frac{y_j^d \mathbf{x}^{\mathbf{q}}}{\mathbf{x}^{\mathbf{r}}},$$

where $q_i = -p_i$ if $p_i < 0$ and $q_i = 0$, otherwise and, in the same way, we put $r_i = p_i$ if $p_i > 0$ and $r_i = 0$ otherwise. There are two possibilities: $\mathbf{q} \neq 0$ or $\mathbf{q} = 0$. Note that we always have that $\mathbf{r} \neq 0$, since $\nu(z_s) > 0$ for all s. Assume first that $\mathbf{q} \neq 0$. Let us choose indices $1 \leq i, s \leq r$ such that $p_i p_s < 0$. We do the (i, s)-blow-up. The sum $|p_i| + |p_s|$ decreases. We continue and one of the independent variables x_i or x_s disappears. In this way we get that $\mathbf{q} = 0$. Now, we consider an index i with $p_i \neq 0$ and we do the (i, j)-blow-up. This blow-up is combinatorial except in the case that $\Phi = y_j / x_i$. If we are not in this case, then $d + p_i$ decreases and finally the variable x_i disappears. We obtain that $\Phi = y_j / x_i$. The only possible j-admissible coordinate blow-up is the (i, j)-blow-up. Moreover, $\nu(y_j) = \nu(x_i)$ and hence it is a coordinate blow-up with translation. □

Remark 3. We are interested in the following features of Puiseux packages. Let us start with $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, \mathbf{y}))$ and assume that $\mathcal{A}' = (\mathcal{O}', \mathbf{z}' = (\mathbf{x}', \mathbf{y}'))$ has been obtained from $\mathcal{A}$ by a j-Puiseux package. Let $\Phi = y_j^d / \mathbf{x}^{\mathbf{p}}$ be the j-contact function and suppose that $\nu(\Phi - c) > 0$. For $s \notin \{i; p_i \neq 0\} \cup \{j\}$ we have that $z_s = z'_s$. Moreover $y'_j = \Phi - c$ and there are monomial expressions

$$z_s = \left(\prod_{i=1}^{r} {x'_i}^{b_i^s} \right) \Phi^{b_j^s}; \ s \in \{i; p_i \neq 0\} \cup \{j\}.$$

This is proved by induction on the number of j-admissible coordinate blow-ups of the j-Puiseux package.

1.3. **Statements in terms of parameterized regular local models.** Consider a foliation by lines $\mathcal{L} \subset \mathrm{Der}_k K$ and a parameterized regular local model $\mathcal{A} = (\mathcal{O}, \mathbf{z})$. The *local foliation induced by* $\mathcal{L}$ *at* $\mathcal{A}$ is defined by

$$\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cap \mathrm{Der}_k \mathcal{O}.$$

Obviously $\mathcal{L}_{\mathcal{A}} = \mathcal{L}_{M,P}$ for any projective model M for K such that $\mathcal{O} = \mathcal{O}_{M,P}$. In the next sections we shall prove the following proposition

Proposition 3. *Assume that $n = 3$. Let ν be a real k-valuation of K with $\kappa_\nu = k$ and take a foliation $\mathcal{L} \subset Der_k K$. Consider a parameterized regular local model $\mathcal{A} = (\mathcal{O}, \mathbf{z})$ for K, ν. There is a finite sequence of coordinate changes and blow-ups such that the parameterized regular local model $\mathcal{A}' = (\mathcal{O}', \mathbf{z}')$ obtained from $\mathcal{A}$ satisfies one of the following properties:*

1. *The foliation $\mathcal{L}_{\mathcal{A}'}$ is log-elementary.*
2. *There is $\hat{f} \in \widehat{\mathcal{O}}'$ having transversal maximal contact with ν.*

This result implies Theorem 3. Indeed, we already know that there is a birational morphism $M \to M_0$, composition of blow-ups with nonsingular centers, such that M is non-singular and the local ring $\mathcal{O}_{M,P}$ of M at the center P of ν supports a parameterized regular local model $\mathcal{A}$. The sequence of blow-ups that gives $\mathcal{A}'$ may be substituted, by Hironaka's reduction of singularities, by another sequence of blow-ups with non-singular centers, since the original blow-ups are non-singular (in fact they are non-singular and two dimensional) at the corresponding centers of the valuation at each projective model.

Next sections are devoted to proving Proposition 3.

2. The combinatorial case ($r = n$)

The following Proposition 4 implies Proposition 3 for the case of maximal rational rank. Let us note that in Proposition 4 there is no assumption about the (arquimedean) rank of the valuation nor on the fact that $\kappa_\nu = k$. Indeed if $n = r$ we know that κ_ν is an algebraic extension of k and thus $\kappa_\nu = k$ since we assume the base field k to be algebraically closed.

Proposition 4. *Let ν be a k-valuation of K with maximal rational rank $r = n$. Take a foliation $\mathcal{L} \subset Der_k K$ and a parameterized regular local model $\mathcal{A}$ for K, ν. There is a parameterized regular local model $\mathcal{A}'$ obtained from $\mathcal{A}$ by a finite sequence of coordinate blow-ups such that $\mathcal{L}_{\mathcal{A}'}$ is elementary.*

Let us recall that $\mathcal{L}_{\mathcal{A}}$ is *elementary* if there is a vector field $\xi \in \mathcal{L}_{\mathcal{A}}$ having a non-nilpotent linear part. If $\xi \in \mathrm{Der}_k \mathcal{O}$ is singular, that is $\xi(\mathcal{O}) \subset \mathcal{M}$, the *linear part* $L\xi$ is intrinsically defined as the $k = \mathcal{O}/\mathcal{M}$-linear map

$$L\xi : \mathcal{M}/\mathcal{M}^2 \to \mathcal{M}/\mathcal{M}^2$$

given by $f + \mathcal{M}^2 \mapsto \xi f + \mathcal{M}^2$. Note that "elementary" implies "log-elementary". Note also that a vector field $\xi \in \mathrm{Der}_k \mathcal{O}$ of the form

$$\xi = \sum_{i=1}^{n} f_i x_i \frac{\partial}{\partial x_i}, \quad b_i \in \mathcal{O}, \tag{1}$$

has a non-nilpotent linear part if and only if one of the f_i is a unit in $\mathcal{O}$.

2.1. **Newton polyhedron.** Note that $\mathbf{z}=\mathbf{x}$, since all the variables have $\mathbb{Q}$-linearly independent values. Any element $f\in\mathcal{O}$ can be expanded in a formal series

$$f=\sum f_{\mathbf{a}}\mathbf{x}^{\mathbf{a}};\quad f_{\mathbf{a}}\in k.$$

The *support of* f is defined by $\mathrm{Supp}(f;\mathbf{x})=\{\mathbf{a};f_{\mathbf{a}}\neq 0\}\subset\mathbb{Z}^n_{\geq 0}$. For a vector field $\xi\in\mathrm{Der}_k\mathcal{O}$ written as in formula (1), the support is

$$\mathrm{Supp}(\xi;\mathbf{x})=\cup_{i=1}^n\mathrm{Supp}(f_i;\mathbf{x}).$$

The *Newton polyhedron* $\mathcal{N}(\xi;\mathbf{x})$ is the convex hull in $\mathbb{R}^n$ of the set $\mathrm{Supp}(\xi;\mathbf{x})+\mathbb{R}^n_{\geq 0}$.

The local foliation $\mathcal{L}_{\mathcal{A}}$ contains a vector field ξ of the form (1) such that the coefficients $f_i\in\mathcal{O}$ have no common factor in $\mathcal{O}$, that we call an $\mathbf{x}$-*generator* of $\mathcal{L}_{\mathcal{A}}$. To see this, take any $\eta\in\mathcal{L}_{\mathcal{A}}$, then $(\prod_{i=1}^n x_i)\eta$ is of the form (1) and now it is enough to divide by the gcd of the f_i.

We define the *Newton polyhedron* $\mathcal{N}(\mathcal{L};\mathbf{x})$ by $\mathcal{N}(\mathcal{L};\mathbf{x})=\mathcal{N}(\xi;\mathbf{x})$, where ξ is an $\mathbf{x}$-generator of $\mathcal{L}_{\mathcal{A}}$.

Remark 4. The Newton polyhedron $\mathcal{N}(\mathcal{L};\mathbf{x})$ has vertices in $\mathbb{Z}^n_{\geq 0}$. Since the coefficients f_i have no common factor (and “a fortiori” they are free of a monomial common factor) the only $\mathbf{v}\in\mathbb{R}^n_{\geq 0}$ such that

$$\mathcal{N}(\mathcal{L};\mathbf{x})\subset\mathbf{v}+\mathbb{R}^n_{\geq 0}$$

is $\mathbf{v}=0$. Thus, if $\mathcal{N}(\mathcal{L};\mathbf{x})$ has only one vertex $\mathbf{v}$, then $\mathbf{v}=0$ and the vector field ξ has a non-nilpotent linear part. This implies that $\mathcal{L}_{\mathcal{A}}$ is elementary.

2.2. **The effect of a blow-up.** Let $\mathcal{A}'=(\mathcal{O},\mathbf{z}')$ be obtained from $\mathcal{A}$ by an (i,s)-*blow-up*. Recall that there are no dependent variables and hence it is a combinatorial blow-up. If $\nu(x_i)<\nu(x_s)$, we have $x'_s=x_s/x_i$ and $x'_s=x_s$, for $s\neq j$. Consider the affine function $\sigma^i_{is}:\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$\sigma^i_{is}(\mathbf{a})_t=\begin{cases} a_i+a_j, & \text{if } t=s\\ a_s, & \text{if } t\neq s\end{cases}$$

Take $\mathbf{v}\in\mathbb{R}^n_{\geq 0}$ such that $\sigma^i_{ij}\left(\mathcal{N}(\mathcal{L};\mathbf{x})\right)$ is inscribed in the orthant $\mathbf{v}+\mathbb{R}^n_{\geq 0}$. Then the Newton polyhedron $\mathcal{N}(\mathcal{L};\mathbf{x}')$ is obtained as

$$\mathcal{N}(\mathcal{L};\mathbf{x}')=\left(\sigma^i_{is}\left(\mathcal{N}(\mathcal{L};\mathbf{x})\right)-\mathbf{v}\right)+\mathbb{R}^n_{\geq 0}.$$

In fact, the behavior of the Newton polyhedron is the same one as the behavior of the Newton polyhedron of the ideal generated by the coefficients f_i. In the case $\nu(x_i)>\nu(x_s)$, we do the same argument with the corresponding affine map σ^s_{is}.

2.3. **End of the proof of Proposition 4.** We can use the same idea as in the proof of Proposition 2. Let N be the number of vertices of $\mathcal{N}(\mathcal{L};\mathbf{x})$. After doing an (i,j)-blow-up, we obtain that $N'\leq N$. If $N=1$, we are done. Assume that $N\geq 2$. Take two distinct vertices $\mathbf{a}$ and $\mathbf{b}$ of $\mathcal{N}(\mathcal{L};\mathbf{x})$ and let $\mathbf{v}$ be the element in $\mathbb{Z}^n_{\geq 0}$ such that the set $\{\mathbf{a},\mathbf{b}\}$ is inscribed in $\mathbf{v}+\mathbb{R}^n_{\geq 0}$. In other terms, the monomial $\mathbf{x}^{\mathbf{v}}$ is the gcd of $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$. Put $\tilde{\mathbf{a}}=\mathbf{a}-\mathbf{v}$ and $\tilde{\mathbf{b}}=\mathbf{b}-\mathbf{v}$. Note that for any index t we have $\tilde{a}_t\tilde{b}_t=0$ and also $\tilde{\mathbf{a}}\neq 0\neq\tilde{\mathbf{b}}$. Choose indices i,s with $\tilde{a}_i\tilde{b}_s\neq 0$. Do the (i,s)-blow-up. Assuming that $N'=N$, the set of indices

$$\{t;\ \tilde{a}_t\neq 0 \text{ or } \tilde{b}_t\neq 0\}$$

is contained in the corresponding one after blow-up. If the two sets coincide, the amount $\tilde{a}_i+\tilde{b}_s$ decreases strictly. This ends the proof.

Remark 5. The same kind of combinatorial game, but using centers of any codimension, with a "permissibility" additional condition, is called the Weak Hironaka's Game [15].

3. The Newton-Puiseux Polygon

Let us assume in this section that $r = n-1$, $\kappa_\nu = k$ and take a parameterized regular local model $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, y))$. Note that since $r = n-1$, there is only one dependent variable y.

Consider an element $f \in y^{-1}\mathcal{O}$, that we write $f = \sum_{s=-1}^{\infty} h_s(\mathbf{x}) y^s$, where $h_s(\mathbf{x})$ is a formal series $h_s(\mathbf{x}) \in k[[\mathbf{x}]] \cap \mathcal{O}$. The *Newton-Puiseux support of* f is the set

$$\mathrm{NPSup}(f; \mathbf{x}, y) = \{(\nu(h_s), s); h_s \neq 0\} \subset \Gamma \times \mathbb{Z}_{\geq -1}.$$

We denote by $\alpha(f; \mathbf{x}, y)$ the minimum abscissa of the Newton Puiseux support, that is $\alpha(f; \mathbf{x}, y) = \min\{(\nu(h_s))\}$. The *main height* $\hbar(f; \mathbf{x}, y)$ is the minimum of the s such that $\nu(h_s) = \alpha(f; \mathbf{x}, y)$. Let $\delta(f; \mathbf{x}, y)$ be the minimum of the values $\nu(h_s) + s\nu(y)$. The *critical segment* $\mathcal{C}(f; \mathbf{x}, y)$ is the set of the s such that

$$\nu(h_s) + s\nu(y) = \delta(f; \mathbf{x}, y).$$

The *main height* $\chi(f; \mathbf{x}, y)$ is the highest s in the critical segment. Let us note that $\chi(f; \mathbf{x}, y) \leq \hbar(f; \mathbf{x}, y)$.

Consider a finite list $\mathbf{f} = (f_1, f_2, \ldots, f_t)$ of elements $f_j \in y^{-1}\mathcal{O}$. The Newton-Puiseux support $\mathrm{NPSup}(\mathbf{f}; \mathbf{x}, y)$ is the set of (u, s), where u is the minimum of the u_j such that $(u_j, s) \in \mathrm{NPSup}(f_j; \mathbf{x}, y)$, for $j = 1, 2, \ldots, t$. We obtain in this way a definition for $\alpha(\mathbf{f}; \mathbf{x}, y)$, $\hbar(\mathbf{f}; \mathbf{x}, y)$,$\delta(\mathbf{f}; \mathbf{x}, y)$ and $\chi(\mathbf{f}; \mathbf{x}, y)$ since these invariants depend only on the Newton-Puiseux support.

3.1. **Newton-Puiseux Polygon of a foliation.** Consider the free $\mathcal{O}$-module $\mathrm{Der}_k\mathcal{O}[\log \mathbf{x}]$ whose elements are the vector fields of the form

$$\xi = \sum_{i=1}^{n-1} f_i(\mathbf{x}, y) x_i \frac{\partial}{\partial x_i} + g(\mathbf{x}, y) \frac{\partial}{\partial y} \tag{2}$$

where $g \in \mathcal{O}$, $f_i \in \mathcal{O}$, $i = 1, 2, \ldots, n-1$. Such vector fields will be called **x**-*logarithmic vector fields*, or simply **x**-*vector fields*. Let us denote $f_n = g/y$ and $\mathbf{f} = (f_1, f_2, \ldots, f_n)$. We define $\mathrm{NPSup}(\xi; \mathbf{x}, y) = \mathrm{NPSup}(\mathbf{f}; \mathbf{x}, y)$ and

$$\alpha(\xi; \mathbf{x}, y) = \alpha(\mathbf{f}; \mathbf{x}, y); \qquad \hbar(\xi; \mathbf{x}, y) = \hbar(\mathbf{f}; \mathbf{x}, y)$$
$$\delta(\xi; \mathbf{x}, y) = \delta(\mathbf{f}; \mathbf{x}, y); \qquad \chi(\xi; \mathbf{x}, y) = \chi(\mathbf{f}; \mathbf{x}, y).$$

Given a foliation $\mathcal{L} \subset \mathrm{Der}_k K$, we consider the local **x**-*logarithmic foliation* $\mathcal{L}_\mathcal{A}[\log \mathbf{x}]$ *at* $\mathcal{A}$ defined by

$$\mathcal{L}_\mathcal{A}[\log \mathbf{x}] = \mathcal{L} \cap \mathrm{Der}_k\mathcal{O}[\log \mathbf{x}]. \tag{3}$$

We define the *main height* $\hbar(\mathcal{L}; \mathcal{A})$, respectively the *critical height* $\chi(\mathcal{L}; \mathcal{A})$, to be the minimum of the $\hbar(\xi; \mathbf{x}, y)$, respectively $\chi(\xi; \mathbf{x}, y)$, where $\xi \in \mathcal{L}_\mathcal{A}[\log \mathbf{x}]$. Note that

$$\hbar(\mathcal{L}; \mathcal{A}) \geq \chi(\mathcal{L}; \mathcal{A}) \geq -1.$$

These ones are the main invariants we shall use to control the singularity of $\mathcal{L}$ after performing a Puiseux package.

3.2. **The initial parts.** Consider an element $h=\sum_{\mathbf{m}}\lambda_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}\in\mathcal{O}\cap k[[\mathbf{x}]]$. Since the values $\nu(x_i)$, $i=1,2,\ldots,n-1$ are $\mathbb{Q}$-linearly independent, there is exactly one exponent $\mathbf{m}_0$ such that $\nu(\lambda_{\mathbf{m}_0}\mathbf{x}^{\mathbf{m}_0})=\nu(h)$. Moreover, if $\tilde{h}=h-\lambda_{\mathbf{m}_0}\mathbf{x}^{\mathbf{m}_0}$ then $\nu(\tilde{h})>\nu(h)$. Take an element $\gamma\in\Gamma$ with $\gamma\leq\nu(h)$. We define the *γ-initial form* $\mathrm{In}_\gamma(h)$ by $\mathrm{In}_\gamma(h)=0$ if $\gamma<\nu(h)$ and $\mathrm{In}_{\nu(h)}(h)=\lambda_{\mathbf{m}_0}\mathbf{x}^{\mathbf{m}_0}$ if $\gamma=\nu(h)$. Given a list $\mathbf{h}=(h_1,h_2,\ldots,h_n)$ of elements $h_j=h_j(\mathbf{x})\in k[[x]]\cap\mathcal{O}$, and $\gamma\in\Gamma$ with $\gamma\leq\min\{\nu(h_j(\mathbf{x}));j=1,2,\ldots,n\}$ we put

$$\mathrm{In}_\gamma(\mathbf{h};\mathbf{x})=(\mathrm{In}_\gamma(h_1;\mathbf{x}),\mathrm{In}_\gamma(h_2;\mathbf{x}),\ldots,\mathrm{In}_\gamma(h_n;\mathbf{x})).$$

If we have a vector field of the form

$$\eta=\sum_{j=1}^{n-1}h_j(\mathbf{x})x_j\frac{\partial}{\partial x_j}+h_n(\mathbf{x})y\frac{\partial}{\partial y}$$

and $\gamma\leq\min\{\nu(h_j(\mathbf{x}));j=1,2,\ldots,n\}$ we put

$$\mathrm{In}_\gamma(\eta;\mathbf{x})=\sum_{j=1}^{n-1}\mathrm{In}_\gamma(h_j;\mathbf{x})x_j\frac{\partial}{\partial x_j}+\mathrm{In}_\gamma(h_n;\mathbf{x})y\frac{\partial}{\partial y}.$$

Take an $\mathbf{x}$-vector field $\xi\in\mathrm{Der}\mathcal{O}[\log\mathbf{x}]$ that we write as in equation (2). Put $f_j=\sum_{s=-1}^{\infty}h_{js}(\mathbf{x})y^s$, $j=1,2,\ldots,n$. We have $\xi=\sum_{s=-1}^{n}y^s\eta_s$, where

$$\eta_s=\sum_{j=1}^{n-1}h_{js}(\mathbf{x})x_j\frac{\partial}{\partial x_j}+h_{ns}(\mathbf{x})y\frac{\partial}{\partial y};\quad s=-1,0,1,\ldots. \tag{4}$$

Put $\delta=\delta(\xi;\mathbf{x},y)=\min_{j,s}\{\nu(y^sh_{js}(\mathbf{x}))\}$. We define the initial form $\mathrm{In}(\xi;\mathbf{x},y)$ as

$$\mathrm{In}(\xi;\mathbf{x},y)=\sum_{s=-1}^{\infty}y^s\mathrm{In}_{\delta-s\nu(y)}(\eta_s;\mathbf{x}).$$

Let us note that if $\tilde{\xi}=\xi-\mathrm{In}(\xi;\mathbf{x},y)$, then $\delta(\tilde{\xi};\mathbf{x},y)>\delta(\xi;\mathbf{x},y)$. Note also that if $\chi=\chi(\xi;\mathbf{x},y)$ is the critical height, then $\mathrm{In}_{\delta-s\nu(y)}(\eta_s;\mathbf{x})=0$ for $s>\chi$ and $\mathrm{In}_{\delta-\chi\nu(y)}(\eta_\chi;\mathbf{x})\neq 0$. In particular $\mathrm{In}(\xi;\mathbf{x},y)$ is a finite sum

$$\mathrm{In}(\xi;\mathbf{x},y)=\sum_{s=-1}^{\chi}y^s\mathrm{In}_{\delta-s\nu(y)}(\eta_s;\mathbf{x}).$$

Now we are going to give a particular expression of $\mathrm{In}(\xi;\mathbf{x},y)$ in terms of the contact rational function $\Phi=y^d/\mathbf{x}^{\mathbf{p}}$.

Let us take an index s such that $\mathrm{In}_{\delta-s\nu(y)}(\eta_s;\mathbf{x})\neq 0$ and in particular $s\leq\chi$. Write $\mathrm{In}_{\delta-s\nu(y)}(\eta_s;\mathbf{x})=\mathbf{x}^{\mathbf{q}(s)}\Lambda_s$, where Λ_s is the linear vector field

$$\Lambda_s=\sum_{j=1}^{n-1}\lambda_{js}x_j\frac{\partial}{\partial x_j}+\lambda_{ns}y\frac{\partial}{\partial y}$$

and $\mathbf{q}(s)\in\mathbb{Z}^{n-1}_{\geq 0}$. Put $\mathbf{r}(s)=\mathbf{q}(s)-\mathbf{q}(\chi)$. We have

$$\nu(\mathbf{x}^{\mathbf{r}(s)})=(\chi-s)\nu(y)=\frac{\chi-s}{d}\nu(\mathbf{x}^{\mathbf{p}}),$$

this implies that $((\chi-s)/d)\mathbf{p}=\mathbf{r}(s)$ and thus $((\chi-s)/d)\mathbf{p}\in\mathbb{Z}^{n-1}$. Since the coefficients $p_1,p_2,\ldots,p_{n-1}$ have no common factor, we have that $(\chi-s)/d\in\mathbb{Z}$.

Put $t=(\chi-s)/d\in\mathbb{Z}_{\geq 0}$; note that $t\leq\varrho$, where $\varrho\in\mathbb{Z}_{\geq 0}$ is the biggest integer bounded above by $(\chi+1)/d$.

We may write $\mathrm{In}(\xi;\mathbf{x},y)$ as follows:

$$\mathrm{In}(\xi;\mathbf{x},y)=\mathbf{x}^{\mathbf{q}(\chi)}y^{\chi}\sum_{s=-1}^{\chi}\frac{1}{\Phi^{(\chi-s)/d}}\Lambda_s=\mathbf{x}^{\mathbf{q}(\chi)}y^{\chi}\sum_{t=0}^{\varrho}\frac{1}{\Phi^{t}}\Lambda_{\chi-dt}. \tag{5}$$

In order to simplify the notation, let us rename $\Delta_t=\Lambda_{\chi-dt}$. Then

$$\mathbf{x}^{-\mathbf{q}(\chi)}y^{-\chi}\Phi^{\varrho}\mathrm{In}(\xi;\mathbf{x},y)=\sum_{t=0}^{\varrho}\Phi^{\varrho-t}\Delta_t. \tag{6}$$

We recall that $\Delta_0\neq 0$.

3.3. **The expression of the derivatives after a Puiseux package.** Assume that $\mathcal{A}'=(\mathcal{O}',\mathbf{z}'=(\mathbf{x}',y'))$ has been obtained from $\mathcal{A}=(\mathcal{O},\mathbf{z}=(\mathbf{x},y))$ by a Puiseux package. Let $\Phi=y^d/\mathbf{x}^{\mathbf{p}}$ be the contact rational function. By remark 3 we have that $y'=\Phi-c$ and there is a matrix $B=(b_i^s)$ with determinant 1 or -1 and positive integer coefficients such that

$$z_s=\left(\prod_{i=1}^{n-1}{x'_i}^{b_i^s}\right)\Phi^{b_n^s};\quad s=1,2,\ldots,n.$$

Moreover if $p_s=0$ we know that $x_s=x'_s$, that is $b_i^s=0$ if $i\neq s$ and $b_s^s=1$. This implies that

$$x'_i\frac{\partial}{\partial x'_i}=\sum_{s=1}^{n-1}b_i^s x_s\frac{\partial}{\partial x_s}+b_i^n y\frac{\partial}{\partial y};\quad i=1,2,\ldots,n-1, \tag{7}$$

$$\Phi\frac{\partial}{\partial y'}=\sum_{s=1}^{n-1}b_n^s x_s\frac{\partial}{\partial x_s}+b_n^n y\frac{\partial}{\partial y} \tag{8}$$

Let $B^{-1}=(\tilde{b}_s^i)$ be the inverse matrix of $B=(b_i^s)$. We obtain

$$x_s\frac{\partial}{\partial x_j}=\sum_{i=1}^{n-1}\tilde{b}_s^i x'_i\frac{\partial}{\partial x'_i}+\tilde{b}_s^n\Phi\frac{\partial}{\partial y'};\quad s=1,2,\ldots,n-1, \tag{9}$$

$$y\frac{\partial}{\partial y}=\sum_{i=1}^{n-1}\tilde{b}_n^i x'_i\frac{\partial}{\partial x'_i}+\tilde{b}_n^n\Phi\frac{\partial}{\partial y'} \tag{10}$$

Note that the $\tilde{b}_i^s$ are integer (may be negative) numbers. Moreover, we have

$$\tilde{b}_n^n=\frac{1}{\Phi}y\frac{\partial}{\partial y}(\Phi)=d\neq 0. \tag{11}$$

Finally, a given linear vector field $\Delta=\sum_{i=1}^{n}\mu_i z_i\partial/\partial z_i$, we have

$$\Delta=\left\{\sum_{i=1}^{n-1}\tilde{\mu}_i x'_j\frac{\partial}{\partial x'_j}+\tilde{\mu}_n y'\frac{\partial}{\partial y'}\right\}+c\tilde{\mu}_n\frac{\partial}{\partial y'}. \tag{12}$$

where $(\tilde{\mu}_1,\tilde{\mu}_2,\ldots,\tilde{\mu}_n)=(\mu_1,\mu_2,\ldots,\mu_n)B^{-1}$.

4. Rational co-rank one

In this section we also assume that $r = n-1$, $\kappa_\nu = k$. We take a parameterized regular local model $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, y))$ and a foliation $\mathcal{L} \subset \mathrm{Der}_k K$. We will prove the following result

Proposition 5. *There is a a parameterized regular local model $\mathcal{A}'$ obtained from $\mathcal{A}$ by a finite sequence of coordinate changes in the dependent variable and coordinate blow-ups with codimension two centers, such that one of the following properties holds:*

1. *There is $\hat{f} \in \widehat{\mathcal{O}}'$ having transversal maximal contact with ν.*
2. *The local foliation $\mathcal{L}_{\mathcal{A}'}$ is non-singular if $n = 2$ and elementary if $n \geq 3$.*

Now, Proposition 5 is a consequence of the following five lemmas.

Lemma 1. *Assume that $\mathcal{A}'$ has been obtained from $\mathcal{A}$ by a coordinate change in the dependent variable. Then $\hbar(\mathcal{L}; \mathcal{A}') = \hbar(\mathcal{L}; \mathcal{A})$.*

Proof. Left to the reader. □

Lemma 2. *Assume that $\mathcal{A}'$ has been obtained from $\mathcal{A}$ by a Puiseux package. Then $\hbar(\mathcal{L}; \mathcal{A}') \leq \chi(\mathcal{L}; \mathcal{A})$. Moreover, we have*

$$\hbar(\mathcal{L}; \mathcal{A}') < \chi(\mathcal{L}; \mathcal{A})$$

if $\chi(\mathcal{L}; \mathcal{A}) \geq 1$ and $d(\mathcal{A}) \geq 2$, where $d(\mathcal{A})$ is the the ramification index of $\mathcal{A}$.

Lemma 3. *Assume that $\hbar(\mathcal{L}; \mathcal{A}) \in \{-1, 0\}$. We have the following properties*

1. *If $\hbar(\mathcal{L}; \mathcal{A}) = -1$, after performing a finite sequence of coordinate blow-ups in the independent variables, we obtain $\mathcal{A}'$ such that $\mathcal{L}_{\mathcal{A}'}$ is non-singular.*
2. *If $\hbar(\mathcal{L}; \mathcal{A}) = 0$, after performing a finite sequence of coordinate blow-ups in the independent variables, we obtain $\mathcal{A}'$ such that $\mathcal{L}_{\mathcal{A}'}$ is elementary.*

Lemma 4. *If $n = 2$, after performing a finite sequence of Puiseux packages we obtain $\mathcal{A}'$ such that either $\mathcal{L}_{\mathcal{A}'}$ is non-singular or there is $\hat{f} \in \widehat{\mathcal{O}}'$ having transversal maximal contact with ν.*

Lemma 5. *Assume that $\hbar(\mathcal{L}_{\mathcal{A}}) \geq 1$ and that the following property holds:*

> *"After any finite sequence of coordinate blow-ups in the independent variables, Puiseux packages and coordinate changes in the dependent variable we have that $d(\mathcal{A}) = 1$ and $\hbar(\mathcal{L}_{\mathcal{A}'}) = \hbar(\mathcal{L}_{\mathcal{A}})$".*

Then there is $\hat{f} \in \widehat{\mathcal{O}}$ having transversal maximal contact with ν.

In order to show that Lemmas 1, 2, 3, 4 and 5 imply Proposition 5, let us only recall that $\chi(\mathcal{L}; \mathcal{A}) < \hbar(\mathcal{L}; \mathcal{A})$. So, unless we have a transversal maximal contact, we arrive to the situation of Lemma 3 by a repeated application of Lemma 2 and we are done.

Let us prove the above lemmas.

4.1. **The effect of a Puiseux package.** Let us consider $\mathcal{A}' = (\mathcal{O}', \mathbf{z}' = (\mathbf{x}', y'))$ obtained from $\mathcal{A}$ by a Puiseux package. Take an $\mathbf{x}$-vector field $\xi \in \mathcal{L}_{\mathcal{A}}[\log \mathbf{x}]$ such that $\chi(\xi; \mathbf{x}, y) = \chi(\mathcal{L}; \mathcal{A})$ and let us write $\xi = \sum_{s=-1}^{n} y^s \eta_s$ as in equations (2) and (4). In order to simplify the notation, put $\chi = \chi(\xi; \mathbf{x}, y)$ and $\delta = \delta(\xi; \mathbf{x}, y)$. Moreover, we denote $d = d(\mathcal{A})$ the ramification index associated to $\mathcal{A}$. Let us write $\tilde{\xi} = \xi - \mathrm{In}(\xi; \mathbf{x}, y)$. We recall that $\delta(\tilde{\xi}; \mathbf{x}, y) > \delta$.

Next we express $\mathrm{In}(\xi;\mathbf{x},y)$ and $\tilde{\xi}$ in terms of the coordinates $\mathbf{z}'=(\mathbf{x}',y')$.

Lemma 6. $\alpha(\tilde{\xi};\mathbf{x}',y')>\delta$.

Proof. Left to the reader. □

Let us consider now $\mathrm{In}(\xi;\mathbf{x},y)$ and let us express it in the coordinates $\mathbf{z}'$. Let us recall equation 6, where $\mathbf{x}^{-\mathbf{q}(\chi)}y^{-\chi}\Phi^{\varrho}\mathrm{In}(\xi;\mathbf{x},y)=\sum_{t=0}^{\varrho}\Phi^{\varrho-t}\Delta_t$ and

$$\Delta_t=\Lambda_{\chi-dt}=\sum_{i=1}^{n}\lambda_{i,\chi-dt}z_i\partial/\partial z_i=\sum_{i=1}^{n}\mu_{it}z_i\partial/\partial z_i,$$

with $\Delta_0\neq 0$. Let us put $\zeta=\sum_{t=0}^{\varrho}\Phi^{\varrho-t}\Delta_t$. We can write $\zeta=\sum_{s\geq\beta'}y'^s\vartheta_s$, where $\vartheta_{\beta'}\neq 0$ and all the ϑ_s are $\mathbf{z}'$-linear vector fields $\vartheta_s=\sum_{j=1}^{n}\alpha_{js}z'_j\partial/\partial z'_j$.

Lemma 7. *We have $\beta'\leq\chi$. If $\chi\geq 1$ and $d\geq 2$, then $\beta'<\chi$.*

Proof. Looking at the equation 12, we see that $\zeta=\sum_{t=0}^{\varrho}(y'+c)^{\varrho-t}\Delta_t$ and

$$\Delta_t=\sum_{j=1}^{n}\tilde{\mu}_{jt}z'_j\frac{\partial}{\partial z'_j}+c\tilde{\mu}_{nt}\frac{\partial}{\partial y'};\quad(\tilde{\mu}_{1t},\tilde{\mu}_{2t},\ldots,\tilde{\mu}_{nt})=(\mu_{1t},\mu_{2t},\ldots,\mu_{nt})B^{-1}.$$

Let $\varsigma=\max\{t;\ \Delta_t\neq 0\}\leq\varrho$. Then $\zeta=\Phi^{\varrho-\varsigma}\sum_{t=0}^{\varsigma}\Phi^{\varsigma-t}\Delta_t$. Recalling that $\Phi=y'+c$, and dividing the above expression by $\Phi^{\varrho-\varsigma}$, we obtain $\beta'\leq\varsigma$. Remember that ϱ is the greatest integer bounded above by $(\chi+1)/d$. Then, if $d\geq 2$ and $\chi\geq 2$, or $d\geq 3$ and $\chi=1$ we obtain $\beta'\leq\varsigma\leq\varrho<\chi$. If $\chi=0$ and $d\geq 2$ we have $\varrho=0$ and then $\beta'\leq 0$. It remains to study the cases with $d=1$, the case $d=2$, $\chi=1$ and the case $\chi=-1$.

The case $\chi=-1$. In this case $\varrho=\varsigma=0$. In particular $\zeta=\Delta_0$. Moreover $\Delta_0=\Lambda_{-1}=\mu_{n0}y\partial/\partial y$. Recalling that $\tilde{b}_n^n=d$ in view of equation 11, we have

$$\zeta=\mu_{n0}y\partial/\partial y=\mu_{n0}\sum_{j=1}^{n}\tilde{b}_n^jz'_j\frac{\partial}{\partial z'_j}+\mu_{n0}dc\frac{\partial}{\partial y'}.$$

This implies that $\alpha_{n,-1}=\mu_{n0}dc\neq 0$ and thus $\beta'=-1$.

Cases with $d=1$, $\chi\geq 0$. We reason by contradiction, assuming that $\beta'\geq\chi+1$. This implies that $\varsigma=\varrho=\chi+1$. In particular, we have $\Delta_{\chi+1}\neq 0$ and $\Delta_{\chi+1}=\Lambda_{-1}$. Note that $\Lambda_{-1}=\mu y\partial/\partial y$, where $\mu=\mu_{n,\chi+1}=\lambda_{n,-1}$. Now, our contradiction hypothesis $\beta'\geq\chi+1$ implies that $\zeta(y')$ is divisible by $y'^{\chi+2}$. We have

$$\begin{aligned}\zeta(y')&=\sum_{t=0}^{\chi+1}\Phi^{\chi+1-t}\Delta_t(y')=\\&=\Phi\left(\Phi^{\chi+1}\tilde{\mu}_{n0}+\Phi^{\chi}\tilde{\mu}_{n1}+\Phi^{\chi-1}\tilde{\mu}_{n2}+\cdots+\Phi\tilde{\mu}_{n\chi}+\tilde{\mu}_{n,\chi+1}\right).\end{aligned}$$

Recall that $\Phi=y'+c$, then we necessarily have that $\zeta(y')=\tilde{\mu}_{n0}y'^{\chi+2}$, since the biggest possible power of y' in the above expression is $y'^{\chi+2}$ and its coefficient is $\tilde{\mu}_{n0}$. Moreover we also have that $\Phi=y'+c$ divides $\zeta'(y')$. The only possibility is that $\zeta(y')=0$ and hence all the coefficients $\tilde{\mu}_{nt}$ are zero, for $t=1,2,\ldots,\chi+1$. This is a contradiction, since $\tilde{\mu}_{n,\chi+1}=\tilde{b}_n^n\mu=d\mu\neq 0$.

Case $d=2$, $\chi=1$. Let us reason by contradiction, assuming that $\beta'\geq\chi$. Then $\varsigma=\varrho=\chi=1$. We have $\zeta=\Phi\Delta_0+\Delta_1$ and y'^2 must divide $\zeta(y')$. That is

$$\zeta(y')=\Phi\left(\Phi\tilde{\mu}_{n0}+\tilde{\mu}_{n1}\right)=y'^2\tilde{\mu}_{n0}.$$

We deduce as above that $\zeta(y')=0$ and thus $\tilde{\mu}_{n1}=\tilde{\mu}_{n0}=0$. Note that $0\neq\Delta_1$, since $\varsigma=1$. Moreover, in our case $\Delta_t=\Lambda_{1-2t}$ and thus $\Delta_1=\Lambda_{-1}=\mu y\partial/\partial y\neq 0$. Now we have $\tilde{\mu}_{n1}=2\mu$ and we obtain that $\tilde{\mu}_{n1}\neq 0$ and $\tilde{\mu}_{n1}=0$ simultaneously, contradiction. □

Lemma 8. $\hbar(\mathcal{L};\mathcal{A}')\leq\beta'$.

Proof. It is enough to show that $\beta'=\hbar(\xi;\mathbf{x}',y')$. We have $\beta'=\hbar(\zeta;\mathbf{x}';y')$ and $\alpha(\zeta;\mathbf{x}';y')=0$. Recall that $\mathrm{In}(\xi;\mathbf{x},y)=\mathbf{x}^{\mathbf{q}(\chi)}y^{\chi}\Phi^{-\varrho}\zeta$, where

$$\nu(\mathbf{x}^{\mathbf{q}(\chi)}y^{\chi}\Phi^{-\varrho})=\nu(\mathbf{x}^{\mathbf{q}(\chi)}y^{\chi})=\delta.$$

Moreover, in view of Remark 3, we have that

$$\mathbf{x}^{\mathbf{q}(\chi)}y^{\chi}\Phi^{-\varrho}=\mathbf{x}'^{\mathbf{q}'}\Phi^{r'},\quad \text{ and } \nu(\mathbf{x}'^{\mathbf{q}'})=\delta.$$

Noting that $\mathrm{In}(\xi;\mathbf{x},y)=\mathbf{x}'^{\mathbf{q}'}\Phi^{r'}\zeta$, we deduce that $\alpha(\mathrm{In}(\xi;\mathbf{x},y);\mathbf{x}',y')=\delta$ and $\hbar(\mathrm{In}(\xi;\mathbf{x},y);\mathbf{x}',y')=\hbar(\zeta;\mathbf{x}',y')=\beta'$. Moreover, by Lemma 6, we have

$$\delta=\alpha(\mathrm{In}(\xi;\mathbf{x},y);\mathbf{x}',y')<\alpha(\tilde{\xi};\mathbf{x}',y').$$

Recalling that $\xi=\mathrm{In}(\xi;\mathbf{x},y)+\tilde{\xi}$, we have that $\alpha(\xi;\mathbf{x}',y')=\delta$ and

$$\hbar(\xi;\mathbf{x}',y')=\hbar(\mathrm{In}(\xi;\mathbf{x},y);\mathbf{x}',y')=\beta'.$$

This ends the proof. □

Remark 6. Lemma 1 follows from Lemma 7, in view of Lemma 8.

Before giving a proof of Lemma 3, we explain the effect of the blow-ups in the independent variables in the following result.

Lemma 9. *Given $\mathcal{A}$ and $\mathcal{L}$, after performing finitely many coordinate blow-ups in the independent variables with centers of codimension two, we can obtain $\mathcal{A}'$ such that $\alpha(\mathcal{L};\mathcal{A}')=0$. Moreover $\hbar(\mathcal{L};\mathcal{A}')\leq\hbar(\mathcal{L};\mathcal{A})$.*

Proof. Write $\xi=\sum_{s=-1}^{\infty}y^s\eta_s$ with $\eta_s=\sum_{j=1}^{n}h_{js}(\mathbf{x})z_j\partial/\partial z_j$. Let us do a blow-up in the independent variables and let $\mathbf{x}',y$ be the obtained variables. Then the same decomposition as above acts in this new set of variables, that is $\xi=\sum_{s=-1}^{\infty}y^s\eta_s$ where we can write

$$\eta_s=\sum_{j=1}^{n}h'_{js}z'_j\frac{\partial}{\partial z'_j};\quad h'_{js}\in k[[\mathbf{x}']].$$

Moreover the ideal $I'_s\subset k[[\mathbf{x}']]$ generated by $\{h'_{js}\}_{j=1}^{n}$ is $I'_s=I_sk[[\mathbf{x}]]$, where I_s is the ideal of $k[[\mathbf{x}]]$ generated by $\{h_{js}\}_{j=1}^{n}$. This already implies that

$$\alpha(\eta_s;\mathbf{x})=\alpha(\eta_s;\mathbf{x}');\quad s=-1,0,1,\ldots.$$

In particular we have that $\hbar(\xi;\mathbf{x},y)=\hbar(\xi;\mathbf{x}',y)$.

Moreover, the ideal $I'=\sum_{s=-1}^{\infty}I'_s\subset k[[\mathbf{x}']]$ is also given by $I'=Ik[[\mathbf{x}']]$, where $I=\sum_{s=-1}^{\infty}I_s\subset k[[\mathbf{x}]]$. Thus, we can apply classical results of reduction of singularities under combinatorial blow-ups, that can be proved as in Proposition 4 (see also [7]) to assure that after a finite number of blow-ups in the independent variables with centers of codimension two, the ideal I is generated by a single monomial, say $\mathbf{x}'^{\mathbf{q}'}$. We obtain an $\mathbf{x}'$-vector field $\xi'=\mathbf{x}'^{-\mathbf{q}'}\xi\in\mathcal{L}_{\mathcal{A}'}[\log\mathbf{x}']$ such that $\alpha(\xi';\mathbf{x}',y)=0$. □

Remark 7. In the above lemma we have $\hbar(\mathcal{L};\mathcal{A}')=\hbar(\mathcal{L};\mathcal{A}')$. Anyway, we do not need to use this fact.

We obtain an immediate proof of Lemma 3. By Lemma 9, we may suppose that there is $\xi\in\mathcal{L}_{\mathcal{A}}[\log\mathbf{x}]$ such that $\hbar(\xi;\mathbf{x},y)=\hbar(\mathcal{L};\mathcal{A})$ and $\alpha(\xi;\mathbf{x},y)=0$. Now, it is evident that

(1) If $\hbar(\xi;\mathbf{x},y)=-1$, then ξ is non-singular.
(2) If $\hbar(\xi;\mathbf{x},y)=0$, then ξ is elementary (or non-singular).

4.2. **Getting a formal hypersurface of transversal maximal contact.** Let us give a proof of Lemma 5. In view of Lemma 9, after performing finitely many blow-ups in the independent variables, we can assume that there is $\xi\in\mathcal{L}_{\mathcal{A}}[\log\mathbf{x}]$ such that $\hbar(\xi;\mathbf{x},y)=\hbar(\mathcal{L};\mathcal{A})$ and $\alpha(\xi;\mathbf{x},y)=0$. Moreover, we also have that

$$\chi(\xi;\mathbf{x},y)=\hbar(\xi;\mathbf{x},y),$$

since otherwise, an application of Lemma 2 allows us to decrease $\hbar(\mathcal{L};\mathcal{A})$. Moreover, in view of our hypothesis, we have $d(\mathcal{A})=1$ and $\hbar(\mathcal{L};\mathcal{A})\geq 1$.

Lemma 10. *Let $\Phi=y/\mathbf{x}^{\mathbf{p}}$ be the contact rational function. We have $\mathbf{p}\in\mathbb{Z}_{\geq 0}^{n-1}$.*

Proof. Let us keep the notations of subsection 3.2. Recall that

$$\mathrm{In}(\xi;\mathbf{x},y)=\sum_{s=-1}^{\chi}y^s\mathbf{x}^{\mathbf{q}(s)}\Lambda_s.$$

Since $\alpha(\xi;\mathbf{x},y)=0$ and $\chi(\xi;\mathbf{x},y)=\hbar(\xi;\mathbf{x},y)$, we have $\mathbf{q}(\chi)=0$. Thus, for any s such that $\Lambda_s\neq 0$ we have

$$\nu(\mathbf{x}^{\mathbf{q}(s)})=(\chi-s)\nu(y)$$

and hence $(\chi-s)\mathbf{p}=\mathbf{q}(s)$. Noting that $\mathbf{q}(s)\in\mathbb{Z}_{\geq 0}^{n-1}$, it is enough to show that there is at least an index $s<\chi$ such that $\Lambda_s\neq 0$. Assume the contrary. Then

$$\mathrm{In}(\xi)=y^{\chi}\Lambda_{\chi},$$

where $\chi=\hbar(\xi;\mathbf{x};y)\geq 1$. Let us do a Puiseux package, taking the notations of the proof of Lemma 7, we obtain $\varsigma=0$ and hence $\chi'\leq\beta'\leq\varsigma=0$. Contradiction. □

In this situation, we have $\nu(y-c\mathbf{x}^{\mathbf{p}})>\nu(y)$. Let us do the coordinate change $y'=y-c\mathbf{x}^{\mathbf{p}}$. The situation repeats. In this way we can produce a sequence of elements $y^{(j)}\in\mathcal{M}\setminus\mathcal{M}^2$, such that $y^{(0)}=y$ and

$$y^{(j)}=y^{(j-1)}-c_j\mathbf{x}^{\mathbf{p}(j)};\quad \nu(y^{(j)})>\nu(y^{(j-1)}),\ j=1,2,\ldots.$$

Taking $\hat f=\lim_j y^{(j)}$, we obtain the desired formal hypersurface.

4.3. **The case of dimension two.** The statement of Lemma 4 is a consequence of Seidenberg's reduction of singularities in dimension two [14]. Let us see this. Assuming that we do not get non-singular points, after finitely Puiseux packages, we obtain a "simple singularity" in the sense of Seidenberg. It is given by an x-vector field of the form

$$\xi=(\lambda+a(x,y))x\frac{\partial}{\partial x}+(\alpha x+\mu y+\tilde b(x,y))\frac{\partial}{\partial x};\quad a(0,0)=0,\ \tilde b(x,y)\in\mathcal{M}^2$$

where, $(\lambda,\mu)\neq(0,0)$ and if $\lambda\neq 0$ then $\mu/\lambda\notin\mathbb{Q}_{>0}$. Such singularity has exactly two formal invariant curves: $x=0$ and $\hat f=0$, where $\hat f=y-\hat\phi(x)$. They are non-singular and transversal one to the other. After doing one more blow-up, the exceptional divisor is invariant and we obtain exactly two simple singularities, one of them corresponds to the strict transform of $x = 0$, it is a *corner*, and the other one is in the strict transform of $\hat{f} = 0$. This shows that blowing-up a corner produces only corners as singularities, thus, since the valuation has rational rank one and we have nontrivial Puiseux packages, we necessarily do blow-ups outside the corners. Hence we follow the infinitely near points of $\hat{f} = 0$. "A fortiori", we obtain that $\hat{f}$ is non-algebraic (otherwise the value of $\hat{f}$ would be infinite) and has maximal contact with ν.

5. Etale Puiseux packages

5.1. **Review on etale neighborhoods.** Let us recall the definition of a local etale morphism as one can see in [1]. Let us fix the local ring $\mathcal{O} = \mathcal{O}_{M,P}$ of a projective model M of K at the center P in M of the k-valuation ν of K and assume that P is a regular point of M. Here we assume that ν is a real valuation with $\kappa_\nu = k$.

Consider a morphism $\mathcal{O} \to \widetilde{\mathcal{O}}$ of local rings. We say that $\mathcal{O} \to \widetilde{\mathcal{O}}$ is *local-etale* or that $\widetilde{\mathcal{O}}$ is a *local-etale extension* of $\mathcal{O}$ if we have the following properties:

(1) The local rings $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ have the same residual field.
(2) $\widetilde{\mathcal{O}}$ is the localization at a prime ideal of an etale $\mathcal{O}$-algebra.

An etale $\mathcal{O}$-algebra is an $\mathcal{O}$-algebra of the type $B = \mathcal{O}[t_1, t_2, \ldots, t_n]/(f_1, f_2, \ldots, f_n)$, where the Jacobian matrix of the f_i is invertible in B. This is equivalent to say that B is a finitely generated A-flat algebra and $\Omega^1_A B = 0$. Note that $\widetilde{\mathcal{O}}$ is also a regular local ring and its fraction field $\widetilde{K}$ is a finitely generated algebraic extension of K. Recall also that $\widetilde{\mathcal{O}} \subset \mathcal{O}^h \subset \widehat{\mathcal{O}}$, where $\mathcal{O}^h$ is the henselian closure of $\mathcal{O}$.

We say that the pair $(\widetilde{\mathcal{O}}, \tilde{\nu})$ is a *local etale extension of* $(\mathcal{O}, \nu)$ if $\widetilde{\mathcal{O}}$ is a local-etale extension of $\mathcal{O}$ and $\tilde{\nu}$ is a k-valuation of $\widetilde{K}$ centered at $\widetilde{\mathcal{O}}$ such that $\tilde{\nu}|_K = \nu$. Note that $\tilde{\nu}$ is a real k-valuation and $\kappa_{\tilde{\nu}} = k$.

In the following proposition, we summarize the properties that allow us to work "up to local-etale extensions".

Proposition 6. *Consider a foliation $\mathcal{L} \subset Der_k K$ and a real k-valuation ν of K such that $\kappa_\nu = k$. Let $(\widetilde{\mathcal{O}}, \tilde{\nu})$ is a be a local etale extension of $(\mathcal{O}, \nu)$ and denote $\widetilde{\mathcal{L}} = \widetilde{K}\mathcal{L} \subset Der_k \widetilde{K}$ the induced foliation on $\widetilde{K}$. Assume that we respectively have:*

(1) *The foliation $\widetilde{\mathcal{L}}$ is log-elementary at $\widetilde{\mathcal{O}}$.*
(2) *There is a formal $\hat{f} \in \widehat{\mathcal{O}}$ with transversal maximal contact relatively to $\widetilde{\mathcal{O}}$.*

Then, up to perform a finite sequence of local blow-ups of $\mathcal{O}$ we respectively have:

(1) *The foliation $\mathcal{L}$ is log-elementary at $\mathcal{O}$.*
(2) *There is a formal $\hat{f} \in \widehat{\mathcal{O}}$ with transversal maximal contact relatively to $\mathcal{O}$.*

Proof. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be a regular system of parameters of $\widetilde{\mathcal{O}}$. Consider $\tilde{h} = \prod_{i=1}^n \tilde{x}_i$. The ideal $\tilde{h}\widetilde{\mathcal{O}}$ gives a principal ideal $h\mathcal{O} = \mathcal{O} \cap \tilde{h}\widetilde{\mathcal{O}}$. We can do the local uniformization of h by using centers that respect the fact that $\widetilde{\mathcal{L}}$ is log elementary (relatively to $\tilde{\mathbf{x}}$) (see [4] to the definition of permissible centers and the needed properties). Finally we get that h is a monomial and we are done. □

5.2. **Etale Puiseux packages.** We introduce here an etale version of Puiseux packages for the case $r = 1$. It has the same effect over a foliation as the Puiseux packages introduced in Section 1, but it will allow us to do an accurate control of the foliation. Indeed, the study of the case $n = 3$, $r = 1$ will be done under the use of etale Puiseux packages.

We assume that ν is a valuation with rational rank $r = 1$ and $\kappa_\nu = k$.

Let $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (x, \mathbf{y}))$ be a regular parameterized model. Consider a dependent variable y_j. Let $\Phi = y_j^d/x^p$ be the contact rational function and $c \in k$ such that $\nu(\Phi - c) > 0$. Recall that d is the *y_j-ramification index of $\mathcal{A}$.*

Remark 8. In the case that $d = 1$, all the blow-ups in a Puiseux package are "in the first chart" in the sense that we always have $\nu(y_j) \geq \nu(x)$.

Let us consider the ring $\mathcal{O}^\natural = \mathcal{O}[T]/(T^d - x) = \mathcal{O}[t]$, where T is an indeterminate and let $\widetilde{K}$ be the fraction field of $\mathcal{O}^\natural$. We know [17] that there are k-valuations $\tilde{\nu}$ of $\widetilde{K}$ such that $R_{\tilde{\nu}} \cap K = R_\nu$. Note that all the $\tilde{\nu}$ have the same group of values. Let us choose one of them, say $\tilde{\nu}$. The ring $\mathcal{O}^\natural$ is a regular local ring that supports a parameterized regular local model

$$\mathcal{A}^\natural = (\mathcal{O}^\natural, \mathbf{z}^\natural = (t, \mathbf{y}))$$

relative to $\tilde{K}$ and $\tilde{\nu}$. We have $k \subset \mathcal{O} \subset \mathcal{O}^\natural$ and $\mathcal{M}^\natural \cap \mathcal{O} = \mathcal{M}$. Moreover, $k = \kappa_{\tilde{\nu}}$ and $t^d = x$. Let us note that $\tilde{\nu}(y/t^{\mathbf{p}}) = 0$. In particular $d(\widetilde{\mathcal{A}}) = 1$. Let $\tilde{c} \in k$ be such that $\tilde{\nu}(y/t^{\mathbf{p}} - \tilde{c}) > 0$, we see that $\tilde{c}^d = c$.

Definition 2. *We say that $(\widetilde{\mathcal{A}}, \tilde{\nu})$ has been obtained from $(\mathcal{A}, \nu)$ by an* etale j-Puiseux package *if and only if $\widetilde{\mathcal{A}}$ has been obtained from $\mathcal{A}^\natural$ by a j-Puiseux package.*

Proposition 7. *Assume that $\widetilde{\mathcal{A}} = (\widetilde{\mathcal{O}}, \tilde{\mathbf{z}} = (t, \tilde{\mathbf{y}}))$ has been obtained from $\mathcal{A}$ by an etale j-Puiseux package. There is $\mathcal{A}' = (\mathcal{O}', \mathbf{z}' = (x', \mathbf{y}'))$ obtained from $\mathcal{A}$ by a j-Puiseux package such that $(\widetilde{\mathcal{O}}, \tilde{\nu})$ is a local-etale extension of $(\mathcal{O}', \nu)$.*

Proof. Consider the j-Puiseux package $\mathcal{A}^\natural \mapsto \widetilde{\mathcal{A}}$. Put $\Phi = y_j^d/x^p$ and $\tilde{\Phi} = y_j/t^p$, the respective contact rational functions for $\mathcal{A}$ and $\mathcal{A}^\natural$. Note that $\tilde{\Phi}^d = \Phi$. Let $c, \tilde{c} \in k$ be such that $\tilde{\nu}(\tilde{\Phi} - \tilde{c}) > 0$ and $\tilde{c}^d = c$. We have

$$\tilde{y}_j = \tilde{\Phi} - \tilde{c}; \quad y'_j = \Phi - c.$$

Moreover $\tilde{y}_j$ is a simple root of a polynomial over $\mathcal{O}'$ as the following relation shows

$$y'_j = (\tilde{y}_j + \tilde{c})^d - c.$$

Now $\tilde{t}$ is of the form $\tilde{t} = x' P(\tilde{y}_j)$, where $P(0) \neq 0$. This is enough to obtain the conclusion. □

Remark 9. If $\widetilde{\mathcal{A}} = (\widetilde{\mathcal{O}}, \tilde{\mathbf{z}} = (t, \tilde{\mathbf{y}}))$ has been obtained from $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (x, \mathbf{y}))$ by an etale Puiseux package, then $\mathcal{O} \subset \widetilde{\mathcal{O}}$ and $t^d = x$.

Definition 3. *We say that $(\mathcal{A}, \nu) \mapsto (\widetilde{\mathcal{A}}, \tilde{\nu})$ is an* etale standard transformation *if and only if $(\widetilde{\mathcal{A}}, \tilde{\nu})$ has been obtained from $(\mathcal{A}, \nu)$ by an etale Puiseux Package or by a coordinate change in the dependent variables.*

6. Rational rank one

We end here the proof of Theorem 3. To do this we consider the following proposition

Proposition 8. *Let $\mathcal{L} \subset Der_k K$ be a foliation over K, where $n = 3$. Take a k-valuation ν of K of rational rank one and such that $\kappa_\nu = k$. Assume that $\mathcal{A}$ is a parameterized regular local model for K and ν. Then, there is a finite sequence of etale standard transformations*

$$(\mathcal{A}, \nu) = (\mathcal{A}_0, \nu_0) \mapsto (\mathcal{A}_1, \nu_1) \mapsto \cdots \mapsto (\mathcal{A}_N, \nu_N) = (\mathcal{A}', \nu')$$

such that either the transformed foliation $\mathcal{L}'$ is log-elementary in $\mathcal{A}'$ or there is $\hat{f} \in \mathcal{O}'$ having transversal maximal contact.

Proposition 8 gives the end of the proof of Proposition 3 and hence it completes the proof of Theorem 3. Indeed, by propositions 4 and 5 we obtain Proposition 3 for rational rank $r = 2, 3$. For the case of rational rank $r = 1$ and $n = 3$, we obtain Proposition 3 from Proposition 8 in view of propositions 6 and 7.

This section is devoted to the proof of Proposition 8. In all this section we assume implicitly that we do not get a formal transversal maximal contact.

Recall that we in this section we have $n = 3$, the rational rank of ν is equal to one and $\kappa_\nu = k$. We start with a parameterized regular local model $\mathcal{A} = (\mathcal{O}; \mathbf{z} = (x, w, y))$ and a foliation $\mathcal{L} \subset \mathrm{Der}_k K$.

6.1. **The independent coefficient.** Let ξ be an $\mathcal{O}$-generator of $\mathcal{L}_\mathcal{A}[\log x]$. Let us put $H = \xi(x)/x \in \mathcal{O}$. Consider an etale standard transformation $(\mathcal{A}, \nu) \mapsto (\mathcal{A}, \nu')$ where

$$\mathcal{A}' = (\mathcal{O}', \mathbf{z}' = (t, w', y')).$$

Recall that $t^d = x$, for $d \geq 1$. We know that $\mathcal{L}'_{\mathcal{A}'}[\log t]$ is generated by a germ of vector field of the form $\xi' = t^q \xi$ where $q \in \mathbb{Z}$. Moreover, we have that

$$\xi(t)/t = \lambda' \xi(x)/x,$$

where $\lambda' = 1/d \in \mathbb{Q}_{>0}$. This implies that

$$H' = \xi'(t)/t = \lambda' t^q H \in \mathcal{O}'. \tag{13}$$

In particular, the coefficient H is transformed essentially "as a function" under the etale standard transformations. This allows us to obtain the following result

Proposition 9. *After finitely many etale standard transformations we can chose an $\mathcal{O}$-generator ξ of $\mathcal{L}_\mathcal{A}[\log x]$ such that $\xi(x)/x = \lambda x^m$, where $\lambda \in \mathbb{Q}_{>0}$.*

Proof. We apply to H the usual local uniformization for functions. We obtain that $H = x^m U$, where U is a unit. Now we divide ξ by U. □

Moreover, the above form of H is persistent under etale standard transformations. This justifies the next definition.

Definition 4. *We say that $\mathcal{L}$ is* x-prepared relatively to $\mathcal{A}$ *if there is an $\mathcal{O}$-generator ξ of $\mathcal{L}_\mathcal{A}[\log x]$ such that $\xi(x)/x = \lambda x^q$, for $0 \neq \lambda \in \mathbb{Q}$. Such generators ξ will be called* x-privileged generators.

In view of Proposition 9 we can obtain that $\mathcal{L}$ is x-prepared after a finite number of etale-standard transformations and this property is persistent under new etale-standard transformations.

6.2. **Invariants from the Newton Puiseux Polygon.** Take $f \in w^{-1}k[[x,w]]$ that we write $f=\sum_{t=-1}^{\infty} w^t f_t(x)$. We put

$$\lambda(f;x,w)=\min_t\{\nu(f_t(x))+t\nu(w)\};\quad \alpha(f;x,w)=\min_t\{\nu(f_t(x))\}. \tag{14}$$

Consider a vector field η of the form

$$\eta=a(x,w)x\frac{\partial}{\partial x}+b(x,w)\frac{\partial}{\partial w}+c(x,w)y\frac{\partial}{\partial y}. \tag{15}$$

We denote

$$\lambda(\eta;x,w) = \min\{\lambda(a;x,w),\lambda(b/w;x,w),\lambda(c;x,w)\} \tag{16}$$

$$\alpha(\eta;x,w) = \min\{\alpha(a;x,w),\alpha(b/w;x,w),\alpha(c;x,w)\} \tag{17}$$

Let us note that $\alpha(b/w;x,w)=\alpha(b;x,w)$. We also write

$$\Lambda(\eta;x,w)=\lambda(\eta;x,w)-\alpha(\eta;x,w). \tag{18}$$

Note that $\Lambda(\eta;x,w)\geq -\nu(w)$.

Remark 10. We can draw a Newton-Puiseux polygon $\mathcal{N}$ for f, or for η, by considering the support $\{(\nu(f_t(x),t))\}\subset\Gamma\times\mathbb{Z}_{\geq -1}\subset\mathbb{R}\times\mathbb{Z}_{\geq -1}$. Then α is the abscissa of the highest vertex and λ corresponds to the smallest value $a+\nu(w^b)$, where (a,b) is in the support. In particular, we have that $\Lambda=-\nu(w)$ if and only if $\mathcal{N}$ has the single vertex $(\alpha,-1)$.

Consider a vector field $\xi=\sum_{s=-1}^{\infty} y^s\eta_s\in \mathrm{Der}_k\mathcal{O}[\log x]$, where

$$\eta_s=a_s(x,w)x\frac{\partial}{\partial x}+b_s(x,w)\frac{\partial}{\partial w}+c_s(x,w)y\frac{\partial}{\partial y}. \tag{19}$$

We denote $\alpha(\xi;\mathcal{A})=\alpha(\xi;x,w,y)=\min_{s=-1}^{\infty}\{\alpha(\eta_s;x,w)\}$. Let us note that $\alpha(\xi;\mathcal{A})=0$ when ξ is a generator of $\mathcal{L}_{\mathcal{A}}[\log x]$, since x is not a common factor of the coefficients. The *main height* $\hbar(\xi;\mathcal{A})$ is the minimum of the s such that $\alpha(\eta_s;x,w)=\alpha(\xi;\mathcal{A})$. When ξ is a generator of $\mathcal{L}$, we put $\hbar(\mathcal{L},\mathcal{A})=\hbar(\xi,\mathcal{A})$.

Denote $\delta(\xi;\mathcal{A})=\min_{s=-1}^{\infty}\{\alpha(\eta_s;x,w)+s\nu(y)\}$. We say that *$s$ belongs to the critical segment* $\mathcal{C}(\xi;\mathcal{A})$ if $\alpha(\eta_s;x,w)+s\nu(y)=\delta(\xi;\mathcal{A})$. The *critical height* $\chi(\xi;\mathcal{A})$ is the greatest $s\in\mathcal{C}(\xi;\mathcal{A})$. Note that $\chi(\xi;\mathcal{A})\leq\hbar(\xi;\mathcal{A})$.

Remark 11. We can draw a Newton-Puiseux polygon $\mathcal{N}(\xi;\mathcal{A})\subset\mathbb{R}\times\mathbb{Z}_{\geq -1}$ by taking as support the set

$$\{(\alpha(\eta_s;x,w),s);\ s=-1,0,1,\ldots\}.$$

Then $(\alpha(\xi;x,w,y),\hbar(\xi;x,w,y))$ is the main vertex of the Newton-Puiseux polygon. We also have that $\delta(\xi;x,w,y)$ is the smallest value $a+\nu(y^b)$, where (a,b) is in the support. Nevertheless, this Newton-Puiseux polygon needs to be prepared by performing preliminary transformations in the variables x,w in order to be a useful tool in the control of the transformations in the variables x,y.

The invariants in three variables make sense also for $f(x,w,y)=\sum_s y^s f_s(x,w)$. Thus, we write

$$\alpha(f;x,w,)=\min_s\{\nu(f_s;x,w)\};\quad \delta(f,x,w,y)=\min_s\{\nu(f_s;x,w)+s\nu(y)\}. \tag{20}$$

6.3. **Prepared situations in two variables.** Take a vector field η as in equation (15). We say that η is (x,w)-*prepared* if there is $q \in \mathbb{Z}_{\geq 0}$ such that

$$(a,b,c) = x^q(\tilde{a}(x,w), \tilde{b}(x,w), \tilde{c}(x,w)); \quad (\tilde{b}(0,0), \tilde{c}(0,0)) \neq (0,0).$$

We say that η is *dominant* if $\tilde{b}_s(0,0) \neq 0$ and *recessive* if $\tilde{b}(0,0) = 0, \tilde{c}(0,0) \neq 0$.

Remark 12. The condition $\Lambda(\eta; x, w) = -\nu(w)$ is equivalent to say that η is prepared-dominant. If η is prepared-recessive, then $0 \geq \Lambda(\eta; x, w) > \nu(w)$.

Definition 5. *Take η as in equation (15). We say that η is* strongly (x,w)-prepared *if there is a decomposition*

$$\eta = x^\rho U(x,w)\theta + x^\tau V(x,w) y\frac{\partial}{\partial y}; \quad \theta = xh(x,w)x\frac{\partial}{\partial x} + \frac{\partial}{\partial w} \tag{21}$$

satisfying the following properties

(1) *$\rho, \tau \in \mathbb{Z} \cup \{+\infty\}$, with $\rho \neq \tau$. Here $\rho = +\infty$, respectively or $\tau = +\infty$, indicates that $U(x,w)$, respectively $V(x,w)$, is identically zero.*
(2) *We can write $U = \lambda + xf(x,w)$ and $V = \mu + xg(x,w)$, where $\lambda, \mu \in k$. Moreover, if $\rho \neq +\infty$ then $\lambda \neq 0$ and if $\tau \neq +\infty$ then $\mu \neq 0$.*

Let us note that "strongly prepared" implies "prepared". The dominant case corresponds to $r < t$ and the recessive case to $r > t$.

6.4. **Effect of etale w-Puiseux packages.** Let us perform an etale w-Puiseux package and let (t, w', y) be the obtained coordinates. Recall that $t^d = x$ and $\nu(w/t^p) = 0$, where p, d are without common factor. Moreover, we have $w' = w/t^p - c$, with $c \neq 0$, and hence

$$x\frac{\partial}{\partial x} = \frac{1}{d}\left\{t\frac{\partial'}{\partial t} - p(w'+c)\frac{\partial'}{\partial w'}\right\}; \quad \frac{\partial}{\partial w} = \frac{1}{t^p}\frac{\partial'}{\partial w'}; \quad \frac{\partial}{\partial y} = \frac{\partial'}{\partial y}. \tag{22}$$

Consider η as in equation (15) and write

$$\eta = a'(t,w')t\frac{\partial'}{\partial t} + b'(t,w')\frac{\partial'}{\partial w'} + c'(t,w')y\frac{\partial'}{\partial y}$$

in the coordinates t, w', y. Then we have

$$\begin{array}{rcccl} a' & = & \eta(t)/t & = & (1/d)a \\ b' & = & \eta(w') & = & (w'+c)\{b/w - (p/d)a\} = t^{-p}\{b - (p/d)t^p(w'+c)a\} \\ c' & = & \eta(y)/y & = & c. \end{array} \tag{23}$$

From these considerations, we obtain the following results:

Lemma 11. *Consider $f = \sum_{\ell=-1}^{\infty} w^\ell f_\ell(x) \in w^{-1}k[[x,w]]$. We have*

$$\alpha(f; t, w') = \lambda(f; x, w).$$

As a consequence, we also have that $\alpha(\eta; t, w') = \lambda(\eta; x, w)$.

Proof. Take a monomial $w^a x^b$. Note that $w^a x^b = t^{ap+bd}(w'+c)^a$ where $(w'+c)^a$ is a unit, hence

$$\begin{array}{rcl} \lambda(w^a x^b; x, w) & = & \nu(w^a x^b) = \\ & = & \nu(t^{ap+bd}(w'+c)^a) = \nu(t^{ap+bd}) = \alpha(w^a x^b; t, w'). \end{array}$$

Note that both $\lambda(\_; x, w)$ and $\alpha(\_; t, w')$ have the usual valuative properties. This gives in particular that $\alpha(f; t, w') \geq \lambda(f; x, w)$, as a consequence of the above property for monomials. Put $\lambda = \lambda(f; x, w)$ and let us decompose $f = \mathrm{L}(f) + f^*$, where $\lambda(f^*; x, w) > \lambda$ and $\mathrm{L}(f)$ is of the form

$$\mathrm{L}(f) = \sum_{\nu(w^a x^b)=\lambda} \mu_{ab} w^a x^b.$$

Now it is enough to prove that $\alpha(\mathrm{L}(f); x', w') = \lambda$. We know that if $\nu(w^a x^b) = \lambda$ then $m = ad + bp$ is independent of (a, b), since $\nu(t^m) = \nu(x^a w^b)$. We have

$$\mathrm{L}(f) = t^m \sum \mu_{ab}(w' + c)^a \neq 0.$$

Then $\alpha(\mathrm{L}; t, w') = \lambda$. The last statement comes from the above arguments and the equations 23. □

Corollary 1. *Consider η as in equation (15). We have*

$$\alpha(\eta; t, w') \geq \alpha(\eta; x, w) - \nu(w),$$

and the equality holds exactly when η is (x, w)-prepared and dominant.

Proof. We know that $\alpha(\eta; t, w') = \lambda(\eta; x, w)$ by Lemma 11. Now, it is a direct consequence of the definitions that $\lambda(\eta; x, w) \geq \alpha(\eta; x, w) - \nu(w)$, and the equality holds exactly when η is (x, w)-prepared and dominant, in view of Remark 12. □

Lemma 12. *Consider η as in equation (15). We have*

1. *If $\Lambda(\eta; x, w) < 0$, then $\Lambda(\eta; t, w') = -\nu(w)$ and hence η is (t, w')-prepared and dominant.*
2. *If η is (x, w)-prepared and dominant, then η is also (t, w')-prepared and dominant.*
3. *If $\Lambda(\eta; x, w) = 0$, then $\Lambda(\eta; t, w') \leq 0$.*
4. *If η is (x, w)-prepared and recessive, then η is (t, w')-prepared.*

Proof. Write $\eta = x^m \eta'$, where $\nu(x^m) = \alpha(\eta; x, w)$. If we substitute η by η' we can assume without loss of generality that $\alpha(\eta; x, w) = 0$ and thus $\Lambda(\eta; x, w) = \lambda(\eta; x, w)$. Let us put $\overline{b} = b/w = \sum_{\ell=-1}^{\infty} \overline{b}_\ell(x) w^\ell$. Consider first the case that $\lambda(\eta; x, w) < 0$. This implies that

$$\eta = \overline{b}_{-1}(x)\frac{\partial}{\partial w} + \eta^* = x^r U(x)\frac{\partial}{\partial w} + \eta^*,\ U(0) \neq 0, \tag{24}$$

where $0 \leq \nu(b_{-1}(x)) = \nu(x^r) = \nu(t^{rd}) < \nu(w) = \nu(t^p)$ and η^* has the form

$$\eta^* = \sum_{\ell=0}^{\infty} w^\ell \left\{ a_\ell(x) x \frac{\partial}{\partial x} + \overline{b}_\ell(x) w \frac{\partial}{\partial w} + c_\ell(x) y \frac{\partial}{\partial y} \right\}$$

By equations (22) we see that $\alpha(\eta^*; t, w') \geq 0$ and

$$x^r U(x)\frac{\partial}{\partial w} = t^{rd-p} U(t^d)\frac{\partial}{\partial w'}.$$

Note that $rd - p < 0$. We obtain

$$\Lambda(\eta; t, w') = -\nu(w')$$

and thus η is (t, w')-prepared and dominant. This proves statement 1. Now, statement 2 is a direct consequence of statement 1.

Assume that $\Lambda(\eta; x, w) = 0$. Write η as in Equation (24), where $\nu(x^r) \geq \nu(w)$ (we accept the case $r = +\infty$ to denote that w divides $\eta(w)$). If $\nu(x^r) = \nu(w)$, by the same argument as above we obtain that $\Lambda(\eta; t, w') = -\nu(w')$, hence η is

(t, w')-prepared and dominant. If $\nu(x^r) > \nu(w)$, we have $\lambda(\eta^*; x, w) = 0$. Thus, we can write

$$\eta = \left(\mu_1 x \frac{\partial}{\partial x} + \mu_2 w \frac{\partial}{\partial w} + \mu_3 y \frac{\partial}{\partial y}\right) + \eta^{**},$$

where $(\mu_1, \mu_2, \mu_3) \neq (0, 0, 0)$ and $\lambda(\eta^{**}; x, w) > 0$. By equations (23) we have

$$\eta - \eta^{**} = \frac{\mu_1}{d} x' \frac{\partial'}{\partial x'} + \frac{d\mu_2 - p\mu_1}{d}(w' + c)\frac{\partial'}{\partial w'} + \mu_3 y \frac{\partial}{\partial y},$$

where $\alpha(\eta^{**}; t, w') = \lambda(\eta^{**}; t, w') > 0$. We obtain

$$0 = \alpha(\eta - \eta^{**}; t, w') = \alpha(\eta; t, w') \tag{25}$$

$$0 \geq \lambda(\eta - \eta^{**}; t, w') = \lambda(\eta; t, w'). \tag{26}$$

This ends the proof of statement 3. Note that if $d\mu_2 - p\mu_1 \neq 0$ then η is (t, w')-prepared and dominant. If $d\mu_2 - p\mu_1 = 0$ and $\mu_3 \neq 0$, we have that η is (t, w')-prepared and recessive. Now, if η where (x, y)-prepared and recessive, then $\mu_3 \neq 0$. This proves statement 4. □

Proposition 10. *Consider η as in equation (15). After performing finitely many etale w-Puiseux packages, either we get transversal formal maximal contact or we obtain one of the following properties:*

a) *The vector field η is strongly (x, w)-prepared dominant and this property persists under new etale w-Puiseux packages.*
b) *The vector field η is strongly (x, w)-prepared recessive and this property persists under new etale w-Puiseux packages.*

Proof. By the two dimensional desingularization for vector fields [14] and since we do not get maximal contact, we can obtain η written down as

$$\eta = f(x, w)\theta + g(x, w) y \frac{\partial}{\partial y}; \quad \theta = xh(x, w)x\frac{\partial}{\partial x} + \frac{\partial}{\partial w}.$$

Under new etale w-Puiseux packages, this form persist. Let us see it. First, we know that

$$\theta = \frac{1}{d}\left\{t^d h'(t, w') t \frac{\partial'}{\partial t} + \left(\frac{d}{t^d} - p(w' + c)t^d h'(t, w')\right)\frac{\partial'}{\partial w'}\right\}$$

where $h'(t, w') = h(t^d, t^p(w' + c))$. This allows us to write $\theta = t^{-d} W(t, w')\theta'$, where

$$W(t, w') = 1 - t^{2d}(p/d)(w' + c)h'(t, w')$$

is a unit and θ' has the same form as θ. Note that $W(t, w') - 1$ is divisible by t. Now, we write

$$\eta = f'(t, w')t^{-d} W(t, w')\theta' + g'(t, w') y \frac{\partial}{\partial y},$$

where $f'(t, w') = f(t^d, t^p(w' + c))$ and $g'(t, w') = g(t^d, t^p(w' + c))$. By the standard desingularization of functions, we can perform new etale w-Puiseux packages to obtain that

$$f = x^\rho U(x, w); \; g = x^\tau V(x, w),$$

where U, V and ρ, τ satisfy to the properties in Definition 5 (note that it is possible that ρ or τ are negative; to recover a non-meromorphic vector field we can multiply by a suitable power of x). By performing new etale w-Puiseux packages, the difference $\nu(x^\tau) - \nu(x^\rho)$ increases the positive amount $\nu(w)$. If this difference is positive, we are in case a), if it is always negative, we obtain case b). If it is zero, in the next step it is positive. □

Remark 13. The above proof also shows that if η is strongly (x,w)-prepared and dominant, it is so with respect to (t,w'). Nevertheless, it is not always true that if η is strongly (x,w)-prepared and recessive the same holds with respect to (t,w'). We start with $\rho>\tau$, by it can happen that $\rho'\leq\tau'$ and in this case, after a new etale w-Puiseux package we would obtain a dominant situation.

Remark 14. Assume that we are in one of the situations a) or b) described in Proposition 10. Let us perform an etale w-Puiseux package. Then we have

$$\alpha(\eta;t,w')=\left\{\begin{array}{ll}\alpha(\eta;x,w)-\nu(w) & \text{dominant case a)}\\ \alpha(\eta;x,w) & \text{recessive case b)}\end{array}\right.$$

To see this, the only difficulty is the recessive case. Note that since the recessive situation is stable under any finite sequence of etale w-Puiseux packages, we have $\nu(x^{\rho-\tau})>\nu(w)$, and this is enough to assure the above formula.

6.5. **Preparations in three variables.** Consider a vector field $\xi\in\mathrm{Der}_k\mathcal{O}[\log x]$, that we write $\xi=\sum_{s=-1}^{\infty}y^s\eta_s$ where the η_s are like in equation (19).

Definition 6. *Let $h=\hbar(\xi;\mathcal{A})$ be the main height. We say that ξ is* main-vertex prepared *with respect to $\mathcal{A}$ when η_h is (x,w)-prepared and dominant. If in addition η_h is strongly (x,w)-prepared, we say that ξ is* strongly main-vertex prepared. *We say that $\mathcal{L}$ is* well prepared *with respect to $\mathcal{A}$ if there is $\xi\in\mathcal{L}_{\mathcal{A}}[\log x]$ that is x-prepared and strongly main-vertex prepared.*

Remark 15. If ξ is main-vertex prepared, we have $\hbar(\xi;\mathcal{A})\geq 0$. Moreover, if $\alpha(\xi;\mathcal{A})=0$ and ξ is main-vertex prepared with $\hbar(\xi;\mathcal{A})=0$, then ξ is a non-singular vector field.

Proposition 11. *Assume that ξ is x-prepared and strongly main-vertex prepared with respect to $\mathcal{A}$. Let us perform an etale w-Puiseux package to obtain $\mathcal{A}'=(\mathcal{O}',(t,w',y))$. Then ξ is t-prepared, strongly main-vertex prepared with respect to $\mathcal{A}'$ and the main height does not vary, that is $\hbar(\xi;\mathcal{A}')=\hbar(\xi;\mathcal{A})$.*

Proof. The fact that ξ is t-prepared has been proved in subsection 6.1. The decomposition $\xi=\sum_{s=-1}^{\infty}y^s\eta_s$ is the same one with respect to x,w,y and with respect to t,w',y. Let us put $h=\hbar(\mathcal{L};\mathcal{A})$. By hypothesis η_h is strongly (x,w)-prepared and dominant and hence it is also strongly (t,w')-prepared and dominant, in view of Remark 13. Now we have only to show that h is also the main height $\hbar(\xi;t,w',y)$ relatively to $\mathcal{A}'$. By Corollary 1 we have

$$\alpha(\eta_h;t,w')=\alpha(\eta_h;x,w)-\nu(w),$$

since η_h is (x,w)-prepared and dominant. For any other index s we have

$$\alpha(\eta_s;t,w')\geq\alpha(\eta_s;x,w)-\nu(w),$$

and this is enough to see that η_h gives the main height for ξ with respect to $\mathcal{A}'$. □

Proposition 12. *Assume that $\xi=\sum_{s=-1}^{\infty}y^s\eta_s$ is x-prepared and strongly main-vertex prepared with respect to $\mathcal{A}$. Let us put $h=\hbar(\xi;x,w,y)$. By performing finitely many etale w-Puiseux packages we have the following properties:*

(1) *For any $s < h$, the vector field η_s is strongly (x, w)-prepared and this is stable, with the same character dominant or recessive, under any new finite sequence of etale w-Puiseux packages.*
(2) *The critical segment $\mathcal{C}(\xi; \mathcal{A})$ does not vary under any new finite sequence of etale w-Puiseux packages, and all the levels $s \in \mathcal{C}(\xi; \mathcal{A})$ have the same character dominant or recessive.*

Proof. The first statement is a corollary of Proposition 10. Let us prove the second statement, assuming that statement 1 holds. Let us perform an etale w-Puiseux package. In view of Remark 14 we have

$$\alpha(\eta_s; t, w') = \begin{cases} \alpha(\eta_s; x, w) - \nu(w) & \text{dominant case} \\ \alpha(\eta_s; x, w) & \text{recessive case} \end{cases}$$

Thus, the critical segment thus not vary if η_s is dominant for all $s \in \mathcal{C}(\xi; \mathcal{A})$. If there is an $s_0 \in \mathcal{C}(\xi; \mathcal{A})$ such that η_{s_0} is dominant, then all the recessive s in the critical segment disappear under a new w-Puiseux package and we are in the first case. Finally, if under any finite sequence of etale w-Puiseux packages there is no dominant η_s that appears in the critical segment, the elements in the critical segment are also stable. □

Definition 7. *We say that $\xi = \sum_{s=-1}^{\infty} y^s \eta_s$ is* completely prepared *with respect to $\mathcal{A}$ if it is x-prepared, strongly main-vertex prepared and the properties 1 and 2 of Proposition 12 hold. We have two possible situations:*

a) Dominant critical segment. *The η_s corresponding to s in the critical segment are strongly (x, w)-prepared and dominant.*
b) Recessive critical segment. *The η_s corresponding to s in the critical segment are strongly (x, w)-prepared and recessive.*

Remark 16. Assume that $\xi = \sum_{s=-1}^{\infty} y^s \eta_s$ is completely prepared with respect to $\mathcal{A}$. Let $\chi = \chi(\xi; \mathcal{A})$ be the critical height and $h = \hbar(\xi; \mathcal{A})$ the main height. We have $\chi \leq h$. Moreover, since η_h is strongly (x, w)-prepared and dominant, in the case of a recessive critical segment we have $\chi \leq h - 1$.

6.6. **Critical initial part and critical polynomial.** Let us consider a vector field $\xi = \sum_{s=-1}^{\infty} y^s \eta_s \in \mathrm{Der}_k(\mathcal{O})[\log x]$ and write it as

$$\xi = \sum_{s=-1}^{\infty} \sum_j y^s x^j \left\{ a_{sj}(w) x \frac{\partial}{\partial x} + b_{sj}(w) \frac{\partial}{\partial w} + c_{sj}(w) y \frac{\partial}{\partial y} \right\}.$$

We know that

$$\delta(\xi; x, w, y) = \min \left\{ \nu(x^j y^s); \quad (a_{sj}(w), b_{sj}(w), c_{sj}(w)) \neq (0, 0, 0) \right\}.$$

Put $\delta = \delta(\xi; x, w, y)$. We define the *critical initial part* of ξ by

$$\mathrm{Crit}(\xi; x, w, y) = \sum_{\nu(x^j y^s) = \delta} y^s x^j \left\{ a_{sj}(w) x \frac{\partial}{\partial x} + b_{sj}(w) \frac{\partial}{\partial w} + c_{sj}(w) y \frac{\partial}{\partial y} \right\}. \tag{27}$$

Obviously, if we put $\xi^* = \xi - \mathrm{Crit}(\xi; x, w, y)$ we have $\delta(\xi^*; x, w, y) > \delta(\xi; x, w, y)$.

Definition 8. *Take $\delta \in \Gamma$. A monic polynomial $P(x, y) \in k[x, y]$ given by*

$$P(x, y) = y^m + \lambda_{m-1} x^{n_1} y^{m-1} + \lambda_{m-2} x^{n_2} y^{m-2} + \cdots + \lambda_0 x^{n_m}$$

is called ν-homogeneous or degree δ *if and only if $\nu(x^{n_j} y^{m-j}) = \nu(y^m) = \delta$ for any j such that $\lambda_j \neq 0$. It is called a* Tchirnhausen polynomial *if $\lambda_{m-1} = 0$.*

Remark 17. Let us perform an etale y-Puiseux package, to obtain coordinates t, w, y' such that $t^d = x$ and $y' = y/t^p - c$. Consider a monic ν-homogeneous polynomial $P = P(x,y)$ or degree δ. Then

$$P = P(t^d, t^p(y'+c)) = t^{q'} P(1, y'+c),$$

where $\nu(t^{q'}) = \delta$ and $P(1, y'+c)$ is a monic polynomial of degree m in the variable y'. Write $P(1, y'+c) = y'^{h'} Q(y')$, where $Q(0) \neq 0$. We have $h' \leq m$. Moreover, the only possibility to have $m = h'$ is that $P(x,y) = (y - cx^{n_1})^m$. This cannot occur when $P(x,y)$ is a Tchirnhausen polynomial. Hence if $P(x,y)$ is a Tchirnhausen polynomial we have $h' < m$. This argument is crucial in most of the procedures of reduction of singularities in characteristic zero.

Lemma 13. *Assume that* $\xi = \sum_{s=-1}^{\infty} y^s \eta_s$ *is completely prepared relatively to* $\mathcal{A}$. *Then, the critical initial part* $\xi_0 = Crit(\xi; x, w, y)$ *satisfies that*

1. $\xi_0(x) = \xi_0(y) = 0$, *in the case of dominant critical segment.*
2. $\xi_0(x) = \xi_0(w) = 0$, *in the case of recessive critical segment.*

More precisely, the critical initial part ξ_0 *takes one of the following forms*

$$\xi_0 = \lambda x^q \sum_{s=0}^{\chi} \lambda_s y^s x^{q_s} \frac{\partial}{\partial w}; \quad \textit{dominant critical segment case} \tag{28}$$

$$\xi_0 = \lambda x^q \sum_{s=-1}^{\chi} \lambda_s y^s x^{q_s} y \frac{\partial}{\partial y}; \quad \textit{recessive critical segment case} \tag{29}$$

where $\lambda \neq 0$, $\lambda_\chi = 1$ *and* $\nu(y^s x^{q_s}) = \delta - \nu(x^q)$ *for each* s *with* $\lambda_s \neq 0$.

Proof. Put $h = \hbar(\xi; x, w, y)$. Recall that for any $s \leq h$ the vector field

$$\eta_s = \sum_j x^j \left\{ a_{sj}(w) x \frac{\partial}{\partial x} + b_{sj}(w) \frac{\partial}{\partial w} + c_{sj}(w) y \frac{\partial}{\partial y} \right\} = \sum_j x^j \eta_{sj}$$

is (x,w)-strongly prepared. Put $\alpha_s = \alpha(\eta_s; x, w)$ and let us take r_s such that $\nu(x^{r_s}) = \alpha_s$. Write $\eta_s = x^{r_s} \tilde{\eta}_s$. In view of definition 5, we have that

$$\tilde{\eta}_s = \begin{cases} \mu_s \partial/\partial w + x \overline{\eta}_s & \text{(dominant case)} \\ \mu_s y \partial/\partial y + x \overline{\eta}_s & \text{(recessive case)} \end{cases}$$

We end by putting $\lambda_s = \mu_s / \lambda$ if s is in the critical segment and $\lambda_s = 0$ otherwise. □

Definition 9. *In the situation of Lemma 13, we define the* critical polynomial $P_\xi(x,y)$ *of* ξ *with respect to* x, w, y *to be*

$$P_\xi(x,y) = \begin{cases} \xi_0(w)/\lambda x^q = \sum_{s=0}^{\chi} \lambda_s y^s x^{q_s} & \textit{(dominant critical segment)} \\ \xi_0(y)/\lambda x^q = \sum_{s=-1}^{\chi} \lambda_s y^{s+1} x^{q_s} & \textit{(recessive critical segment)} \end{cases}$$

(It is a ν-homogeneous monic polynomial of ν-degree $\chi\nu(y)$, respectively $(\chi+1)\nu(y)$, in the case of a dominant, respectively recessive critical segment.)

Remark 18. The critical initial part is obtained from the critical polynomial by the formula

$$\mathrm{Crit}(\xi; x, w, y) = \begin{cases} \lambda x^q P_\xi(x,y) \partial/\partial w & \text{(dominant critical segment)} \\ \lambda x^q P_\xi(x,y) \partial/\partial y & \text{(recessive critical segment)} \end{cases}$$

6.7. **Stability of the main height. Dominant critical segment.** In this subsection we start to study the effect of an etale y-Puiseux package on the main height. Let us consider $\xi \in \mathrm{Der}_k \mathcal{O}[\log x]$ and denote

$$h = \hbar(\xi; x, w, y); \quad \chi = \chi(\xi; x, w, y); \quad \delta = \delta(\xi; x, w, y); \quad \xi_0 = \mathrm{Crit}(\xi; x, w, y).$$

Let us perform an etale y-Puiseux package, to obtain t, w, y' such that $t^d = x$ and $y' = y/t^p - c$. We recall that

$$x\frac{\partial}{\partial x} = \frac{1}{d}\left\{t\frac{\partial'}{\partial t} - p(y'+c)\frac{\partial'}{\partial y'}\right\}; \ \frac{\partial}{\partial w} = \frac{\partial'}{\partial w}; \ y\frac{\partial}{\partial y} = (y'+c)\frac{\partial'}{\partial y'}. \tag{30}$$

Lemma 14. $\alpha(\xi; t, w, y') \geq \delta = \delta(\xi; x, w, y)$.

Proof. In view of the valuative behavior of the invariant $\alpha(\_; x, w)$ and because of the "monomial" definition of $\delta(\_; x, w)$, it is enough to verify the case that ξ is of one of the following monomial types

$$\xi = y^s x^m w^n x\frac{\partial}{\partial x}; \ \xi = y^s x^m w^n \frac{\partial}{\partial w}; \xi = y^s x^m w^n y\frac{\partial}{\partial y},$$

where $\nu(y^s x^m) \geq \delta$. Note that

$$y^s x^m w^n = x'^{sp+dm}(y'+c)w^n,$$

where $\nu(x'^{sp+dm}) = \nu(y^s x^m) \geq \delta$. Now, in view of the equations 30 we have that $\xi = x'^{sp+dm}\xi^*$, where $\alpha(\xi^*; t, w, y') \geq 0$ and we are done. □

Proposition 13. *Assume that ξ is completely prepared with a dominant critical segment and $h \geq 1$. Let us perform an etale y-Puiseux package. After performing finitely many subsequent etale w-Puiseux packages, we obtain $\mathcal{A}'$ such that ξ is completely prepared with respect to $\mathcal{A}'$ and $h' = h(\xi; \mathcal{A}') \leq \chi$. Moreover, if the critical polynomial $P_\xi(x, y)$ is a Tchirnhausen polynomial, we have $h' < \chi \leq h$.*

Proof. Denote $\xi = \xi_0 + \xi^*$. We know that $\delta(\xi^*; x, w, y) > \delta$ and hence, by Lemma 14 we have $\alpha(\xi^*; t, w, y) > \delta$. On the other hand $\xi_0 = \lambda x^q P_\xi(x, y)\frac{\partial}{\partial w}$. After performing the etale y-Puiseux package, we obtain

$$\xi_0 = \lambda x'^{q'} P_\xi(1, y'+c)\frac{\partial'}{\partial w}$$

where $\nu(t^{q'}) = \delta$. If $\xi^{*\prime} = \lambda^{-1}t^{-q'}\xi^*$, we have $\alpha(\xi^{*\prime}; t, w, y') > 0$. Write

$$\xi' = \lambda^{-1}t^{-q'}\xi = P_\xi(1, y'+c)\frac{\partial'}{\partial w} + \xi^{*\prime}, \tag{31}$$

Then $\alpha(\xi'; t, w, y') = 0$. Let $h' \leq \chi$ be such that

$$P_\xi(1, y'+c) = y'^{h'}\sum_{s=h'}^{\chi}\lambda'_s y'^{s-h'}; \ \lambda'_{h'} \neq 0.$$

It is obvious that $h' \leq \chi \leq h$ and, in view of Remark 17, we have that $h' < \chi \leq h$ in the case that $P_\xi(x, y)$ is a Tchirnhausen polynomial. Moreover, we see that $h' = \hbar(\xi'; \mathcal{A}') = \hbar(\xi; \mathcal{A}')$. Write $\xi = \sum_{s=-1}^{\infty} y'^s \eta'_s$, as usual, with

$$\eta'_s = a'_s(t, w)x'\frac{\partial}{\partial' t} + b'_s(t, w)\frac{\partial'}{\partial w} + c'_s(t, w)y'\frac{\partial'}{\partial y'}.$$

Then $\eta'_{h'}$ is (t, w)-prepared and dominant in view of Equation 31. By performing new etale w-Puiseux packages to obtain a completely prepared ξ, the main height h' is not modified and we are done. □

6.8. **Stability of the main height. Recessive critical segment.** Take here the situation and notations of the previous Subsection 6.7.

Let us assume that ξ is completely prepared with respect to $\mathcal{A}$ with a recessive critical segment and $h \geq 1$. Recall that $\chi \leq h-1$ in view of Remark 16. We also have $\xi = \xi_0 + \xi^*$ where

$$\xi_0 = \lambda x^q P_\xi(x,y)\frac{\partial}{\partial y} = \lambda x^q \frac{P_\xi(x,y)}{y} y\frac{\partial}{\partial y},$$

where $P_\xi(x,y) = y^{\chi+1} + \sum_{s=-1}^{\chi-1} \lambda_s y^{s+1}x^{q_s}$ is the critical polynomial. After performing an etale y-Puiseux package, we have

$$\xi_0 = \lambda t^{q'} P_\xi(1, y'+c)\frac{\partial'}{\partial y'}$$

where $\nu(t^{q'}) = \delta$. Write $\xi^{*\prime} = \lambda^{-1}t^{-q'}\xi^*$, as in the proof of Proposition 13. We have $\alpha(\xi^{*\prime}; t, w, y') > 0$ and

$$\xi' = \frac{1}{\lambda t^{q'}}\xi = P_\xi(1, y'+c)\frac{\partial}{\partial y'} + \xi^{*\prime} = \xi'_0 + \xi^{*\prime}. \tag{32}$$

Let $-1 \leq h' \leq \chi$ be such that

$$P_\xi(1, y'+c) = y'^{h'+1}\sum_{s=h'}^{\chi} \lambda'_s y'^{s-h'} = y'^{h'+1}Q(y'); \quad Q(0) \neq 0.$$

It is obvious that $h' \leq \chi \leq h-1$. Moreover, if $P_\xi(x,y)$ is a Tchirnhausen polynomial we have $h' < \chi \leq h-1$, in view of Remark 17.

Remark 19. We have that $h' = \hbar(\xi'; t, w, y')$, but the main vertex is not dominant. For this reason, we will do a coordinate change in the dependent variables of the type $w'' = w + y'$.

Let us do a coordinate change $w'' = w + y'$ to obtain $\mathcal{A}'' = (\mathcal{O}', \mathbf{z}'' = (t, w'', y'))$. We have

$$\xi'_0 = P_\xi(1, y'+c)\frac{\partial'}{\partial y'} = y'^{h'+1}Q(y')\left\{\frac{\partial''}{\partial w''} + \frac{\partial''}{\partial y'}\right\}. \tag{33}$$

Let us write $\xi' = \sum_{s=-1}^{\infty} y'^s \eta''_s$, where

$$\eta''_s = a''_s(x', w'')x'\frac{\partial''}{\partial x'} + b''_s(x', w'')\frac{\partial''}{\partial w''} + c''_s(x', w'')y'\frac{\partial''}{\partial y'}.$$

Recalling that $\xi' = \xi'_0 + \xi^{*\prime}$ and $\alpha(\xi^{*\prime}; x', w'', y') > 0$, we see from Equations 33 that $\alpha(\eta''_{h'+1}; x', w'', y') = 0$ and $\eta''_{h'+1}$ is dominant and prepared with respect to (t, w'', y'). In particular $\hbar(\xi'; t, w'', y') \leq h'+1$.

As a consequence, by performing new etale w''-Puiseux packages, we obtain $\tilde{\mathcal{A}}$ such that ξ' is completely prepared and $h(\xi'; \tilde{\mathcal{A}}) \leq h'+1$. Now, recalling that $h' \leq \chi \leq h-1$ and in the case of a Tchirnhausen critical polynomial we have $h' < \chi \leq h-1$, we have proved the following statement:

Proposition 14. *Let ξ be completely prepared with a recessive critical segment and assume $h = \hbar(\xi; \mathcal{A}) \geq 1$. Let us perform an etale y-Puiseux package. After performing a coordinate change in the dependent variables and finitely many subsequent etale w-Puiseux packages, we obtain $\tilde{\mathcal{A}}$ such that ξ is completely prepared with respect to $\tilde{\mathcal{A}}$ and $\tilde{h} = \tilde{\hbar}(\xi; \tilde{\mathcal{A}}) \leq \chi + 1 \leq h$. Moreover, if the critical polynomial $P_\xi(x,y)$ is a Tchirnhausen polynomial, we have $\tilde{h} < \chi + 1 \leq h$.*

6.9. **The condition of Tchirnhaus.** Let $\xi \in \mathrm{Der}_k\mathcal{O}[\log x]$ be completely prepared with respect to $\mathcal{A}$. Put $h = h(\xi;\mathcal{A})$, $\chi = \chi(\xi;\mathcal{A})$ and assume $h \geq 1$. We have the following possible cases:

A) The critical polynomial is not Tchirnhausen and the critical segment is dominant with $\chi = h$.
B) The critical polynomial is not Tchirnhausen and the critical segment is recessive with $\chi = h - 1$.
C) We have one of the following properties:
   (a) The critical polynomial is Tchirnhausen.
   (b) The critical segment is recessive and $\chi < h - 1$.
   (c) The critical segment is dominant and $\chi < h$.

The last case C corresponds to a winning situation in the sense of the following proposition

Proposition 15. *Assume we are in case C above. Let us perform an etale y-Puiseux package. By performing a subsequent coordinate change in the dependent variables (if it is necessary) and finitely many etale w-Puiseux packages, we obtain* $\tilde{\mathcal{A}}$ *such that* ξ *is completely prepared with respect to* $\tilde{\mathcal{A}}$ *and* $\tilde{h} = h(\tilde{\mathcal{L}};\tilde{\mathcal{A}}) < h$.

Proof. Direct consequence of Propositions 13 and 14. □

Next subsections are devoted to the study of situations B and A.

6.10. **Tchirnhausen preparation. Recessive case.** In this subsection we introduce a *recessive Tchirnhausen preparation algorithm* in order to deal with the case B of the preceding subsection. This algorithm is based on the following two definitions.

Definition 10. *Let* $\varsigma > 0$ *be a positive element of the value group* $\Gamma \subset \mathbb{R}$. *Consider* $\mathcal{A} = (\mathcal{O},(x,w,y))$. *We say that* (x,w) *is* recessive *for* ς *if and only if we have* $\varsigma > \sum_{i=0}^{N} \nu(w_i)$ *for any finite sequence of etale w-Puiseux Packages*

$$\mathcal{A} = \mathcal{A}_0 \mapsto \mathcal{A}_1 \mapsto \cdots \mathcal{A}_N,$$

where (x_i, w_i, y) *is the coordinate system in* $\mathcal{A}_i$, $i = 0, 1, \ldots, N$.

An example of this situation is obtained if we are in the case B of Subsection 6.9. More generally, let ξ be completely prepared with recessive critical segment and put $h = \hbar(\xi;x,w,y)$, $\chi = \chi(\xi;x,w,y)$. Let (α,h) be the main vertex and (β,χ) the "critical" vertex. Then (x,w) is recessive for $\varsigma = (h-\chi)\nu(y) - \beta + \alpha$.

Remark 20. Assume that ξ and $\mathcal{A}$ are in the situation of case B of Subsection 6.9. Then, there is an integer number $p \in \mathbb{Z}_{>0}$ such that $\nu(y) = \nu(x^p)$. Indeed, this is always true when we have a ν-homogeneous Tchirnhausen polynomial, that we write

$$P(x,y) = y^m + \lambda_1 x^{n_1} y_{m-1} + \cdots + \lambda_m x^{n_m}$$

since $\lambda_1 \neq 0$ implies that $\nu(y) = \nu(x^{n_1})$.

Definition 11. *Consider a vector field* $\xi \in Der_k(\mathcal{O})[\log x]$ *and* $\mathcal{A}$ *with coordinates* (x,w,y). *Write* $\xi = x^{\tilde{q}}\xi'$, *where* $\alpha(\xi';x,w,y) = 0$. *Consider two elements* ϵ, γ_0 *of the value group* Γ *and take* $h \in \mathbb{Z}_{\geq 1}$. *Let us do the decomposition* $\xi' = \sum_{s=-1}^{\infty} y^s \eta'_s$ *associated to* (x,w,y). *We say that* $(\xi;\mathcal{A};\epsilon,\gamma_0,h,p)$ *is a* recessive preparation step of order $p \in \mathbb{Z}_{\geq 0}$ *if the following properties hold*

(1) *$h=\hbar(\xi;x,w,y)$ and $\gamma_0\le\nu(y)$.*
(2) *There is $q\in\mathbb{Z}_{\ge 1}$ such that $\epsilon=\nu(x^q)$ and (x,w) is recessive for $\gamma_0-\epsilon$.*
(3) *$\gamma_0\le\nu(x^p)$*
(4) *There are units $U(x,w),V(x,w)$ such that the levels η'_h, η'_{h-1}, and η'_{h-2} take the forms*

$$
\begin{array}{rcl}
\eta'_h &=& U(x,w)\{\partial/\partial w + xc_h(x,w)y\partial/\partial y\},\\
\eta'_{h-1} &=& x^qV(x,w)\{xa_{h-1}(x,w)x\partial/\partial x + xb_{h-1}(x,w)\partial/\partial w + y\partial/\partial y\},\\
\eta'_{h-2} &=& a_{h-2}(x,w)x\partial/\partial x + b_{h-2}(x,w)\partial/\partial w + x^{q+p}c_{h-2}(x,w)y\partial/\partial y,
\end{array}
\tag{34}
$$

We say that $(\xi;\mathcal{A};\epsilon,\gamma_0,h,p)$ is a final recessive step *if in addition we have that $\nu(y)<\nu(x^p)$.*

Remark 21. Assume that ξ and $\mathcal{A}$ are in the situation of case B of Subsection 6.9. Take $p\in\mathbb{Z}_{>0}$ such that $\nu(y)=\nu(x^p)$ and $\gamma_0=\nu(y)$. Let (α,h) be the main vertex and $(\beta,h-1)$ the critical vertex and put $\epsilon=\beta-\alpha$. Then $(\xi;\mathcal{A};\epsilon,\gamma_0,h,p)$ is a (non-final) recessive preparation step or order p.

Proposition 16. *Assume that $(\xi,\mathcal{A};\epsilon,\gamma_0,h,p)$ is a recessive preparation step of order p. There is a coordinate change $y^*=y-x^pg(x,w)$ such that $(\xi,\mathcal{A}^*;\epsilon,\gamma_0,h,p^*)$ is a recessive preparation step of order $p^*>p$.*

Proof. Take $g(x,w)$ in the Hensel closure of $\mathcal{O}$ and let us write $y^*=y-x^pg(x,w)$. Note that $\gamma_0\le\nu(y^*)$ since $\gamma_0\le\nu(y)$ and $\gamma_0\le\nu(x^pg(x,y))$. The property that (x,w) is recessive for $\gamma_0-\epsilon$ does not depend on y^*. We have

$$x\frac{\partial}{\partial x} = x\frac{\partial^*}{\partial x}+x^p\left(pg(x,w)+x\frac{\partial g(x,w)}{\partial x}\right)\frac{\partial^*}{\partial y^*} \tag{35}$$

$$\frac{\partial}{\partial w} = \frac{\partial^*}{\partial w}+x^p\frac{\partial g(x,w)}{\partial w}\frac{\partial^*}{\partial y^*} \tag{36}$$

$$y\frac{\partial}{\partial y} = y^*\frac{\partial^*}{\partial y^*}+x^p\frac{\partial^*}{\partial y^*} \tag{37}$$

Let us decompose $\xi'=\sum_{s=-1}^{\infty}y^{*s}{\eta'}^*_s$ as usual with respect to (x,w,y^*). Noting that $q<p$ since $\epsilon=\nu(x^q)<\gamma_0\le\nu(x^p)$, it is a straightforward computation from equations 35, 36 and 35 that $\hbar(\xi';x,w,y^*)=h$ and ${\eta'}^*_h$, ${\eta'}^*_{h-1}$ and ${\eta'}^*_{h-2}$ take the forms

$$
\begin{array}{rcl}
{\eta'}^*_h &=& U^*(x,w)\{\partial/\partial w + xc^*_h(x,w)y\partial/\partial y\},\\
{\eta'}^*_{h-1} &=& x^qV^*(x,w)\{xa^*_{h-1}(x,w)x\partial/\partial x + xb^*_{h-1}(x,w)\partial/\partial w + y\partial/\partial y\},\\
{\eta'}^*_{h-2} &=& a^*_{h-2}(x,w)x\partial/\partial x + b^*_{h-2}(x,w)\partial/\partial w + x^{q+p}c^*_{h-2}(x,w)y\partial/\partial y
\end{array}
$$

where $U^*(0,0)\neq 0$ and $V^*(0,0)\neq 0$. In order to end our proof it is enough to show that $g(x,w)$ may be chosen in such a way that x divides c^*_{h-2}.Let us put

$$
\begin{array}{rcccccl}
F &=& \xi'(y) &=& \sum_{s=0}^{\infty}y^sF_s(x,w) &=& \sum_{s=0}^{\infty}y^{*s}F^*_s(x,w)\\
H &=& \xi'(x^pg(x,w)) &=& \sum_{s=0}^{\infty}y^sH_s(x,w) &=& \sum_{s=0}^{\infty}y^{*s}H^*_s(x,w)\\
G &=& \xi'(y^*) &=& F-H &=& \sum_{s=0}^{\infty}y^{*s}G^*_s(x,w)
\end{array}
\tag{38}
$$

We have to prove that $G^*_{h-1}(x,w)$ is divisible by x^{q+p+1} after a suitable choice of $g(x,w)$. Let us decompose

$$
\begin{array}{rcll}
F &=& \tilde{F}+\overline{F} & ;\tilde{F}=y^{h-1}F_{h-1}+y^hF_h\\
H &=& \tilde{H}+\overline{H} & ;\tilde{H}=y^{h-1}H_{h-1}+y^hH_h
\end{array}
\tag{39}
$$

We have that $\overline{F}^*_{h-1}$ and $\overline{H}^*_{h-1}$ are divisible by x^{q+p+1}, since they are is divisible by x^{2p} and $2p \geq p+q+1$. Note also that

$$\tilde{H} = J + K; \quad J = y^{h-1}\eta'_{h-1}(x^p g(x,w)),\ K = y^h \eta'_h(x^p g(x,w)).$$

Moreover, J is divisible by x^{q+p+1} in view of form of η'_{h-1} in Definition 11. We also have that x^{2p} divides K^*_{h-1} and $2p > p+q+1$.

Thus, we have only to prove that after a suitable choice of $g(x,w)$ we can obtain that $\tilde{F}^*_{h-1}$ is divisible by x^{q+p+1}. Recall that $\tilde{F} = y^{h-1}(yF_h + F_{h-1})$ where

$$\begin{aligned} yF_h &= \eta'_{h-1}(y) = yV(x,w)x^q \\ F_{h-1} &= \eta'_{h-2}(y)/y = x^{q+p}c_{h-2}(x,w). \end{aligned} \tag{40}$$

Now, write $c_{h-1}(x,w) = f_0(w) + xf_1(x,w)$. If we put $g(x,w) = -f_0(w)/V(x,w)$ we are done. □

Let us show how to obtain a final recessive step. We start with ξ completely prepared with respect to $\mathcal{A}$ in the case B of Subsection 6.9. Thus we have a recessive preparation step $(\xi, \mathcal{A}; \epsilon, \gamma_0, h, p)$ of order p, where $\gamma_0 = \nu(y) = \nu(x^p)$. Since $\nu(x^p) = \nu(y)$, it is not a final recessive step. We do a coordinate change $y_1 = y - x^p g_1(x,w)$ as in Proposition 16 to obtain a new recessive preparation step $(\xi, \mathcal{A}_1; \epsilon, \gamma_0, h, p_1)$ with $p_1 > p$. We repeat to obtain

$$y_{j+1} = y_j - x^{p_j} g_j(x,w),$$

where (x, w, y_j) are the coordinates for a recessive preparation step $(\xi; \mathcal{A}_j; \epsilon, \gamma_0, h, p_j)$ of order p_j and $p_{j+1} > p_j$. There are two possibilities:

1. We have $\nu(y_j) \geq \nu(x^{p_j})$ for all j. In this case we obtain a transversal formal maximal contact element $\hat{f} \in \hat{\mathcal{O}}$ as the limit of the y_j.
2. There is an index j_0 such that $\nu(y_{j_0}) < \nu(x^{p_{j_0}})$. In this case we obtain a final recessive step $(\xi; \mathcal{A}_{j_0}; \epsilon, \gamma_0, h, p_{j_0})$.

Proposition 17. *Assume that we have a final recessive step $(\xi; \mathcal{A}; \epsilon, \gamma_0, h, p)$. After performing finitely many w-Puiseux packages we obtain $\mathcal{A}'$ such that ξ is completely prepared with respect to $\mathcal{A}'$ and we are in the winning situation C of Subsection 6.9.*

Proof. Note that if $U(x,w,y)$ is a unit, then $(U(x,w,y)\xi; \mathcal{A}; \epsilon, \gamma_0, h, p)$ is still a final recessive step.

Let us perform an etale w-Puiseux package to obtain $\mathcal{A}_1$ whose coordinates are (t, w_1, y), where $t^d = x$ and $w_1 = w/t^{\tilde{p}} - c$. In view of Equations 34 we see that ξ is main-vertex prepared with respect to (x,w,y) and hence the main height h is not changed under the etale w-Puiseux package. The form of Equations 34 persists, with the following observations:

1. The parameter ϵ is transformed into $\epsilon_1 = \epsilon + \nu(w)$. Anyway, we still have that (t, w_1) is recessive for $\gamma_0 - \epsilon_1$ (see Definition 10).
2. The order p is transformed into $p_1 = pd$.

In particular $(\xi; \mathcal{A}_1; \epsilon_1, \gamma_0, h, p_1)$ is a final recessive preparation step.

Thus, we can multiply by a unit ξ an do successive etale w-Puiseux packages in order to obtain that in addition ξ is completely prepared with respect to $\mathcal{A}$. Let us look at Equations 34. Let us put

$$\alpha_{h-1} = \alpha(\eta'_{h-1}; x, w) \quad \alpha_{h-2} = \alpha(\eta'_{h-2}; x, w).$$

From the form of η'_{h-1} in Equations 34 we have that $\alpha_{h-1} = \nu(x^q) = \epsilon$. Since $\epsilon < \gamma_0 \leq \nu(y)$ we see that $\chi < h$. In particular, if we are in the case of a dominant critical segment, we are in one of the winning situations C. Assume that $\chi = h-1$ and we have a recessive critical segment. Recall that

$$\eta'_{h-2} = a_{h-2}(x,w)x\partial/\partial x + b_{h-2}(x,w)\partial/\partial w + x^{q+p}c_{h-2}(x,w)y\partial/\partial y$$

and we are assuming moreover that η'_{h-2} is prepared. If this level $h-2$ is dominant we are in a winning situation C, since it cannot be in the critical segment and thus the critical polynomial is a Tchirnhausen polynomial. If the level $h-2$ is recessive, from the above form of η'_{h-1} we deduce that $\alpha_{h-2} = \nu(x^{q+p}) = \epsilon + \nu(x^p)$. But we know that $\nu(x^p) > \nu(y)$ and thus the level $h-2$ cannot be in the critical segment. This ends the proof. □

6.11. **Tchirnhausen preparation. Dominant case.** In this Subsection we assume we are in case A of Subsection 6.9. That is, we have ξ completely prepared with respect to $\mathcal{A}$, the critical segment is dominant with $\chi = h$ and the critical polynomial is not Tchirnhausen. We also assume that $h \geq 2$ since the cases $h \leq 1$ correspond to log-elementary singularities.

Proposition 18. *We can perform a coordinate change $y^* = y - x^p g(x,w)$, with $\nu(x^p) = \nu(y)$ to obtain $\mathcal{A}^*$ is such a way that after performing finitely many etale w-Puiseux packages, we get $\mathcal{A}'$ such that ξ is completely prepared with respect to $\mathcal{A}'$ with $h = \hbar(\xi;\mathcal{A}')$ and we are in one of the situations B or C of 6.9.*

Proof. Since the critical polynomial is not Tchirnhaus, we have that $\nu(y) = \nu(x^p)$ for some $p \in \mathbb{Z}_{\geq 0}$. Up to multiply ξ by a power of x, let us assume without loss of generality that $\alpha(\xi; x, w, y) = 0$. Denote $F = \xi(w) = \sum_{s=0}^{\infty} y^s F_s(x,w)$. We know that $F_h(0,0) \neq 0$. Moreover, in view of our hypothesis, we have $F_{h-1}(x,w) = x^p G_{h-1}(x,w)$, where $G_{h-1}(0,0) \neq 0$. By an argument like in Proposition 16, we can find a coordinate change of the form $y^* = y - x^p g(x,w)$ such that

$$F^*(x,w,y^*) = F(x,w,y^* + x^p g(x,w)) = \sum_{s=0}^{\infty} {y^*}^s F^*_s(x,w)$$

satisfies that $F^*_{h-1} = 0$. This condition eliminates the level $h-1$ from the critical segment if we persist in the situation A after subsequent etale w-Puiseux packages. □

7. Maximal contact

In this section we prove Theorem 4. Recall that we consider the case when $n = 3$ and ν is a valuation of arquimedean rank one with $\kappa_\nu = k$. We have a projective model M_0 of K, where P_0 is the center of ν at M_0. We assume that P_0 is a regular point of M_0 and there is $\hat{f} \in \widehat{\mathcal{O}}_{M_0,P_0}$ that has transversal maximal contact with ν. The rational rank r can be supposed to be $r = 1$ or $r = 2$, since if $r = 3$ the definition of transversal maximal contact makes no-sense (see the Introduction).

The computations in this section are essentially contained in the paper [5], but we include them for the sake of completeness.

7.1. **Maximal contact with rational rank two.** Take a regular system of parameters (x_1, x_2, y) of $\mathcal{O}_{M_0,P_0}$ such that $\nu(x_1), \nu(x_2)$ are $\mathbb{Z}$-linearly independent and

$$\hat{f} = y + \sum_{i,j} \lambda_{ij} x_1^i x_2^j.$$

Since ν is arquimedian, we may write $\hat{f}$ as the Krull limit $\hat{f} = \lim_{\mu\to\infty} f_\mu$, where

$$f_\mu = y + \sum_{i\nu(x_1)+j\nu(x_2)\leq\mu} \lambda_{ij} x_1^i x_2^j \in \mathcal{O}_{M_0,P_0}.$$

Note that $\nu(f_\mu) > \mu$ and, more precisely we have

$$\nu(f_\mu) = \min\{\nu(x_1^i x_2^j); \lambda_{ij} \neq 0,\, \nu(x_1^i x_2^j) > \mu\}.$$

In this paragraph we denote $Y_0 = \{P_0\}$, $Y_1 = \{x_1 = \hat{f} = 0\}$ and $Y_2 = \{x_2 = \hat{f} = 0\}$.

The next Lemma 15 may be proved by standard computations in terms of blow-ups and valuations and we leave the verification to the reader:

Lemma 15. *Let $\pi : M' \to M_0$ be the blow-up of M_0 with one of the centers Y_0, Y_1 or Y_2 and assume that if we use Y_1, respectively Y_2, as a center, then $\hat{f}(0, x_2, y) \in \mathcal{O}_{M_0,P_0}$, respectively $\hat{f}(x_1, 0, y) \in \mathcal{O}_{M_0,P_0}$. Let $P' \in M'$ be the center of ν at M'. Then P' belongs to the strict transform of $\hat{f} = 0$. More precisely , we have the following cases:*

T-01: *The center is Y_0 and $\mu = \nu(x_1) < \nu(x_2)$. In this case P_1 is in the strict transform of the formal curve $x_2 = \hat{f} = 0$ and there is a regular system of parameters (x_1', x_2', y^*) at $\mathcal{O}_{M',P'}$ such that $x_1' = x_1$, $x_2' = x_2/x_1$, $y^* = \hat{f}_\mu/x_1$. Moreover*

$$\hat{f}' = \hat{f}/x_1 = y^* + \sum_{i\nu(x_1)+j\nu(x_2)>\mu} \lambda_{ij} {x'}_1^{i+j-1} {x'}_2^j \in \mathcal{O}_{M',P'}$$

has transversal maximal contact with ν.

T-02: *The center is Y_0 and $\mu = \nu(x_2) < \nu(x_1)$. Similar to T-01.*

T-1: *The center is Y_1, where $\mu = \nu(x_1)$. In this case P_1 is the only point over P_0 in the strict transform of $\hat{f} = 0$ and there is a regular system of parameters (x_1', x_2', y^*) at $\mathcal{O}_{M',P'}$ such that $x_1' = x_1$, $x_2' = x_2$, $y^* = (f_\mu + \hat{h}_\mu(0, x_2))/x_1$, where $\hat{h}_\mu(x_1, x_2) = \hat{f} - f_\mu$. Moreover*

$$\hat{f}' = \hat{f}/x_1 = y^* + \sum_{\nu(x_1^i x_2^j)>\mu;\, i\geq1} \lambda_{ij} {x'}_1^{i-1} {x'}_2^j \in \mathcal{O}_{M',P'}$$

has transversal maximal contact with ν.

T-2: *The center is Y_2, where $\mu = \nu(x_2)$. Similar to T-1.*

Take a generator ξ_0 of $\mathcal{L}_{M_0,P_0}[\log x_1 x_2]$. Define the formal vector field $\hat{\xi}$ to be $\hat{\xi} = \xi_0$ if $\hat{f}$ divides $\xi_0(\hat{f})$ (this corresponds to saying that $\hat{f} = 0$ defines a formal invariant hypersurface) and $\hat{\xi} = \hat{f}\xi_0$ if $\hat{f}$ does not divide $\xi_0(\hat{f})$. Let us write

$$\hat{\xi} = \hat{a}_1 x_1 \frac{\partial}{\partial x_1} + \hat{a}_2 x_2 \frac{\partial}{\partial x_2} + \hat{b}\hat{f} \frac{\partial}{\partial \hat{f}}.$$

Note that $\hat{a}_1, \hat{a}_2, \hat{b}$ have no common factors. The *adapted (or logarithmic) order* of $\mathcal{L}$ at P_0 with respect to $x_1x_2\hat{f}$ is

$$\mathrm{LogOrd}(\mathcal{L}, \mathcal{O}_{M_0,P_0}; x_1x_2\hat{f}) = \mathrm{ord}_{\widehat{\mathcal{M}}_{M_0,P_0}}(\hat{a}_1, \hat{a}_2, \hat{b}) \in \mathbb{Z}_{\geq 0},$$

where $\mathrm{ord}_{\widehat{\mathcal{M}}_{M_0,P_0}}(-)$ means the $\widehat{\mathcal{M}}_{M_0,P_0}$-adic order (see also [4]).

Put $\zeta = \mathrm{LogOrd}(\mathcal{L}, \mathcal{O}_{M_0,P_0}; x_1x_2\hat{f})$. We say that Y_1 is *permissible* for $\mathcal{L}$ adapted to x_1x_2 if the two following properties hold:

(1) $\hat{f}(0, x_2, y) \in \mathcal{O}_{M_0,P_0}$. (Hence Y_1 is a subvariety of M_0)
(2) $\mathrm{ord}_{(x_1,\hat{f})}(\hat{a}_1, \hat{a}_2, \hat{b}) = \zeta$.

We give a symmetric definition for Y_2 being permissible. By definition Y_0 is always permissible.

Remark 22. If $\zeta \geq 2$, the condition 2 above implies condition 1, since in this case, the curve Y_1 must be contained in the locus

$$\xi_0(x_1)/x_1 = \xi_0(x_1)/x_2 = \xi_0(y) = 0.$$

If $\zeta = 1$ and $\hat{\xi} = \xi_0$, the same argument holds.

Lemma 16. *Let $\pi : M' \to M_0$ be the blow-up of M_0 with a permissible center Y_0, Y_1 or Y_2. Let $P' \in M'$ be the center of ν at M'. Then*

$$\mathrm{LogOrd}(\mathcal{L}, \mathcal{O}_{M_0,P_0}; x_1x_2\hat{f}) \geq \mathrm{LogOrd}(\mathcal{L}, \mathcal{O}_{M',P'}; x_1'x_2'\hat{f}').$$

Proof. We may assume that either the center of the blow-up is Y_1 or it is Y_0 and $\nu(x_1) < \nu(x_2)$ (the other cases follow from these by interchanging the roles of x_1, x_2). Then we have $\hat{f}' = \hat{f}/x_1$ and $\hat{\xi}' = x_1^{-\zeta}\hat{\xi}$ where

$$\hat{a}_1' = x_1^{-\zeta}\hat{a}_1;\ \hat{b}' = x_1^{-\zeta}(\hat{b} - \hat{a}_1'),$$

and $\hat{a}_2' = x_1^{-\zeta}(\hat{a}_2 - \hat{a}_1)$ if Y_0, $\hat{a}_2' = x_1^{-\zeta}\hat{a}_2$ if Y_1. The rest of the proof is given by the standard results on the blow-up of equimultiple centers. □

We proceed by induction on ζ. First, consider the case $\zeta \geq 2$.

We now define Hironaka's characteristic polygons (see for instance [8]). Take an element $\hat{g} = \sum_s \hat{f}^s g_{ijs} x_1^i x_2^j \in \widehat{\mathcal{O}}_{M_0,P_0} = k[[x_1, x_2, \hat{f}]]$ and an integer $\eta \in \mathbb{Z}_{>0}$. The Hironaka's *characteristic polygon* $\Delta(\hat{g}; x_1, x_2, \hat{f}; \eta)$ is the positive convex hull in $\mathbb{R}^2_{\geq 0}$ of the points of the form $(i/(\eta - s), j/(\eta - s))$, where $g_{ijs} \neq 0$ and $s < \eta$. Given a list $\{\hat{g}_l\}$ we define $\Delta(\{\hat{g}_l\}; x_1, x_2, \hat{f}; \eta)$ to be the convex hull of the union of the $\Delta(\hat{g}_l; x_1, x_2, \hat{f}; \eta)$. Now, we define

$$\Delta(\mathcal{L}; x_1, x_2, \hat{f}; \eta) = \Delta(\{\hat{a}_1, \hat{a}_2, \hat{b}\}; x_1, x_2, \hat{f}; \eta).$$

Let us list the properties of $\Delta_\eta = \Delta(\mathcal{L}; x_1, x_2, \hat{f}; \eta)$, similar to those used by Hironaka in his Bowdoin College Memoir [8]:

(1) $\Delta_\eta \neq \emptyset$. Otherwise $\hat{a}_1, \hat{a}_2, \hat{b}$ would be divisible by $\hat{f}$.
(2) $\Delta_\eta \subset \{(u, v);\ u + v \geq 1\}$ iff $\zeta \geq \eta$.
(3) $\Delta_\zeta \subset \{(u, v);\ u \geq 1\}$ iff condition 2 of permissibility holds for Y_1.
(4) $\Delta_\zeta \subset \{(u, v);\ v \geq 1\}$ iff condition 2 of permissibility holds for Y_2.

The characteristic polygon behaves under blow-up as in the classical case of varieties, as we show in the next Lemma 17. To see this, let us introduce the linear mappings $\sigma_{01}, \sigma_{02}, \sigma_1, \sigma_2$ defined as follows

$$\sigma_{01}(u,v) = (u+v-1, v), \quad \sigma_1(u,v) = (u-1, v),$$
$$\sigma_{02}(u,v) = (u, u+v-1), \quad \sigma_2(u,v) = (u, v-1).$$

Lemma 17. *Keep notations as in Lemma 15. Let $\pi : M' \to M_0$ be the blow-up of M_0 with a permissible center Y_0, Y_1 or Y_2. Let $P' \in M'$ be the center of ν at M'. Put $\Delta = \Delta(\mathcal{L}; (x_1, x_2, \hat{f}); \zeta)$. Then the characteristic polygon $\Delta(\mathcal{L}; (x_1', x_2', \hat{f}'); \zeta)$ is the positive convex hull of*

$$\sigma_{01}(\Delta), \sigma_{02}(\Delta), \sigma_1(\Delta), \sigma_2(\Delta)$$

*if we are respectively in the cases **T-01**, **T-02**, **T-1** and **T-2** of Lemma 15.*

Proof. Let $I \subset \widehat{\mathcal{O}}_{M_0,P_0}$ be the ideal generated by $\hat{a}_1, \hat{a}_2, \hat{f}$. Then the ideal $I' \subset \widehat{\mathcal{O}}'_{M_0,P_0}$ generated by $\hat{a}_1', \hat{a}_2', \hat{f}'$ is $I' = x_1^{-\zeta} I$, respectively $I' = y_1^{-\zeta} I$ if we are in the cases $(01), (1)$, respectively $(02), (2)$. Now we apply the classical remarks of Hironaka in his Bowdoin College seminar [8]. □

Now, we choose the following strategy to blow up. We select the blow-up center Y_0 until the characteristic polygon has only one vertex, this occurs after finitely many steps. Then, since we are in the case $\zeta \geq 2$, at least one of the centers Y_1, Y_2 is permissible, since it is equimultiple. Blow-up this curve. After finitely many operations the characteristic polygon intersects $\{(u,v);\ u+v < 1\}$ and hence the logarithmic order drops. We arrive in this way to the case $\zeta \leq 1$.

Assume now that $\zeta \leq 1$. If $\zeta = 0$ and $\hat{\xi} = \xi_0$, we get an elementary singularity and if $\hat{\xi} = \hat{f}\xi_0$ the foliation is in fact non-singular. Assume that $\zeta = 1$. By Remark 22, the case $\hat{\xi} = \xi_0$ can be handled as before. So we consider only the case $\hat{\xi} = \hat{f}\xi_0$. Blowing-up the origin (that is, we take the center Y_0 each time), we get as above that the characteristic polygon has exactly one vertex of integer coordinates, say $(\alpha, \beta) \in \mathbb{Z}^2_{\geq 0}$, where $\alpha + \beta \geq 1$. Assume that $\alpha + \beta \geq 2$; since $\zeta = 1$, we have either $\mathrm{ord}_{x_1,x_2,\hat{f}}(\xi_0(x_1), \xi_0(x_2)) = 0$ or $\hat{b}(0,0,\hat{f}) = \hat{f}U(\hat{f})$, with $U(0) \neq 0$. In both cases ξ_0 is non-nilpotent and we obtain an elementary singularity. It remains to study the case $\alpha + \beta = 1$. We have two possibilities $(\alpha, \beta) = (1,0)$ and $(\alpha, \beta) = (0,1)$, that can be treated in a similar way. Consider, for instance, the case $(\alpha, \beta) = (1,0)$, if $\nu(x_1) > \nu(x_2)$, we are done by blowing-up the origin, since we get $\zeta = 0$; if $\nu(x_1) < \nu(x_2)$ the situation repeats itself, but this cannot occur infinitely many times, since we are dealing with an arquimedian valuation ν.

This ends the proof of Theorem 4 in the case of rational rank $r = 2$.

7.2. **Maximal contact with rational rank one.** Take a regular system of parameters (x, w, y) of $\mathcal{O}_{M_0,P_0}$ such that

$$\hat{f} = y + \sum_{i,j} \lambda_{ij} x^i w^j.$$

In this paragraph we denote $Y_0 = \{P_0\}$ and $Y_1 = \{x_1 = \hat{f} = 0\}$. The next Lemma 18 may be proved by standard computations in terms of blow-ups and valuations and we left the verification to the reader:

Lemma 18. *Let $\pi : M' \to M_0$ be the blow-up of M_0 with one of the centers Y_0 or Y_1 and assume that if we use Y_1 as a center, then $\hat{f}(0, w, y) \in \mathcal{O}_{M_0,P_0}$. Then, the center $P' \in M'$ of ν at M' belongs to the strict transform of $\hat{f} = 0$. More precisely, we have the following cases:*

T_{01}**:** *The center is Y_0 and $\mu = \nu(x) < \nu(w)$. In this case P_1 is in the strict transform of the formal curve $w = \hat{f} = 0$ and there is a regular system of parameters (x', w', y^*) of $\mathcal{O}_{M',P'}$ such that $x' = x$, $w' = w/x$, $\hat{f}' = \hat{f}/x$ has transversal maximal contact with ν and has the form*

$$\hat{f}' = y^* + \sum_{i,j} \lambda'_{ij} x'^i w'^j.$$

T_{02}**:** *The center is Y_0 and $\mu = \nu(w) < \nu(x)$. Similar to the previous case, by the roles of x, w interchanged.*

T_{01}, c**:** *The center is Y_0 and $\mu = \nu(x) = \nu(w)$. Take a parameter $c \in k$ such that $\nu(w - cx) > \nu(x)$. We do the coordinate change $w^* = w - cx$ and we proceed as in the case* (01).

T_1**:** *The center is Y_1. In this case P_1 is in the strict transform of $\hat{f} = 0$ and there is a regular system of parameters (x', w', y^*) of $\mathcal{O}_{M',P'}$ such that $x' = x$, $w' = w$, $\hat{f}' = \hat{f}/x$ has transversal maximal contact with ν and it is written as*

$$\hat{f}' = y^* + \sum_{i,j} \lambda'_{ij} x'^i w'^j.$$

We define an $\{x, w, y, \hat{f}\}$-*formal Puiseux package* to be a sequence of blow-ups

$$M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_N = M'$$

such that:

(1) Each blow-up has center at the center $P_i \in M_i$ of the valuation in the projective model M_i.
(2) We get $(x_i, w_i, y_i, \hat{f}_i)$ at each P_i, obtained as in Lemma 18, starting from $(x_0, w_0, y_0, \hat{f}_0) = (x, w, y, \hat{f})$.
(3) Each blow-up is given by T_{01} or T_{02}, except the last blow-up that is given by (T_{01}, c), with $c \neq 0$.

A $\{x, w, y, \hat{f}\}$-formal Puiseux package exists and is unique. More precisely, if we put $\nu(w^d/x^p) = 0$ and c is such that $\nu(w^d/x^p - c) > 0$, the sequence of blow-ups is the reduction of singularities of the formal curve $w^d - cx^p = \hat{f} = 0$.

Now, let us consider a generator ξ_0 of $\mathcal{L}_{M_0,P_0}[\log x]$, that we write a follows:

$$\xi_0 = a(x, w, \hat{f}) x \frac{\partial}{\partial x} + b(x, w, \hat{f}) \frac{\partial}{\partial w} + \hat{h}(x, w, \hat{f}) \frac{\partial}{\partial \hat{f}},$$

where $a = \xi_0(x)/x$, $b = \xi_0(w)$, $\hat{h} = \xi_0(\hat{f})$. Note that a, b and $\hat{h}$ have no common factors. There are two cases that we will consider separately

(1) The formal hypersurface $\hat{f} = 0$ is invariant by ξ_0. Then $\hat{f}$ divides $\hat{h}$ and we can put $\hat{h} = \hat{g}\hat{f}$.
(2) The formal hypersurface $\hat{f} = 0$ is not invariant by ξ_0. Then $\hat{f}$ does not divide $\hat{h}$ and thus $\hat{f}a, \hat{f}b, \hat{h}$ have no common factors.

Let us put $\hat{\xi}_0 = \hat{\xi}_0$ if $\hat{f} = 0$ is invariant and $\hat{\xi}_0 = \hat{f}\xi_0$ if $\hat{f}$ is not invariant. In both cases we denote

$$\hat{a}_0 = \hat{\xi}_0(x)/x;\ \hat{b}_0 = \hat{\xi}_0(w); \hat{g}_0 = \hat{\xi}_0(\hat{f})/\hat{f}.$$

Then $\hat{a}_0, \hat{b}_0, \hat{g}_0$ have no common factors. Define the *logarithmic order* as

$$\mathrm{LogOrd}(\mathcal{L}; x\hat{f}) = \mathrm{ord}_{\widehat{\mathcal{M}}_0}(\hat{a}_0, \hat{b}_0, \hat{g}_0).$$

Lemma 19. *Let $\pi : M' \to M_0$ be given by the $\{x, w, y, \hat{f}\}$-formal Puiseux package and let $(x', w', y', \hat{f}')$ be the resulting list at the center P' of ν in M'. Then*

$$\mathrm{LogOrd}(\mathcal{L}; x'\hat{f}') \leq \mathrm{LogOrd}(\mathcal{L}; x\hat{f}).$$

Proof. The result is true under each of the blow-ups of the sequence given by the $\{x, w, y, \hat{f}\}$-formal Puiseux package. This is a standard verification which is also a part of the proof of the vertical stability of the adapted order given in [4]. □

Consider an expansion $\hat{\xi} = \sum_{s\geq 0} \hat{f}^s \hat{\eta}_s(x, w)$, where

$$\hat{\eta}_s(x, w) = \hat{a}_s(x, w)x\frac{\partial}{\partial x} + \hat{b}_s(x, w)\frac{\partial}{\partial w} + \hat{g}_s(x, w)\hat{f}\frac{\partial}{\partial \hat{f}}.$$

We say that $\hat{\eta}_s$ is *formally strongly prepared* if we can write

$$\hat{\eta}_s = x^{\rho}\hat{U}(x, w)\theta + x^{\tau}\hat{V}(x, w)\hat{f}\frac{\partial}{\partial \hat{f}}; \quad \theta = x\hat{h}(x, w)x\frac{\partial}{\partial x} + \frac{\partial}{\partial w} \tag{41}$$

satisfying the same properties as in Definition 5, that is

(1) $\rho, \tau \in \mathbb{Z} \cup \{+\infty\}$, with $\rho \neq \tau$.
(2) $\hat{U} = \lambda + x(\cdots)$ and $\hat{V} = \mu + x(\cdots)$, where $\lambda, \mu \in k\setminus\{0\}$. (Except if $\rho = +\infty$ or $\tau = \infty$, that indicates that $\hat{U}$, respectively $\hat{V}$ is identically zero)

By the same proof as in 10, we have

Proposition 19. *Assume that $\hat{\eta}_s \neq 0$, then after finitely many formal Puiseux packages we obtain $\hat{\eta}_s$ that is formally strongly prepared.*

Let us work by induction on $\varrho = \mathrm{LogOrd}(\mathcal{L}; x\hat{f})$. If $\varrho \leq 1$ we have a log-elementary singularity. Assume that $\varrho \geq 2$. By Proposition 19, after finitely many formal Puiseux packages, the vector field $\hat{\xi}$ can be written as $\hat{\xi} = \sum_{0\leq s} \hat{f}^s \hat{\eta}'_s(x, w)$, where

$$\hat{\eta}_s = x^{\rho_s}\hat{U}_s(x, w)\theta_s + x^{\tau_s}\hat{V}_s(x, w)\hat{f}\frac{\partial}{\partial \hat{f}}; \quad \theta_s = x\hat{h}_s(x, w)x\frac{\partial}{\partial x} + \frac{\partial}{\partial w}$$

is formally strongly prepared for any $s \leq \varrho$. Let us put $m_s = \min\{\rho_s, \tau_s\}$. let us also define

$$\delta = \min\left\{\frac{m_s}{\varrho - s};\ s < \varrho\right\}.$$

It is clear that $1 \leq \delta < \infty$, since the adapted order is ϱ.

Consider the ideal $(x, \hat{f})$. Since $\varrho \geq 2$, this ideal gives a curve in the singular locus of $\mathcal{L}$. Thus we can blow-up it. After blowing-up, we get that $\varrho' \leq \varrho$ and $\delta' = \delta - 1$ if $\varrho' = \varrho$. This ends the proof of Theorem 4.

Part 2. **Higher rank and higher dimensional valuations**

In this part we complete the proof of Theorem 1 by considering valuations of higher arquimedean rank or of dimension bigger than zero. In fact these cases correspond to situations simpler than in Part 1, since they are "essentially" of ambient dimension two.

8. Higher rank valuations

In this section we assume that $n = 3$ and $\kappa_\nu = k$ but ν has rank bigger than one, that is, the value group Γ is not arquimedean. If the rational rank $r = 3$, there is no difference with the computations in the case of an arquimedean valuation done in Section 2. The only remaining situation is $r = 2$. Let us consider this situation.

We can work in terms of parameterized regular local models $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, y))$ as in the case of a real valuation of rational rank two (Sections 3-4). Let us consider the following statement

> **TRI: Trivial ramification index assumption:** After performing any finite sequence of y-Puiseux packages, coordinate blow-ups in the independent variables and coordinate changes in the dependent variable, we obtain $\mathcal{A} = (\mathcal{O}, \mathbf{z} = (\mathbf{x}, y))$ such that the ramification index is equal to one. That is $\nu(y) = \nu(x_1^{p_1} x_2^{p_2})$ for $(p_1, p_2) \in \mathbb{Z}^2$.

Following the same arguments as in Sections 3-4 we obtain

Proposition 20. *Assume that the* Trivial Ramification Index Assumption does not hold *after performing any finite sequence of y-Puiseux packages, coordinate blow-ups in the independent variables and coordinate changes in the dependent variable. Then we can obtain a log-elementary $\mathcal{L}_{\mathcal{A}}$ after performing such a finite sequence of transformations.*

Thus, we assume that TRI holds. We can work by induction on the main height $h = \hbar(\xi; \mathbf{x}, y)$ of a generator of $\mathcal{L}$. By the same arguments in Sections 3-4, if the critical polynomial is not Thchirnhaus, we can win. So, we find an element at the level $h-1$ corresponding to the critical polynomial. This means that $(p_1, p_2) \in \mathbf{Z}^2_{\geq 0}$, since this point of the support is associated to a monomial $x_1^{p_1} x_2^{p_2}$ appearing in the coefficients of ξ. Now, we can do the coordinate change

$$y_1 = y - c_1 x_1^{p_1} x_2^{p_2}; \quad \nu(y_1) > \nu(y).$$

The situation repeats. We obtain a formal element $\hat{f} = y - \sum c_i x_1^{p_{i1}} x_2^{p_{i2}}$. Now we can apply to $\hat{f}$ the same arguments as in Subsection 7.1.

9. Higher dimensional valuations

In this section we assume that $n = 3$ and $\kappa_\nu \neq k$. We look for a projective model M of K and a birational morphism $M \to M_0$ such that the center Y of ν at M has dimension ≥ 1 and a generic point of Y is a regular point of M which is log-elementary for $\mathcal{L}$. Since k is algebraically closed, the assumption $\kappa_\nu \neq k$ implies that $\dim \nu \geq 1$, where $\dim \nu$ is the transcendence degree of κ_ν / k. Applying Hironaka's reduction of singularities to M_0, we may assume that all the points in M_0 are nonsingular. Also by classical results on reduction of singularities, we obtain the following statement:

Lemma 20. *There is a birational morphism $M \to M_0$ such that the center Y of ν at M has dimension equal to* $\dim \nu$.

Proof. See for instance Vaquié's paper [17]. □

Thus we may assume that M_0 is non-singular and the center Y_0 of ν at M_0 has dimension equal to $\dim \nu$. If $\dim \nu = 2$, then Y_0 is a hypersurface and a generic point of Y_0 is always nonsingular for $\mathcal{L}$, since the singular locus of $\mathcal{L}$ has codimension at least 2 in any nonsingular ambient space.

Consider the case $\dim \nu = \dim Y_0 = 1$. We blow-up M_0 with center Y_0 to get $M_1 \to M_0$. The new center Y_1 of ν at M_1 is a curve that applies surjectively over Y_0. We repeat the procedure to get an infinite sequence

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

where the center Y_i of ν at M_i is a curve that applies surjectively over Y_{i-1}. In this situation we can apply the equireduction arguments in [5], (see also [13]) to obtain an elementary $\mathcal{L}$ at a generic point of Y_i for $i >> 0$. These arguments are actually of two-dimensional nature and the invoked equireduction results are very similar to the original Seidenberg's result in [14].

Part 3. Globalization

In this Part 3 we prove the global result stated in Theorem 2. To do this we will apply the axiomatic version of the Zariski's Patching of Local Uniformizations [19] that has been developed by O. Piltant in [12].

Let us state the axiomatic version of the patching of local uniformizations that we need to use. Fix a field of rational functions K/k of transcendence degree three over k. We take k an algebraically closed field of characteristic zero, even if Piltant's result is more general than that. Assume that we have an assignation

$$M \mapsto \mathrm{RegP}(M) \subset M$$

that chooses a nonempty Zariski open subset $\mathrm{RegP}(M) \subset M$ for each projective model M of K. This map can be thought of by saying that $\mathrm{RegP}(M)$ is the *set of points of M that satisfy the property "P"*. Let us introduce now a list of axioms for globalization.

> *Axiom I.* For each projective model M of K the set $\mathrm{RegP}(M)$ is a nonempty Zariski open set contained in the set of regular points $\mathrm{Reg}(M)$ of M. Moreover the definition of $\mathrm{RegP}(M)$ is local in the sense that given two projective models M and M', two Zariski open sets $U \subset M$ and $U' \subset M'$ and an isomorphism $\phi : U \to U'$, then $\phi(\mathrm{RegP}(M) \cap U) = \mathrm{RegP}(M') \cap U'$.

The next axiom says that $\mathrm{RegP}(M)$ has a good behavior under blow-up.

> *Axiom II.* Let $Y \subset M$ be an irreducible algebraic subvariety of M such that $Y \cap \mathrm{RegP}(M) \neq \emptyset$. Let $\pi : M' \to M$ be the blow-up with center Y. There is a nonempty Zariski open subset V_Y of $Y \cap \mathrm{RegP}(M)$ defined by the property that $\pi^{-1}(V_Y) \subset \mathrm{RegP}(M')$.

In what follows we take V_Y to be the largest possible between the subsets $V \subset Y \cap \mathrm{RegP}(M)$ such that $\pi^{-1}(V) \subset \mathrm{RegP}(M')$. This determines V_Y uniquely. As a consequence of Axiom II, if we blow-up a point $P \in \mathrm{RegP}(M)$ then we have $\pi^{-1}(P) \subset \mathrm{RegP}(M')$. The open set V_Y is called *set of permissibility for Y*. We say that Y is *permissible if* $V_Y = Y$. We need also the notion of *strong permissibility*. If $Y \subset M$ is a point or a hypersurface that cuts $\mathrm{RegP}(M)$, we define the open set of strong permissibility W_Y as $W_Y = V_Y$. Assume that Y is an irreducible curve and let $P \in V_Y$. We say that Y is *strongly permissible at* P iff the following property holds

> Let $M = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_N$ be a finite sequence of blow-ups centered at points $P_i \in M_i$, such that $P_0 = P$ and P_i projects over P_{i-1} and is in the strict transform Y_i of Y. Then Y_N is permissible at P_N (that is $P_N \in V_{Y_N}$).

We denote by $W_Y \subset V_Y$ the set of points where Y is strongly permissible and we say that Y is *strongly permissible* iff $W_Y = Y$.

> *Axiom III.* Let Y be a curve in M such that $Y \cap \mathrm{RegP}(M) \neq \emptyset$. There is a finite sequence
>
> $$M = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_N = M'$$
>
> of blow-ups with center in closed points such that the strict transform Y' of Y is strongly permissible.

In fact, the centers in Axiom III can be chosen in $Y - W_Y$ at each step; this also shows that W_Y is a nonempty open set of Y.

We also need another axiom (of principalization), that can be seen as a result on *conditionated desingularization*

> *Axiom IV [Principalization].* Given a (normal) projective model M_0 of K and an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{M_0}$, there is a projective birational morphism $\pi : M \to M_0$ such that
> (1) $\mathcal{I}\mathcal{O}_M$ is locally principal in $\pi^{-1}(\mathrm{RegP}(M_0))$.
> (2) $\pi^{-1}(\mathrm{RegP}(M_0)) \subset \mathrm{RegP}(M)$.
> (3) The induced map $\pi^{-1}(\mathrm{RegP}(M_0) \cap U) \to \mathrm{RegP}(M_0) \cap U$ is an isomorphism where U is the open set of the points p of M_0 such that $\mathcal{I}_p$ is principal.

The last axiom states the existence of Local Uniformization.

> *Axiom V [Local Uniformization].* Let ν be a k-valuation of K. There is a projective model M of K such that the center Y of ν in M cuts $\mathrm{RegP}(M)$, that is $Y \cap \mathrm{RegP}(M) \neq \emptyset$.

With this axioms, it is possible to reproduce Zariski's arguments in [20] for the patching of local uniformizations and we can state the following result

Theorem 5 (Piltant)**.** *Assume that the assignation $M \mapsto RegP(M)$ satisfies to the axioms I,II,III, IV and V above. Consider a projective model M_0 of K. Then there is a birational projective morphism $M \to M_0$ such that $RegP(M) = M$.*

This statement is slightly more restrictive that the result proved by Piltant in [12]. It is the result we need to get our global statement for the case of foliations. Now, in order to prove Theorem 2, we just need to prove the following statement

Proposition 21. *Let us consider a foliation $\mathcal{L} \subset Der_k K$. The assignation*

$$M \mapsto RegLog_{\mathcal{L}}(M) = \{P \in M; P \text{ is log-elementary } \}$$

satisfies to the axioms I,II,III,IV and V.

The Local Uniformization Axiom is given by Theorem 1. The first axiom is evident from the local definition of log-elementary points, let us just point that $\mathrm{RegLog}_{\mathcal{L}}(M)$ is non-empty since the non singular points of $\mathcal{L}$ in $\mathrm{Reg}(M)$ are in the complement of a closed subset of codimension bigger or equal than two.

The axioms II and III come from the general computations done in [4] concerning the definition and properties of permissible centers in terms of the adapted multiplicity. More precisely, in theorem 3.1.4. of [4] is proved the stability of the adapted order under blow-up (the log-elementary singularities are defined to have adapted order less or equal to one). The permissibilyzing and permissibility properties come from the results on stationary sequences in the section 3.3. of [4].

Finally axiom IV of principalization has been explicitly proved for the case $\mathrm{RegP}(M) = \mathrm{RegLog}_{\mathcal{L}}(M)$ by Piltant in [12], Proposition 4.2.

Now, Theorem 2 is a consequence of Proposition 21 and Theorem 5.

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