An inequality for sums of binary digits, with application to Takagi functions

Pieter C. Allaart
University of North Texas *

September 15, 2010

Abstract

Let $\phi(x) = 2\inf\{|x-n| : n \in \mathbf{Z}\}$, and define for $\alpha > 0$ the function

$$f_\alpha(x) = \sum_{j=0}^{\infty} \frac{1}{2^{\alpha j}} \phi(2^j x).$$

Tabor and Tabor [*J. Math. Anal. Appl.* **356** (2009), 729–737] recently proved the inequality

$$f_\alpha\left(\frac{x+y}{2}\right) \le \frac{f_\alpha(x)+f_\alpha(y)}{2} + |x-y|^\alpha,$$

for $\alpha \in [1,2]$. By developing an explicit expression for f_α at dyadic rational points, it is shown in this paper that the above inequality can be reduced to a simple inequality for weighted sums of binary digits. That inequality, which seems of independent interest, is used to give an alternative proof of the result of Tabor and Tabor, which captures the essential structure of f_α.

AMS 2000 subject classification: 26A27 (primary); 26A51 (secondary)

Key words and phrases: Takagi function, Approximate convexity, Digital sum inequality.

*Address: Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203-5017, USA; E-mail: allaart@unt.edu, kiko@unt.edu

1 Introduction

Let $\phi(x) = 2\inf\{|x-n| : n \in \mathbf{Z}\}$ be the so-called "tent-map," and define for $\alpha > 0$ the function

$$f_\alpha(x) = \sum_{j=0}^{\infty} \frac{1}{2^{\alpha j}} \phi(2^j x).$$

Observe that f_1 is Takagi's continuous nowhere differentiable function; see [5]. For $0 < \alpha < 1$, the graph of f_α is a fractal whose Hausdorff dimension was calculated by Ledrappier [3]. For $\alpha > 1$, the function f_α is Lipschitz, so that it is differentiable almost everywhere. This paper concerns the following inequality, proved recently by Tabor and Tabor [4].

Theorem 1. (Tabor and Tabor [4, Corollary 2.1]). *For every $1 \le \alpha \le 2$ and for all $x, y \in [0,1]$,*

$$f_\alpha\left(\frac{x+y}{2}\right) \le \frac{f_\alpha(x) + f_\alpha(y)}{2} + |x-y|^\alpha. \tag{1}$$

This inequality plays an important role in the study of approximate convexity of continuous functions, where f_α occurs naturally in a best possible upper bound; see [4]. For the case $\alpha = 1$, the inequality had previously been proved by Boros [1]. Both proofs, while cleverly devised, fail to bring out the essential structure of the function f_α. The aim of this note is to show how (1) can be reduced to a simple inequality concerning weighted sums of binary digits, thereby providing a simpler proof for the inequality (1) that emphasizes the basic structure of f_α.

We need the following notation. For a nonnegative integer n and a real number p, write n in binary as $n = \sum_{j=0}^{\infty} 2^j \varepsilon_j$ with $\varepsilon_j \in \{0,1\}$, and define

$$s_p(n) = \sum_{j=0}^{\infty} 2^{pj} \varepsilon_j.$$

Let

$$S_p(n) = \sum_{m=0}^{n-1} s_p(m).$$

It turns out that (1) is equivalent to the simple inequality

$$S_p(m+n) + S_p(m-n) - 2S_p(m) \le n^{p+1}, \tag{2}$$

for $0 \le p \le 1$ and $0 \le n \le m$. This inequality, which seems to be of independent interest, is proved in Section 2. There we also specify the cases when equality holds in (2). Note that when $p = 0$, $S_p(n)$ is the number of binary 1's needed to express the numbers $0, \ldots, n-1$. This function has been well-studied in the literature; see, for instance, Trollope [6] for a precise expression and asymptotics. When $p = 1$, $S_p(n)$ is simply the sum of the first $n-1$ positive integers, from which it follows readily that (2) holds with equality for all n and m. It seems that for $0 < p < 1$ the inequality may be new. In fact, even for the case $p = 0$ the author has not been able to find a reference.

The key to showing that (1) boils down to (2) is the following formula for the values of f_α at dyadic rational points.

Proposition 2. *For $n = 0, 1, \ldots$ and $m = 0, 1, \ldots, 2^n$,*

$$f_\alpha\left(\frac{m}{2^n}\right) = \sum_{k=0}^{m-1}\sum_{i=0}^{n-1} \frac{(-1)^{\varepsilon_i(k)}}{2^{(n-i-1)\alpha+i}}, \tag{3}$$

where $\varepsilon_i(k) \in \{0,1\}$ is determined by $\sum_{i=0}^{n-1} 2^i \varepsilon_i(k) = k$.

For $\alpha = 1$, this formula simplifies to a well-known expression for the Takagi function; see, for instance, Krüppel [2, eq. (2.4)]. Proposition 2 is proved in Section 3. It is then used, in combination with (2), to give a short proof of Theorem 1.

2 A digital sum inequality

This section gives a proof of the inequality (2), and specifies in which cases equality holds.

Theorem 3. *Let $0 \le p \le 1$. Then (2) holds for all $0 \le n \le m$. Moreover, if $n \ge 1$ and k is the integer such that $2^{k-1} < n \le 2^k$, then equality holds in (2) if and only if*

(i) $p = 1$*; or*

(ii) $0 < p < 1$*,* $n = 2^k$*, and* $m \equiv n \bmod 2^{k+1}$ *; or*

(iii) $p = 0$*, and either* $m \equiv n \bmod 2^{k+1}$ *or* $m \equiv -n \bmod 2^{k+1}$*.*

The main idea behind the proof is illustrated in Figure 1. In this example, $m = 47$ and $n = 5$. The braces to the right of the binary representations indicate a division of the list $m - n, \ldots, m + n - 1$ into four groups of consecutive integers, in such a way that the first number in the last group differs from the first number in the first group by a power of 2 (in this case, 2^3), and these groups contain an equal number of integers. Because of this, each number in the fourth group has the same last three digits as the corresponding number in the first group. (These digits are boxed in the figure.) The digits to the left of the two boxes form numbers that are exactly one apart (in the example, 5 and 6). Thus, assuming that the theorem is true for

$$
\begin{array}{r cccccc l}
m-n & 1 & 0 & 1 & 0 & 1 & 0 & \multirow{2}{*}{$\}\ \Sigma(m-n, m+n-2^k)$} \\
 & 1 & 0 & 1 & 0 & 1 & 1 & \\
m+n-2^k & 1 & 0 & 1 & 1 & 0 & 0 & \\
 & 1 & 0 & 1 & 1 & 0 & 1 & \Big\}\ \Sigma(m+n-2^k, m) \\
 & 1 & 0 & 1 & 1 & 1 & 0 & \\
\hline
m & 1 & 0 & 1 & 1 & 1 & 1 & \\
 & 1 & 1 & 0 & 0 & 0 & 0 & \Big\}\ \Sigma(m, m-n+2^k) \\
 & 1 & 1 & 0 & 0 & 0 & 1 & \\
m-n+2^k & 1 & 1 & 0 & 0 & 1 & 0 & \}\ \Sigma(m-n+2^k, m+n) \\
 & 1 & 1 & 0 & 0 & 1 & 1 &
\end{array}
$$

Figure 1: The proof of inequality (2) illustrated for $n = 5$ and $m = 47$.

$n = 1$, the summation of $s_p(i)$ over the fourth group minus the summation over the first group is bounded by 2^{3p} times the cardinality of the fourth group (in this case, 2). As for the two middle groups, we may proceed inductively and assume that the summation of $s_p(i)$ over the third group minus the summation over the second group is at most 3^{p+1}, since these groups are adjacent, and (in this example) each group includes 3 integers. The induction step is completed by appealing to an elementary inequality from calculus; see (8) below.

This is the basic idea. A formal proof now follows.

Proof of Theorem 3. **Part 1: Inequality.** Fix $p \in [0, 1]$. For brevity, write

$$\Delta(m, n) := S_p(m + n) + S_p(m - n) - 2S_p(m).$$

The proof proceeds by induction on n. First, let $n = 1$, and note that in this case,

$$\Delta(m, 1) = s_p(m) - s_p(m - 1).$$

Consider two cases regarding the parity of m. If m is odd, then $\varepsilon_0(m - 1) = 0$ and $\varepsilon_0(m) = 1$, while $\varepsilon_j(m - 1) = \varepsilon_j(m)$ for all $j \geq 1$. Hence, $\Delta(m, 1) = 1$.

Assume then that m is even. In this case, there is $j_0 \geq 1$ such that:

(1) $\varepsilon_{j_0}(m - 1) = 0$ and $\varepsilon_{j_0}(m) = 1$;

(2) $\varepsilon_j(m - 1) = 1$ and $\varepsilon_j(m) = 0$ for $0 \leq j < j_0$; and

(3) $\varepsilon_j(m - 1) = \varepsilon_j(m)$ for all $j > j_0$.

This gives

$$\Delta(m,1) = 2^{pj_0} - \sum_{j=0}^{j_0-1} 2^{pj}. \tag{4}$$

If $p = 0$, it follows immediately that $\Delta(m,1) < 1$. If $0 < p \leq 1$, we may put $\lambda = 2^p$ and obtain that

$$\Delta(m,1) - 1 = \lambda^{j_0} - 1 - \frac{\lambda^{j_0} - 1}{\lambda - 1} = (\lambda^{j_0} - 1)\frac{\lambda - 2}{\lambda - 1} \leq 0. \tag{5}$$

Thus, (2) holds for $n = 1$ and all $m \geq 1$. In fact, if $n = 1$ and $p < 1$, it is clear from (4) and (5) that equality obtains in (2) if and only if m is odd.

Next, let $n > 1$, and assume that $\Delta(m,l) \leq l^{p+1}$ for all $l < n$ and all m. For ease of notation, put

$$\Sigma(t,u) := \sum_{r=t}^{u-1} s_p(r) = S_p(u) - S_p(t), \qquad t, u \in \mathbb{N}, \quad t < u.$$

Let k be the integer such that $2^{k-1} < n \leq 2^k$. The idea is to write

$$\begin{aligned} \Delta(m,n) = {} & \Sigma(m + 2^k - n, m + n) - \Sigma(m - n, m + n - 2^k) \\ & + \Sigma(m, m + 2^k - n) - \Sigma(m + n - 2^k, m) \end{aligned}$$

(see Figure 1 and the heuristic explanation preceding the proof). Since $2^k - n < n$, the induction hypothesis implies that

$$\Sigma(m, m + 2^k - n) - \Sigma(m + n - 2^k, m) = \Delta(m, 2^k - n) \leq (2^k - n)^{p+1}. \tag{6}$$

On the other hand, for each $r = 0, 1, \ldots, 2n - 2^k - 1$, we have

$$s_p(m + 2^k - n + r) - s_p(m - n + r) = 2^{kp}\{s_p(t+1) - s_p(t)\},$$

where t is the greatest integer in $(m - n + r)/2^k$. Therefore, by the case $n = 1$,

$$\Sigma(m + 2^k - n, m + n) - \Sigma(m - n, m + n - 2^k) \leq (2n - 2^k) \cdot 2^{kp}, \tag{7}$$

the inequality being strict when $p < 1$ and t is odd. Combining (6) and (7) we obtain

$$\Delta(m,n) \leq (2n - 2^k) \cdot 2^{kp} + (2^k - n)^{p+1} = 2^{k(p+1)} \left[2\left(\frac{n}{2^k}\right) - 1 + \left(1 - \frac{n}{2^k}\right)^{p+1} \right].$$

Put $x = n/2^k$. Then $1/2 < x \leq 1$, and it will follow that $\Delta(m,n) \leq n^{p+1}$ provided that

$$2x - 1 + (1-x)^{p+1} \leq x^{p+1}. \tag{8}$$

But this last inequality follows since the function $g_p(x) = 2x - 1 + (1-x)^{p+1} - x^{p+1}$ is convex on $[1/2, 1]$ for $p \in [0,1]$, with $g_p(1/2) = g_p(1) = 0$. This concludes the inductive proof of the inequality (2).

Part 2: Equality. We now turn to the question of equality. It was noted in the introduction that if $p = 1$, then $s_p(n) = n$ and so $\Delta(m,n) = n^2$ for all m and all n. Suppose $0 < p < 1$. If $n = 2^k$ and $m \equiv n \bmod 2^{k+1}$, then

$$\Delta(m,n) = n \cdot 2^{kp} = n^{p+1}.$$

On the other hand, if $n = 2^k$ but $m \not\equiv n \bmod 2^{k+1}$, then strict inequality obtains in (7) in the induction step, as the greatest integer in $(m-n+r)/2^k$ is odd for at least one r. Finally, if $n < 2^k$, then with $x = n/2^k$ we have strict inequality in (8), since the function g_p is strictly convex on $[1/2, 1]$ when $0 < p < 1$.

The case $p = 0$ is the most involved. We will show inductively that $\Delta(m,n) = n$ if and only if $m \equiv \pm n \bmod 2^{k+1}$. Note that this equivalence holds for the case $n = 1$ by the remark following (5).

Let $n \geq 2$, and assume that whenever $l < n$ and j is the integer such that $2^{j-1} < l \leq 2^j$, the equivalence

$$m \equiv \pm l \bmod 2^{j+1} \quad \Longleftrightarrow \quad \Delta(m,l) = l \tag{9}$$

holds. Let k be the integer such that $2^{k-1} < n \leq 2^k$, and put

$$n' := 2^k - n.$$

Suppose $m \equiv \pm n \bmod 2^{k+1}$. Then either the binary representation of $m-n$ ends in k zeros, or that of $m+n-1$ ends in k ones. In both cases,

$$\varepsilon_k(m-n+2^k+r) = 1 \quad \text{and} \quad \varepsilon_k(m-n+r) = 0, \quad \text{for } 0 \leq r < 2n - 2^k, \tag{10}$$

so

$$\Sigma(m-n+2^k, m+n) - \Sigma(m-n, m+n-2^k) = 2n - 2^k = n - n'. \tag{11}$$

If $n' = 0$ the two middle groups vanish, so $\Delta(m,n) = n - n' = n$. Assume then that $n' > 0$. Let j be the integer such that $2^{j-1} < n' \leq 2^j$. If $m \equiv n \bmod 2^{k+1}$, then $m + n' \equiv 2^k \bmod 2^{k+1}$ and hence $m + n' \equiv 0 \bmod 2^{j+1}$, since $j < k$. Similarly, if $m \equiv -n \bmod 2^{k+1}$, then $m - n' \equiv 0 \bmod 2^{j+1}$. Thus, by the induction hypothesis,

$$\Sigma(m+n-2^k, m) - \Sigma(m, m-n+2^k) = \Delta(m,n') = n'.$$

Combining this with (11) yields $\Delta(m,n) = n$.

Conversely, suppose $\Delta(m,n)=n$. Then necessarily

$$s_0(m-n+2^k+r)-s_0(m-n+r)=1, \qquad \text{for } 0\le r<2n-2^k,$$

which implies (10). If $n=2^k$, this immediately yields that $m\equiv n \bmod 2^{k+1}$. Otherwise, $n'>0$, and we let j be the integer such that $2^{j-1}<n'\le 2^j$. Since

$$\Delta(m,n')=\Sigma(m,m+n')-\Sigma(m-n',m)=n',$$

the induction hypothesis implies that $m\equiv \pm n' \bmod 2^{j+1}$. Thus, either the binary representation of $m-n'$ ends in $j+1$ zeros, or that of $m+n'-1$ ends in $j+1$ ones. Since $j<k$ and there are $2n'\le 2^{j+1}$ numbers in the list $m-n',\dots,m+n'-1$, these numbers must, in both cases, have their kth binary digit (ε_k) in common. If this common digit is a "0", then $\varepsilon_k(m-n+r)=0$ for $0\le r<2^k$ in view of (10), and so $m-n\equiv 0 \bmod 2^{k+1}$. If the common digit is a "1", then $\varepsilon_k(m+n-r)=1$ for $1\le r\le 2^k$, and so $m+n\equiv 0 \bmod 2^{k+1}$. This completes the proof. □

3 Application to Takagi functions

This section gives a proof of Proposition 2, and shows how the expression given in the proposition can be used, in conjunction with the inequality (2), to give a more straightforward proof of the theorem of Tabor and Tabor.

Proof of Proposition 2. Observe first that the definition of f_α immediately gives

$$f_\alpha\left(\frac{2j+1}{2^{n+1}}\right)=\frac{1}{2}\left\{f_\alpha\left(\frac{j}{2^n}\right)+f_\alpha\left(\frac{j+1}{2^n}\right)\right\}+\frac{1}{2^{\alpha n}},$$

for $n=0,1,\dots,$ and $j=0,1,\dots,2^n-1$. From this, it follows that

$$f_\alpha\left(\frac{k+1}{2^{n+1}}\right)-f_\alpha\left(\frac{k}{2^{n+1}}\right)=\frac{1}{2}\left\{f_\alpha\left(\frac{j+1}{2^n}\right)-f_\alpha\left(\frac{j}{2^n}\right)\right\}+\frac{(-1)^k}{2^{\alpha n}}, \qquad (12)$$

where $j=[k/2]$ is the greatest integer in $k/2$. A straightforward induction argument using (12) yields

$$f_\alpha\left(\frac{k+1}{2^{n+1}}\right)-f_\alpha\left(\frac{k}{2^{n+1}}\right)=\sum_{i=0}^{n}\frac{(-1)^{\varepsilon_i}}{2^{(n-i)\alpha+i}},$$

where $k=\sum_{i=0}^n 2^i\varepsilon_i$. Replacing n with $n-1$ and summing over $k=0,\dots,m-1$ gives (3), as $f_\alpha(0)=0$. □

Proof of Theorem 1. Since f_α is continuous, it suffices to prove (1) for dyadic rational points x and y. Thus, we may assume that there exist nonnegative integers n, m and l such that $x = (m-l)/2^n$ and $y = (m+l)/2^n$. It is to be shown that

$$\Delta_2^{(n)}(m,l) := f_\alpha\left(\frac{m}{2^n}\right) - \frac{1}{2}\left\{f_\alpha\left(\frac{m-l}{2^n}\right) + f_\alpha\left(\frac{m+l}{2^n}\right)\right\} \le \left(\frac{2l}{2^n}\right)^\alpha.$$

Proposition 2 gives

$$\begin{aligned}\Delta_2^{(n)}(m,l) &= \sum_{i=0}^{n-1} \frac{1}{2^{(n-i-1)\alpha+i}} \left(\sum_{k=0}^{m-1} (-1)^{\varepsilon_i(k)} - \frac{1}{2}\sum_{k=0}^{m-l-1} (-1)^{\varepsilon_i(k)} - \frac{1}{2}\sum_{k=0}^{m+l-1} (-1)^{\varepsilon_i(k)}\right) \\ &= \frac{1}{2^{(n-1)\alpha}} \sum_{i=0}^{n-1} 2^{(\alpha-1)i-1} \left(\sum_{k=m-l}^{m-1} (-1)^{\varepsilon_i(k)} - \sum_{k=m}^{m+l-1} (-1)^{\varepsilon_i(k)}\right).\end{aligned}$$

Since $(-1)^\varepsilon = 1 - 2\varepsilon$ for $\varepsilon \in \{0,1\}$, we can write

$$\begin{aligned}2^{(n-1)\alpha}\Delta_2^{(n)}(m,l) &= \sum_{i=0}^{n-1} 2^{(\alpha-1)i} \sum_{r=1}^{l} \{\varepsilon_i(m+r-1) - \varepsilon_i(m-r)\} \\ &= \sum_{r=1}^{l} \{s_{\alpha-1}(m+r-1) - s_{\alpha-1}(m-r)\} \\ &= S_{\alpha-1}(m+l) + S_{\alpha-1}(m-l) - 2S_{\alpha-1}(m).\end{aligned}$$

Thus, by Theorem 3,

$$\Delta_2^{(n)}(m,l) \le \frac{l^\alpha}{2^{(n-1)\alpha}} = \left(\frac{2l}{2^n}\right)^\alpha,$$

as required. □

In fact, it is not difficult to use Theorem 3 to determine for which dyadic points x and y equality holds in (1).

References

[1] Z. Boros, An inequality for the Takagi function. *Math. Inequal. Appl.* **11** (2008), no. 4, 757–765.

[2] M. Krüppel, On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function, *Rostock. Math. Kolloq.* **62** (2007), 41–59.

[3] F. Ledrappier, On the dimension of some graphs, *Contemp. Math.* **135** (1992), 285–293.

[4] J. Tabor and J. Tabor, Takagi functions and approximate midconvexity. *J. Math. Anal. Appl.* **356** (2009), no. 2, 729–737.

[5] T. Takagi, A simple example of the continuous function without derivative, *Phys.-Math. Soc. Japan* **1** (1903), 176-177. *The Collected Papers of Teiji Takagi*, S. Kuroda, Ed., Iwanami (1973), 5–6.

[6] J. R. Trollope, An explicit expression for binary digital sums, *Math. Mag.* **41** (1968), 21-25.