A SIMPLIFIED, BOHR QUANTUM THEORETICAL DERIVATION OF THE UNRUH TEMPERATURE, ENTROPY AND EVAPORATION

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Abstract

In this work we reproduce Unruh temperature for a spherical physical system using a simplified rule, very similar to Bohr angular momentum quantization postulate interpreted via de Broglie relation, and using a thermal equilibrium stability condition. (Our rule, by a deeper analysis that goes over basic intentions of this work, corresponds, for Schwarzschild black hole, to a closed string loop theory of Copeland and Lahiri.) Firstly, we suppose that at the surface of a large system gravitational field of this system can generate some quantum excitations, i.e. small quantum systems. Mass spectrum of this small quantum system is determined using a rule that states that circumference of the large system holds integer numbers of the reduced Compton wavelengths of this small quantum system. Secondly, we suppose that absolute value of the classical gravitational interaction between large system and small quantum system in the ground state is equivalent to thermal kinetic energy of the small quantum system interacting with large system as thermal reservoir. It is very similar to virial theorem of the ideal gas and it yields directly and exactly Unruh temperature. Finally, using Unruh temperature and thermodynamical law, we originally propose corresponding Unruh entropy and Unruh evaporation of the system and demonstrate that for Schwarzschild black hole Unruh temperature, entropy and evaporation can be exactly reduced in the Hawking temperature, Bekenstein-Hawking entropy and Hawking evaporation.

As it is well-known, Unruh radiation [1]-[4] represents an important, general relativistic and quantum effect whose exact derivation needs a relatively complex theoretical formalism of the quantum gravity. Some simplifications of the derivation of Unruh radiation can be done too [5]. In this work "without knowing the details of quantum gravity" (we paraphrase Fursaev [6]) we shall reproduce exactly final form of the remarkable Unruh temperature for a spherical physical system. Firstly, we shall use a simplified rule, very similar but not identical to Bohr angular momentum quantization postulate interpreted via de Broglie relation and by a thermal equilibrium stability condition. (This rule, by a deeper analysis that goes over basic intentions of this work, corresponds, for Schwarzschild black hole, to a closed string loop theory of Copeland and Lahiri [7].) Concretely, firstly, we suppose that at the surface of a large system gravitational field of this system (eventually interacting with quantum vacuum) can generate some quantum excitations, or simply small quantum systems. Mass (energy) spectrum of this small quantum system is determined so that circumference of the large system holds integer numbers of the reduced Compton wavelengths of this small quantum system. Secondly, we suppose that absolute value of the classical gravitational interaction between large system and small quantum system in the ground state is equivalent to thermal kinetic energy of the small quantum system interacting with large system as thermal reservoir. It is very similar but not identical to virial theorem of the ideal gas and it yields directly and exactly Unruh temperature. Moreover, using Unruh temperature and thermodynamical law, we originally propose corresponding Unruh entropy (proportional to product of the system surface and logarithm of the system mass) and Unruh evaporation of the system (during time interval approximately inversely proportional to third degree of the Planck mass. Finally, we demonstrate that in the especial case when physical system represents a Schwarzschild black hole Unruh temperature, entropy and evaporation can be exactly reduced in the Hawking temperature, Bekenstein-Hawking entropy and Hawking evaporation (time).

Consider a large, massive spherical system with mass M, characteristic radius R and surface gravity

$$a = \frac{GM}{R^2} \tag{1}$$

where G represents the Newtonian gravitational constant. Suppose that in the general case there is no functional dependence between R and M.

Firstly, suppose that at distance R, i.e. at large system surface, gravitational field of large system (eventually interacting with quantum vacuum) can create some quantum excitations, or simply - small quantum systems, whose quantized masses (energies) are determined by the following condition

$$m_n c R = n\frac{\hbar}{2\pi}, \quad \text{for} \;\; m_n \ll M \quad \text{and} \;\; n = 1, 2, ... \tag{2}$$

where m_n and $E_n = m_n c^2$for $m_n \ll M$ and $n = 1, 2,$ represent the mass and energy spectrum of mentioned small quantum system, while $\hbar$ represents the reduced Planck constant.

It implies

$$2\pi R = n\frac{\hbar}{m_n c} = n\lambda_{rn} \quad \text{for} \;\; m_n \ll M \quad \text{and} \;\; n = 1, 2, ... \tag{3}$$

where $2\pi R$ represents the circumference of the large system, i.e. circumference of a great circle at large system surface, while

$$\lambda_{rn} = \frac{\hbar}{m_n c} \tag{4}$$

represents n-th reduced Compton wavelength of the small quantum system with mass m_n for $n = 1, 2, .$ Expression (3) simply means that circumference of the large system holds exactly n corresponding n-th reduced Compton wave lengths of the small quantum system with mass m_n for $n = 1, 2, $.

All this it is very similar to Bohr angular momentum quantization postulate interpreted via de Broglie relation. But, of course, (2) cannot be completely consistently interpreted as angular momentum by simple rotation of the small system along great circle at large system surface. (Such simple interpretation of (2) as a classical rotation of the small quantum system leads to rotation speed 2π times smaller than c. Namely, if we suppose $m_1 c = \frac{\hbar\omega_1}{c}$ and $v_1 = \omega_1 R$, then, according to (2), it follows $v_1 = \frac{c}{2\pi}$.) Simply speaking (2) represents a postulate on the quantum field theoretical characteristic of gravitational field of the large system (eventually interacting with quantum vacuum) at surface of the system. Or, expression (2) represents a postulate on the effective reduction of the quantum field theory of gravitational field of large system at its surface. (Detailed discussion of this postulate goes over basic intention of this work. We can only point out that Copeland and Lahiri [7] showed that basic results of the Schwarzchild black hole thermodynamics can be obtained by consideration of the small excitations of a closed string loop which is full agreement with our postulate.)

According to (2) it follows too

$$m_n = n\frac{\hbar}{2\pi Rc} \equiv nm_1 \quad \text{for} \quad m_n \ll M \quad \text{and} \quad n = 1, 2, ... \tag{5}$$

$$m_1 = \frac{\hbar}{2\pi Rc} \tag{6}$$

represents the minimal, i.e. ground mass of the small quantum system. Obviously, mass spectrum (5), i.e. corresponding energy spectrum $E_n = m_n c^2 = nm_1 c^2 = nE_1$, for $n = 1, 2,$ represent practically mass, i.e. energy spectrum of an linear harmonic oscillator.

Secondly, suppose now that potential energy gravitational interaction between large system and small quantum system in the ground state

$$V = \frac{Gm_1 M}{R} = \frac{G\hbar M}{2\pi Rc} \tag{7}$$

is equivalent to the statistical average kinetic energy of the small system kT in the contact with large system as thermal reservoir, i.e. that the following is satisfied

$$kT = V = \frac{Gm_1 M}{R} \tag{8}$$

where k represents the Boltzmann constant. It, according to (1), (6), implies

$$T = \frac{1}{k}\frac{Gm_1 M}{R} = \frac{1}{k}\frac{\hbar}{2\pi c}\frac{GM}{R^2} = \frac{1}{k}\frac{\hbar}{2\pi c}a \tag{9}$$

representing exactly the Unruh temperature in the general case.

It can be observed that all this is conceptually very similar to theory of the ideal gas by virial expansion. In this analogy Unruh temperature corresponds to the temperature of the gas, while small perturbations of the thermodynamical equilibrium can be described only by second virial coefficient analogous to Planck law of the black body radiation or to Bose-Einstein distribution (which implies that here Stefan-Boltzmann law can be satisfied too).

In the especial case of the Schwarzschild black hole, for which there is an especial functional dependence between R and M

$$R = \frac{2GM}{c^2} \tag{10}$$

representing the Schwarzschild radius, expressions (6), (9), turn out in

$$m_1 = \frac{\hbar c}{4\pi GM} \tag{11}$$

$$T = \frac{\hbar c^3}{8\pi GM}. \tag{12}$$

Last expression represents the Hawking temperature.

Further, Unruh radiation must satisfy thermodynamical law

$$dMc^2 = TdS \tag{13}$$

where Mc^2 represents total energy of the large system and S - corresponding Unruh entropy. It, implies

$$dS = \frac{1}{T}c^2 dM \tag{14}$$

or, according to (6), (9),

$$dS = k\frac{2\pi c^3)}{\hbar}\frac{R^2}{GM}dM = \frac{k}{2}\frac{4\pi R^2}{\frac{\hbar G}{c^3}}d(\ln[\frac{M}{M_0}]) = \frac{k}{2}\frac{A}{L_P^2}d(\ln[\frac{M}{M_0}]) \tag{15}$$

and

$$S = \frac{k}{2}\frac{4\pi R^2}{\frac{\hbar G}{c^3}}\ln[\frac{M}{M_0}] = \frac{k}{2}\frac{A}{L_P^2}\ln[\frac{M}{M_0}]. \tag{16}$$

Here

$$A = 4\pi R^2 \tag{17}$$

represents the surface of the large system, $L_P = (\frac{\hbar G}{c^3})^{\frac{1}{2}}$ - the Planck length and M_0 - some constant mass for which it can be supposed that it represents Planck mass $M_P = (\frac{\hbar c}{G})^{\frac{1}{2}}$. In this way Unruh entropy is pratically proportional to the product of the large system surface and logarithm of the large system mass. It represents an original and interesting result.

In the especial case of the Schwarzschild black hole thermodynamical law, (14), according to (12), implies

$$dS = \frac{k8\pi GM}{\hbar c}dM = d[k4\pi GM^2\hbar c] \tag{18}$$

and

$$S = k4\pi GM^2\hbar c = \frac{k}{4}\frac{A}{L_P^2} \tag{19}$$

where A represents the Schwarzschild black hole horizon surface that satisfy (17) for Schwarzschild radius (10). Expression (19) represents, of course, the Bekenstein-Hawking entropy of the Schwarzschild black hole.

Suppose that differential form of (14) is changed by corresponding finite difference form, which according to (6), (9), and (19) yields

$$\Delta S_n = \frac{1}{T}c^2\Delta M = \frac{1}{T}c^2 m_n = n\frac{1}{T}c^2 m_1 = nk\frac{Rc^2}{GM} \quad \text{for} \quad m_n \ll M \quad \text{and} \quad n = 1, 2, ... \tag{20}$$

$$\Delta A_n = \frac{4}{k}L_P^2\Delta S_n = n4\frac{Rc^2}{GM}L_P^2 \quad \text{for} \quad m_n \ll M \quad \text{and} \quad n = 1, 2, ... \tag{21}$$

Expressions (20), (21) can be considered as the Unruh entropy and surface quantization.

In the especial case of the Schwarzschild black hole, according to (10)-(12), (20), (21) turn out in

$$\Delta S_n = n2k \quad \text{for} \quad m_n \ll M \quad \text{and} \quad n = 1, 2, ... \tag{22}$$

$$\Delta A_n = \frac{4}{k}L_P^2\Delta S_n = n8L_P^2 \quad \text{for} \quad m_n \ll M \quad \text{and} \quad n = 1, 2, ... \tag{23}$$

representing Bekenstein quantization of the black hole entropy and horizon surface.

Suppose now that large system radiates analogously to the black body at the Unruh temperature so that Unruh-Stefan-Boltzmann law is satisfied

$$-\frac{1}{A}\frac{dM}{dt}c^2 = \sigma T^4 \tag{24}$$

where σ represents the Stefan-Boltzmann constant. It according to (9), (17), implies

$$-\frac{dM}{dt} = \frac{\sigma A}{c^2}(\frac{1}{k}\frac{\hbar}{2\pi c}\frac{GM}{R^2})^4 = \frac{\sigma A}{c^2}(\frac{1}{k}\frac{\hbar}{2\pi c}\frac{G}{R^2})^4 M^4 \equiv \alpha M^4 \tag{25}$$

where $\alpha = \frac{\sigma A}{c^2}(\frac{1}{k}\frac{\hbar}{2\pi c}\frac{G}{R^2})^4$. After simple integration, where t increases from initial, zero time moment to final time moment of the Unruh evaporation τ, while M decrease from initial mass M_{in} to Planck mass M_P, (25) yields

$$\tau = \frac{1}{3\alpha}\frac{M_{in}^3 - M_P^3}{M_{in}^3 M_P^3} \simeq \frac{1}{3\alpha M_P^3} for M \gg M_P. \tag{26}$$

(It can be pointed out that it is chosen that final mass cannot be smaller than Planck mass, especially that it cannot be zero, since then Unruh evaporation time tends toward infinity. All this is full agreement with quantum field theory.) In this way for practically all macroscopic large system, with mass much larger than Planck mass, Unruh evaporation time is the same and it is inversely proportional to the third degree of the Planck mass. It represents an original and interesting result.

In the especial case of the Schwarzschild black hole, according to (10), (12), (17), and Hawking-Stefan-Boltzmann law, analogous to the general form of Unruh-Stefan-Boltzmann law (24) with changed sign, it follows

$$\frac{1}{A}\frac{dM}{dt}c^2 = \sigma T^4 \tag{27}$$

which implies

$$\frac{dM}{dt} = \frac{\sigma}{c^2}16\pi\frac{G^2M^2}{c^4}(\frac{\hbar c^3}{k8\pi G})^4 M^{-4} = \beta M^{-2} \tag{28}$$

where $\beta = \frac{\sigma}{c^2} 16\pi \frac{G^2 M^2}{c^4} (\frac{\hbar c^3}{k 8\pi G})^4$. After simple integration, where t increases from initial, zero time moment to final time moment of the Hawking evaporation τ, while M decrease from initial mass M_{in} to final mass that here can be formally used to be zero, (28) yields

$$\tau = \frac{1}{3\beta} M_{in}^3. \tag{29}$$

It means that Hawking evaporation time is proportional to the third degree of the initial black hole mass. This final result is principally different from the final result obtained for Unruh evaporation in the general case (26), but both results are deduced in the formally completely analogous way.

In conclusion we can shortly repeat and point out the following. In this work we reproduce exactly final form of the remarkable Unruh temperature for a spherical physical system using a simplified rule, very similar but not identical to Bohr angular momentum quantization postulate interpreted via de Broglie relation and using a thermal equilibrium stability condition. (Our rule, by a deeper analysis that goes over basic intentions of this work, corresponds, for Schwarzschild black hole, to a closed string loop theory of Copeland and Lahiri.) Concretely, firstly, we suppose that at the surface of a large system gravitational field of this system (eventually interacting with quantum vacuum) can generate some quantum excitations, or simply small quantum systems. Mass (energy) spectrum of this small quantum system is determined using a rule that states that circumference of the large system holds integer numbers of the reduced Compton wavelengths of this small quantum system. Secondly, we suppose that absolute value of the classical gravitational interaction between large system and small quantum system in the ground state is equivalent to thermal kinetic energy of the small quantum system interacting with large system as thermal reservoir. It is very similar but not identical to virial theorem of the ideal gas and it yields directly and exactly Unruh temperature. Moreover, using Unruh temperature and thermodynamical law, we originally propose corresponding Unruh entropy (proportional to product of the system surface and logarithm of the system mass) and Unruh evaporation of the system (during time interval approximately inversely proportional to third degree of the Planck mass. Finally, we demonstrate that in the especial case when physical system represents a Schwarzschild black hole Unruh temperature, entropy and evaporation can be exactly reduced in the Hawking temperature, Bekenstein-Hawking entropy and Hawking evaporation (time).

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