ON THE LENGTH OF CHAINS OF PROPER SUBGROUPS COVERING A TOPOLOGICAL GROUP

TARAS BANAKH, DUŠAN REPOVŠ, AND LYUBOMYR ZDOMSKYY

Abstract. We prove that if an ultrafilter $\mathcal{L}$ is not coherent to a Q-point, then each analytic non-σ-bounded topological group G admits an increasing chain $\langle G_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ of its proper subgroups such that: (i) $\bigcup_\alpha G_\alpha = G$; and (ii) For every σ-bounded subgroup H of G there exists α such that $H \subset G_\alpha$. In case of the group $\mathrm{Sym}(\omega)$ of all permutations of ω with the topology inherited from ω^ω this improves upon earlier results of S. Thomas.

1. Introduction

A theorem of Macpherson and Neumann [13] states that if the group $\mathrm{Sym}(\omega)$ can be written as a union of an increasing chain $\langle G_i : i < \lambda\rangle$ of proper subgroups G_i, then $\lambda > \omega$. Throughout this paper the minimal λ with this property will be denoted by $\mathrm{cf}(\mathrm{Sym}(\omega))$. For every increasing function $f \in \omega^\omega$ we denote by S_f the subgroup of $\mathrm{Sym}(\omega)$ generated by $\{\pi \in \mathrm{Sym}(\omega) : \pi, \pi^{-1} \leq^* f\}$, where $x \leq^* y$ means that $x(n) \leq y(n)$ for all but finitely many $n \in \omega$. If we additionally require that for every $f \in \omega^\omega$ there exists $i \in \lambda$ such that $S_f \subset G_i$, then the minimal length of such a chain will be denoted by $\mathrm{cf}^*(\mathrm{Sym}(\omega))$. It is clear that $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \geq \max\{\mathrm{cf}(\mathrm{Sym}(\omega)), \mathfrak{b}\}$. The consistency of $\mathrm{cf}^*(\mathrm{Sym}(\omega)) > \mathrm{cf}(\mathrm{Sym}(\omega))$ and the inequality $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \leq \mathrm{cf}(\mathfrak{d})$ were established in [18, Proposition 2.5]. The initial aim of this paper was to sharpen the latter upper bound on $\mathrm{cf}^*(\mathrm{Sym}(\omega))$. This led us to consider increasing chains of proper submonoids of topological monoids.

We recall that a *semigroup* is a set with a binary associative operation $\cdot : X \times X \to X$. A semigroup with a two-sided unit 1 is called a *monoid*. It is clear that each group is a monoid. By a *topological monoid* we understand a monoid X with a topology τ making the binary operation $\cdot : X \times X \to X$ of X continuous.

Definition 1.1. Let X be a topological monoid (resp. group). The minimal length of an increasing chain $\langle X_i : i < \lambda\rangle$ of proper submonoids (resp. subgroups) X_i of X such that $X = \bigcup_{i<\lambda} X_i$ and for every σ-bounded subset H of X there exists $i \in \lambda$ such that $H \subset X_i$ will be denoted by $\mathrm{cf}^*_m(X)$ (resp. $\mathrm{cf}^*_g(X)$).

We recall that a subset B of a topological monoid X is said to be *totally bounded*, if for every open neighborhood U of the identity 1 of X there exists a finite subset F of X such that $X \subset FU \cap UF$. A subset B is said to be *σ-bounded*, if it can

Date: November 5, 2010.

2000 *Mathematics Subject Classification.* Primary: 03E17, 54H11; Secondary: 54D20.

Key words and phrases. Q-point, P_κ-point, σ-bounded group, ω-bounded group, Menger property, $[\mathcal{F}]$-Menger property.

This research was supported in part by the Slovenian Research Agency grants P1-0292-0101, J1-2057-0101 and BI-UA/09-10-005. The third author acknowledges the support of the FWF grant P19898-N18. We thank the referees for comments and suggestions.

be written as a countable union of totally bounded subsets. A direct verification shows that $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ as defined in [18] and $\mathrm{cf}^*_g(\mathrm{Sym}(\omega))$ in the sense of our Definition 1.1 coincide.

It is clear that $\mathrm{cf}^*_m(X) \le \mathrm{cf}^*_g(X)$ for every topological group X. We do not know whether these cardinals can be different. Probably the most interesting case is the group $\mathrm{Sym}(\omega)$.

Let R be a relation on ω and $x, y \in \omega^\omega$. We denote by $[x\,R\,y]$ the set $\{n \in \omega : x(n)\,R\,y(n)\}$. For an ultrafilter $\mathcal{F}$ the notation $x \le_{\mathcal{F}} y$ means $[x \le y] \in \mathcal{F}$. Let $\mathfrak{b}(\mathcal{F})$ be the cofinality of the linearly ordered set $(\omega^\omega, \le_{\mathcal{F}})$.

Following [2] we define a point $x \in X$ of a topological monoid X to be *left balanced* (resp. *right balanced*) if for every neighborhood $U \subset X$ of the unit 1 of X there is a neighborhood $V \subset X$ of 1 such that $Vx \subset xU$ (resp. $xV \subset Ux$). Observe that x is left balanced if the left shift $l_x : X \to X$, $l_x : y \mapsto xy$, is open at 1. Let B_L and B_R denote respectively the sets of all left and right balanced points of the monoid X. A topological monoid X is defined to be *left balanced* (resp. *right balanced*) if $X = B_L \cdot U$ (resp. $X = U \cdot B_R$) for every neighborhood $U \subset X$ of the unit 1 in X. If a topological monoid X is both left and right balanced, then we say that X is *balanced*.

We define a topological monoid X to be a Menger monoid[1], if for every sequence $\langle U_n : n \in \omega\rangle$ of open neighborhoods of 1 there exists a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of X such that $X = \bigcup_{n\in\omega} F_nU_n \cap U_nF_n$. A topological monoid X is said to be *ω-bounded*, if for every neighborhood U of 1 there exists a countable $C \subset X$ such that $X = C \cdot U$.

The following two theorems are the principal results of this paper.

Theorem 1.2. *Let X be a first countable ω-bounded balanced topological monoid such that one of its finite powers is not a Menger monoid. Then $\mathrm{cf}^*_m(X) \le \mathfrak{b}(\mathcal{L})$ for every ultrafilter $\mathcal{L}$ which is not coherent to any Q-point.*

Theorem 1.3. *Let G be an ω-bounded topological group such that one of its finite powers is not a Menger monoid. Then $\mathrm{cf}^*_g(G) \le \mathfrak{b}(\mathcal{L})$ for every ultrafilter $\mathcal{L}$ which is not coherent to any Q-point.*

Applying [2, Proposition 7.5] we conclude that the Baire space ω^ω with the operation of composition is a balanced topological monoid, and σ-bounded subsets of this topological monoid are exactly those which are contained in the σ-compact subsets of ω^ω. It is easy to see that ω^ω is not a Menger monoid. Thus we get the following

Corollary 1.4. *Let $\mathcal{L}$ be an ultrafilter coherent to no Q-point. Then ω^ω can be written as the union of an increasing chain of its proper subsets of length $\le \mathfrak{b}(\mathcal{L})$, each of which is closed under composition, and such that every σ-compact subset of ω^ω is contained in one of the elements of this chain.*

A metrizable space X is said to be *analytic*, if it is a continuous image of ω^ω. A topological group G is called *analytic* if such is the underlying topological space. Theorem 1.3 implies the following:

Corollary 1.5. *Let G be an analytic group which is not σ-bounded. Then $\mathrm{cf}^*_g(G) \le \mathfrak{b}(\mathcal{L})$ for every ultrafilter $\mathcal{L}$ which is not coherent to any Q-point.*

---

[1] In terms of [2] this means that $(X, \mu_L \wedge \mu_R)$ is a Menger monoid.

$\mathrm{Sym}(\omega)$ is easily seen to be a G_δ-subset of ω^ω and the composition as well as the inversion are continuous with respect to the topology inherited from ω^ω. Therefore $\mathrm{Sym}(\omega)$ with this topology is a Polish topological group. A direct verification also shows that it is not σ-bounded.

Corollary 1.6. $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \leq \mathfrak{b}(\mathcal{L})$ *for every ultrafilter* $\mathcal{L}$ *which is not coherent to a* Q*-point.*

Combined with the following consequence of [12, Theorem 2.8], Corollary 1.6 yields the upper bound for $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ obtained earlier in [18].

Proposition 1.7. *There exists an ultrafilter* $\mathcal{L}$ *which is not coherent to any* Q*-point and such that* $\mathfrak{b}(\mathcal{L}) = \mathrm{cf}(\mathfrak{d})$.

We recall from [5] that ultrafilters $\mathcal{F}$ and $\mathcal{U}$ on ω are said to be *nearly coherent*, if there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of natural numbers such that $\bigcup_{n\in I}[k_n, k_{n+1}) \in \mathcal{F}$ if and only if $\bigcup_{n\in I}[k_n, k_{n+1}) \in \mathcal{U}$ for every subset I of ω. In what follows we shall drop "near" and simply say that two ultrafilters are coherent. In other words, $\mathcal{F}$ and $\mathcal{U}$ are coherent if and only if $\phi(\mathcal{F}) = \phi(\mathcal{U})$ for some increasing surjection $\phi : \omega \to \omega$. The coherence relation is an equivalence relation. NCF is the statement that all ultrafilters are coherent. Its consistence was established in [7].

An ultrafilter $\mathcal{L}$ is called:

- *a (pseudo-)* P_κ*-point*, where κ is a cardinal, if for every $\mathcal{L}' \in [\mathcal{L}]^{<\kappa}$ there exists $L \in \mathcal{L}$ (resp. $L \in [\omega]^\omega$) such that $L \subset^* L'$ for all $L' \in \mathcal{L}'$. P_{ω_1}-points are also called P-points;
- *a simple* P_κ*-point*, if there exists a sequence $\langle L_\alpha : \alpha < \kappa \rangle$ of infinite subsets of ω such that $L_\alpha \subset^* L_\beta$ for all $\kappa > \alpha > \beta$ and $\mathcal{L} = \{X \subset \omega : L_\alpha \subset X$ for some $\alpha < \kappa\}$;
- *a* Q*-point*, if for every increasing surjection $\phi : \omega \to \omega$ there exists $L \in \mathcal{L}$ such that $\phi \restriction L$ is injective;
- *a Ramsey ultrafilter*, if it is simultaneously both a P- and a Q-point.

Corollary 1.6 implies the following statements.

Corollary 1.8. *Suppose that there exists a pseudo-*$P_{\mathfrak{b}^+}$*-point. Then* $\mathrm{cf}^*(\mathrm{Sym}(\omega)) = \mathfrak{b}$.

Corollary 1.9. *Suppose that* $\mathfrak{u} < \mathrm{cf}^*(\mathrm{Sym}(\omega))$. *Every two ultrafilters that are not coherent to* Q*-points are coherent. In particular, if there is no* Q*-point, then NCF holds.*

Corollary 1.8 can be compared to the following theorem: If $\lambda < \kappa$ are regular uncountable cardinals such that there exists a simple P_λ-point $\mathcal{U}$ and a P_κ-point $\mathcal{F}$, then $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \leq \lambda$ (cf. [18, Theorem 3.4]). The assumption of this theorem (whose consistency was conjectured in [7]) clearly implies that $\mathfrak{u} < \mathfrak{s}$ and $\mathcal{U}$ is not coherent to $\mathcal{F}$, and hence there are exactly two coherence classes of ultrafilters (cf. [6, Corollary 13]). The question whether there can be exactly n coherence classes of ultrafilters for $1 < n < \omega$ remains open.

On the other hand, given any ground model of GCH and a regular cardinal ν in it, the forcing from [8] with $\delta = \omega_1$ and $\nu = \kappa$ (δ and ν are the two parameters there) yields a model of "there exists a simple P_κ-point $\mathcal{U}$ and $\mathfrak{b} = \omega_1 \leq 2^\omega = \kappa$". Combined with Theorem 1.3 this gives the consistency of the statement "there exists a simple P_κ-point $\mathcal{U}$ and $\omega_1 = \mathfrak{b} = \mathrm{cf}^*(\mathrm{Sym}(\omega)) = \mathfrak{b}(\mathcal{U}) < \kappa$".

We shall denote the set of all unbounded nondecreasing elements of ω^ω by $\omega^{\uparrow\omega}$. We call a set $F \subset \omega^{\uparrow\omega}$ *finitely dominating*, if for every $x \in \omega^\omega$ there exists a finite subset $\{f_0, \dots, f_n\}$ of F such that $x \le^* \max\{f_0, \dots, f_n\}$. Following [14] we denote the minimal size of a family of non-finitely dominating sets covering $\omega^{\uparrow\omega}$ by $\mathrm{cov}(\mathfrak{D}_{fin})$.

As the next theorem shows, NCF implies that $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ is maximal possible.

Theorem 1.10. $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \ge \mathrm{cov}(\mathfrak{D}_{fin})$. *Moreover, NCF implies that* $\mathrm{cf}^*(\mathrm{Sym}(\omega)) = \mathfrak{d}$.

Shelah and Tsaban [17] proved that $\max\{\mathfrak{b}, \mathfrak{g}\} \le \mathrm{cov}(\mathfrak{D}_{fin})$, and the strict inequality is consistent (cf. [14]). Thus Theorem 1.10 improves the lower bound in $\mathfrak{g} \le \mathrm{cf}^*(\mathrm{Sym}(\omega))$ [18, Theorem 2.6]. Combining Corollary 1.9 and the fact that there are no Q-points under $\mathfrak{u} < \mathfrak{s}$ (cf. [3, Theorems 13.6.2, 13.8.1]), we get the following:

Corollary 1.11. *If* $\mathfrak{u} < \min\{\mathfrak{s}, \mathrm{cf}^*(\mathrm{Sym}(\omega))\}$, *then NCF holds.*

We do not know whether the inequality $\mathfrak{u} < \mathrm{cf}^*(\mathrm{Sym}(\omega))$ (or even $\mathfrak{u} < \mathrm{cf}(\mathrm{Sym}(\omega))$) implies NCF. This would be true if $\mathrm{cf}(\mathrm{Sym}(\omega)) \le \mathfrak{mcf} = \min\{\mathfrak{b}(\mathcal{F}) : \mathcal{F}$ is an ultrafilter$\}$ (in particular, if $\mathfrak{mcf}$ is attained at some ultrafilter not coherent to a Q-point). It would also be interesting to establish whether NCF implies $\mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{d}$.

This work is a continuation of our previous paper [2]. We refer the reader to [19] for the definitions and basic properties of small cardinals which are used but not defined in this paper. All filters are assumed to be non-principal.

2. Proofs

The main technical tool for the proofs of Theorems 1.2 and 1.3 was developed in [2]. This will allow us to prove some stronger technical statements in this section, namely Propositions 2.5 and 2.6. In order to formulate them we need to recall some definitions.

Let $\mathcal{F}$ be a filter. Following [4] (our definition of an $[\mathcal{F}]$-cover differs slightly from the one given in [2, 4], however, by [3, 5.5.2, 5.5.3] the two versions are equivalent), we define an indexed cover $\langle B_n : n \in \omega\rangle$ of a set X to be an $[\mathcal{F}]$-*cover* if there is an increasing surjection $\phi : \omega \to \omega$ such that $\phi(\{n \in \omega : x \in B_n\}) \in \mathcal{F}$ for every $x \in X$.

A subset X of a topological monoid M is defined to be $[\mathcal{F}]$-*Menger* if for every sequence $\langle U_n : n \in \omega\rangle$ of neighborhoods of 1 in M there is a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of M such that $\langle U_n \cdot F_n \cap F_n \cdot U_n : n \in \omega\rangle$ is an $[\mathcal{F}]$-cover of X. The latter happens if and only if

$$X \subset \bigcup_{F \in \mathcal{F}} \bigcap_{n \in \phi(F)} U_n \cdot F_n \cap F_n \cdot U_n$$

for some monotone surjection $\phi : \omega \to \omega$.

Definition 2.1. For a topological monoid (group) X and a free filter $\mathcal{F}$ on ω by $\mathrm{cf}^{\mathcal{F}}_m(X)$ (resp. $\mathrm{cf}^{\mathcal{F}}_g(X)$) we denote the minimal length of an increasing chain $\langle X_i : i < \lambda\rangle$ of proper submonoids (subgroups) X_i of X such that $X = \bigcup_{i<\lambda} X_i$ and for every $[\mathcal{F}]$-Menger subset H of X there exists $i \in \lambda$ such that $H \subset X_i$.

If no such chain exists, then we say that $\mathrm{cf}^{\mathcal{F}}_m(X)$ (resp. $\mathrm{cf}^{\mathcal{F}}_g(X)$) is undefined.

It is easy to check that $\mathrm{cf}^*_m(X)$ (resp. $\mathrm{cf}^*_g(X)$) is $\mathrm{cf}^{\mathfrak{Fr}}_m(X)$ (resp. $\mathrm{cf}^{\mathfrak{Fr}}_g(X)$), where $\mathfrak{Fr}$ denotes the Fréchet filter consisting of all cofinite subsets of ω.

Let $\mathcal{F}$ be an ultrafilter. A sequence $\langle b_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\rangle$ of increasing elements of ω^ω is called a $\mathfrak{b}(\mathcal{F})$-*scale*, if it is cofinal with respect to $\le_\mathcal{F}$ and $b_\alpha \le_\mathcal{F} b_\beta$ for all $\alpha \le \beta < \mathfrak{b}(\mathcal{F})$.

Let us denote the family of all monotone surjections from ω to ω by $\mathcal{S}$. Following [3, §10.1] (see also [9]) we denote for an ultrafilter $\mathcal{F}$ by $\mathfrak{q}(\mathcal{F})$ the minimal size of a subfamily Φ of $\mathcal{S}$ such that for every $\psi \in \mathcal{S}$ there exists $\phi \in \Phi$ such that $[\phi \le \psi] \in \mathcal{F}$. It is clear that there exists a sequence $\langle \phi_\alpha : \alpha < \mathfrak{q}(\mathcal{F})\rangle \in \mathcal{S}^{\mathfrak{q}(\mathcal{F})}$ such that $[\phi_\beta < \phi_\alpha] \in \mathcal{F}$ for all $\beta > \alpha$ and for every $\psi \in \mathcal{S}$ there exists α with the property $[\phi_\alpha < \psi] \in \mathcal{F}$. Such a family will be called a $\mathfrak{q}(\mathcal{F})$-*scale*.

Cardinals $\mathfrak{b}(\mathcal{F})$ and $\mathfrak{q}(\mathcal{F})$ are the cofinality and the coinitiality of the linearly ordered set $(\omega^{\uparrow\omega}, \le_\mathcal{F})$, which in a certain sense makes them dual.

If an ultrafilter $\mathcal{F}$ is not coherent to any Q-point then $\mathfrak{b}(\mathcal{F}) = \mathfrak{q}(\mathcal{F})$, for a proof see [12, 10] or [3, 10.2.5]. On the other hand, there can be ultrafilters $\mathcal{F}$ with $\mathfrak{b}(\mathcal{F}) \neq \mathfrak{q}(\mathcal{F})$, see [9]. As we shall see later, this means that $\mathrm{cf}^\mathcal{F}_g(X)$ and $\mathrm{cf}^\mathcal{F}_m(X)$ are not always well-defined.

Theorem 2.2. *Let $\mathcal{F}$ be an ultrafilter and X a first countable ω-bounded balanced topological monoid (resp. first countable topological group) and suppose that one of its finite powers is not a Menger monoid.*

(1) *If the cardinal $\mathrm{cf}^\mathcal{F}_m(X)$ (resp. $\mathrm{cf}^\mathcal{F}_g(X)$) exists, then it is equal to $\mathfrak{b}(\mathcal{F})$ and $\mathfrak{b}(\mathcal{F}) = \mathfrak{q}(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is not coherent to any Q-point, then the cardinal $\mathrm{cf}^\mathcal{F}_m(X)$ (resp. $\mathrm{cf}^\mathcal{F}_g(X)$) exists and hence it is equal to $\mathfrak{b}(\mathcal{F}) = \mathfrak{q}(\mathcal{F})$.*
(3) *For the group $X = \mathrm{Auth}(\mathbb{R}_+)$ of the homeomorphisms of the half-line the cardinal $\mathrm{cf}^\mathcal{F}_m(X)$ exists if and only if $\mathrm{cf}^\mathcal{F}_g(X)$ exists if and only if $\mathcal{F}$ is not coherent to a Q-point.*

We postpone the proof of Theorem 2.2 for the moment. It is clear that for a topological group X the existence of $\mathrm{cf}^\mathcal{F}_g(X)$ implies the existence of $\mathrm{cf}^\mathcal{F}_m(X)$, and in this case $\mathrm{cf}^\mathcal{F}_m(X) \le \mathrm{cf}^\mathcal{F}_g(X)$.

Question 2.3. Is the existence of $\mathrm{cf}^\mathcal{F}_g(X)$ equivalent to the existence of $\mathrm{cf}^\mathcal{F}_m(X)$ (at least for the group $\mathrm{Sym}(\omega)$)? Are these cardinals always equal (if they exist)?

The following result was established in [2].

Lemma 2.4. *A topological group (resp. balanced topological monoid) H is $[\mathcal{L}]$-Menger for some ultrafilter $\mathcal{L}$ coherent to no Q-point if and only if H is algebraically generated by an $[\mathcal{L}]$-Menger subspace $X \subset H$.*

The condition in Lemma 2.4 that $\mathcal{L}$ is not coherent to any Q-point is essential by [2, Theorem 6.4]. However, we do not know whether it can be omitted from Theorem 1.2, Theorem 1.3 or Corollary 1.6.

Theorem 1.2 is a special case of the following result:

Proposition 2.5. *Let X be a first countable ω-bounded balanced topological monoid such that one of its finite powers is not a Menger monoid, and let $\mathcal{F}$ be a filter on ω. If there exists an ultrafilter $\mathcal{L} \supset \mathcal{F}$ that is not coherent to any Q-point, then $\mathrm{cf}^\mathcal{F}_m(X)$ is well-defined and is less than or equal to $\mathfrak{b}(\mathcal{L})$.*

Proof. Let $\mathcal{L} \supset \mathcal{F}$ be an ultrafilter that is not coherent to any Q-point, $\langle b_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ be a $\mathfrak{b}(\mathcal{L})$-scale, and $\langle \phi_\alpha : \alpha < \mathfrak{q}(\mathcal{L}) = \mathfrak{b}(\mathcal{L})\rangle$ be a $\mathfrak{q}(\mathcal{L})$-scale. Assume that X^k is not a Menger monoid for some $k \in \omega$. Let $\{U_n : n \in \omega\}$ be a local base at the neutral element 1 of X. Without loss of generality, we may assume that $U^3_{n+1} \subset U_n$ for all $n \in \omega$. Applying [2, Proposition 7.1], we can additionally assume that there exists a sequence $\langle C_n : n \in \omega\rangle$ of countable subsets of X such that $U_n \cdot C_n = C_n \cdot U_n = X$ for all n, and for every $F \in [X]^{<\omega}$ there exists $F' \in [C_n]^{<\omega}$ such that $FU_{n+1} \cap U_{n+1}F \subset F'U_n \cap U_nF'$. Fix an enumeration $\{c_{n,m} : m \in \omega\}$ of C_n. For a pair $(\phi, b) \in \mathcal{S} \times \omega^\omega$ we set

$$Y_{\phi,b} = \bigcup_{L\in\mathcal{L}} \bigcap_{n\in L} U_{\phi(n)} \cdot \{c_{k,m} : \phi(n) \le k \le n,\ m \le b(n)\} \cap \cap\{c_{k,m} : \phi(n) \le k \le n,\ m \le b(n)\} \cdot U_{\phi(n)}$$

and denote by X_α the submonoid of X generated by Y_{ϕ_α,b_α}. A direct verification shows that $Y_{\phi,b}$ is an $[\mathcal{L}]$-Menger subset of X for arbitrary pair $(\phi, b) \in \mathcal{S} \times \omega^\omega$ (cf. e.g., the proof of [2, Lemma 3.2]), and hence by Lemma 2.4 X_α is an $[\mathcal{L}]$-Menger submonoid of X. Thus $\langle X_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ is an increasing sequence of $[\mathcal{L}]$-Menger submonoids of X. Since X^k is not a Menger monoid and the $[\mathcal{L}]$-Menger property is preserved by finite powers [2, Corollary 3.5], each X_α is a proper submonoid of X.

It suffices to show that each $[\mathcal{F}]$-Menger submonoid H of X is contained in some X_α. Given such H let us find an increasing $f \in \omega^\omega$ and $\phi \in \mathcal{S}$ such that

$$H \subset \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} U_{\phi(n)} \cdot \{c_{\phi(n),m} : m \le f(n)\} \cap \{c_{\phi(n),m} : m \le f(n)\} \cdot U_{\phi(n)}.$$

(Such f and ϕ can be easily constructed by the definition of the $[\mathcal{F}]$-Menger property.)

Choose α such that $f \le_{\mathcal{L}} b_\alpha$ and $\phi_\alpha \le_{\mathcal{L}} \phi$. We claim that $H \subset X_\alpha$. Indeed, let us fix $h \in H$ and pick $F_0 \in \mathcal{F}$ such that

$$h \in \bigcap_{n\in F_0} U_{\phi(n)} \cdot \{c_{\phi(n),m} : m \le f(n)\} \cap \{c_{\phi(n),m} : m \le f(n)\} \cdot U_{\phi(n)}.$$

Set $A = [\phi_\alpha \le \phi]$, $B = [f \le b_\alpha]$, and observe that $A, B \in \mathcal{L}$. Then

$$\begin{aligned} h \in \bigcap_{n\in F_0} U_{\phi(n)} \cdot \big\{c_{\phi(n),m} : m \le f(n)\big\} \cap \big\{c_{\phi(n),m} : m \le f(n)\big\} \cdot U_{\phi(n)} \subset \\ \subset \bigcap_{n\in F_0\cap A} U_{\phi_\alpha(n)} \cdot \big\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le f(n)\big\} \cap \\ \cap\big\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le f(n)\big\} \cdot U_{\phi_\alpha(n)} \subset \\ \subset \bigcap_{n\in F_0\cap A\cap B} U_{\phi_\alpha(n)} \cdot \big\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le b_\alpha(n)\big\} \cap \\ \cap\big\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le b_\alpha(n)\big\} \cdot U_{\phi_\alpha(n)} \subset X_\alpha, \end{aligned}$$

which completes our proof. □

Theorem 1.3 is a consequence of the following:

Proposition 2.6. *Let G be an ω-bounded topological group such that one of its finite powers is not a Menger monoid and let $\mathcal{F}$ be a filter on ω. If there exists an* *ultrafilter $\mathcal{L} \supset \mathcal{F}$ that is not coherent to any Q-point, then $\mathrm{cf}_g^{\mathcal{F}}(G)$ is well-defined and is less than or equal to $\mathfrak{b}(\mathcal{L})$.*

Proof. By a result of Guran [11], G is topologically isomorphic to a subgroup of a product $\prod_{i\in I} Q_i$, where each Q_i is a second countable group. There exists $J \in [I]^\omega$ with the property that one of the finite powers of $H := \mathrm{pr}_J(G)$ is not a Menger monoid. Indeed, let $k \in \omega$ be such that G^k is not a Menger monoid. There exists a sequence $\langle U_n : n \in \omega\rangle$ of open neighbourhoods of the neutral element of G such that $G^k \neq \bigcup_{n\in\omega} F_n\, U_n^k \cap U_n^k\, F_n$ for any sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of G^k. Shrinking U_n, if necessary, we may additionally assume that $U_n = \prod_{i\in J_n} W_{i,n} \times \prod_{i\in I\setminus J_n} Q_i$, where J_n is a finite subset of I and $W_{i,n}$ is an open neighbourhood of the neutral element of Q_i. Set $J = \bigcup_{n\in\omega} J_n$, $H = \mathrm{pr}_J(G)$, and $V_n = \prod_{i\in J_n} W_{i,n} \times \prod_{i\in J\setminus J_n} Q_i$. It follows from the above that $H^k \neq \bigcup_{n\in\omega} K_n V_n^k \cap V_n^k\, K_n$ for any sequence $\langle K_n : n \in \omega\rangle$ of finite subsets of H^k, which means that H^k is not a Menger monoid.

By applying the same argument as in the proof of Proposition 2.5 to the (first countable) group H, we conclude that there exists an appropriate increasing chain $\langle H_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ of proper subgroups of H such that $H = \bigcup_\alpha H_\alpha$. Now $\langle \mathrm{pr}_J^{-1}(H_\alpha) : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ is a witness for $\mathrm{cf}_g^{\mathcal{F}}(G) \le \mathfrak{b}(\mathcal{L})$, which completes our proof. □

Proof of Theorem 2.2. (1) Suppose that $\kappa := \mathrm{cf}_m^{\mathcal{F}}(X)$ exists and $\kappa < \mathfrak{q}(\mathcal{F})$. All other cases ($\kappa > \mathfrak{q}(\mathcal{F})$, $\kappa < \mathfrak{b}(\mathcal{F})$, $\kappa > \mathfrak{b}(\mathcal{F})$, or X is a topological group, $\mathrm{cf}_g^{\mathcal{F}}(X)$ exists and $\mathrm{cf}_g^{\mathcal{F}}(X) < \mathfrak{q}(\mathcal{F})$, $\mathrm{cf}_g^{\mathcal{F}}(X) > \mathfrak{q}(\mathcal{F})$, $\mathrm{cf}_g^{\mathcal{F}}(X) < \mathfrak{b}(\mathcal{F})$, or $\mathrm{cf}_g^{\mathcal{F}}(X) > \mathfrak{b}(\mathcal{F})$) are analogous.

We use the notations from the proof of Proposition 2.5. For every $\alpha < \mathfrak{q}(\mathcal{F})$ let

$$Z_\alpha = \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} U_{\phi_\alpha(n)} \cdot \{c_{\phi_\alpha(n),m} : m \le n\} \cap \{c_{\phi_\alpha(n),m} : m \le n\} \cdot U_{\phi_\alpha(n)}$$

and observe that $\langle Z_\alpha : \alpha < \mathfrak{q}(\mathcal{F})\rangle$ is an increasing sequence of $[\mathcal{F}]$-Menger subspaces of X covering X. Let $\langle X_\xi : \xi < \kappa\rangle$ be a sequence of proper submonoids of X witnessing for $\mathrm{cf}_m^{\mathcal{F}}(X) = \kappa$. Since $\mathfrak{q}(\mathcal{F})$ is regular and for every $\alpha < \mathfrak{q}(\mathcal{F})$ there exists $\xi < \kappa$ with $Z_\alpha \subset X_\xi$, we conclude that there exists ξ such that $X_\xi \supset Z_\alpha$ for cofinally many $\alpha \in \mathfrak{q}(\mathcal{F})$, which means $X_\xi = X$ and thus contradicts the assumption that X_ξ is a proper submonoid of X.

(2) The existence of $\mathrm{cf}_m^{\mathcal{F}}(X)$ (resp. $\mathrm{cf}_g^{\mathcal{F}}(X)$) follows from Proposition 2.5 (resp. Proposition 2.6.) The rest is a consequence of the previous item.

(3) This item follows directly from [2, Theorem 6.4]. □

A sequence $\langle U_n : n \in \omega\rangle$ is called an *ω-cover* of a set X if for every finite $F \subset X$ there exists $n \in \omega$ such that $F \subset U_n$. If, moreover, there exists an increasing sequence $\langle n_k : k \in \omega\rangle$ of integers such that for every finite $F \subset X$ and for all but finitely many $k \in \omega$ there exists $n \in [n_k, n_{k+1})$ such that $F \subset U_n$, then the cover $\langle U_n : n \in \omega\rangle$ is called *ω-groupable.*

Proof of Corollary 1.5. In light of Theorem 1.3 it is enough to verify the following:

Claim 2.7. *If all finite powers of an analytic topological group G are Menger monoids, then G is σ-bounded*[2].

Proof. Suppose that all finite powers of G are Menger monoids. By applying [21, Lemma 17] and [2, Prop. 3.1, Lemma 3.2], we can conclude that G is $[\mathcal{U}]$-Menger for some ultrafilter $\mathcal{U}$. Given a decreasing base $\langle U_n : n \in \omega\rangle$ at the identity of G we can find a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of G such that $\langle B_n = F_nU_n \cap U_nF_n : n \in \omega\rangle$ is an $[\mathcal{U}]$-cover of G. For every $g \in G$ denote the set $\{n \in \omega : g \in B_n\}$ by $\mathcal{N}_g$.

It follows that there exists an increasing number sequence $\langle n_k : k \in \omega\rangle$ such that $\bigcup_{\mathcal{N}_g \cap [n_k, n_{k+1}) \neq \emptyset} [n_k, n_{k+1}) \in \mathcal{U}$ for all $g \in G$ (if ϕ is a finite-to-one surjection witnessing for $\langle B_n : n \in \omega\rangle$ being an $[\mathcal{U}]$-cover, then the sequence $\langle \min \phi^{-1}(k)\rangle_{k\in\omega}$ is as required.) Let F'_k be a finite subset of G such that $D_k := U_kF'_k \cap F'_kU_k \supset \bigcup_{n\in[n_k,n_{k+1})} B_n$. $\langle D_k : k \in \omega\rangle$ is clearly an ω-cover of G. Applying [16, Theorem 4.5] (see also [20, Theorem 7]), we conclude that $\langle D_k : k \in \omega\rangle$ is ω-groupable.

Let $\langle k_m : m \in \omega\rangle$ be an increasing number sequence witnessing for this. Set $Y_m = \bigcap_{l\geq m} \bigcup_{k\in[k_m,k_{m+1})} D_k$. A direct verification shows that each Y_m is totally bounded and $G = \bigcup_{m\in\omega} Y_m$. □ □

Proof of Corollary 1.8. Suppose that $\mathcal{U}$ is a pseudo-$P_{\mathfrak{b}^+}$-point. Since $\phi(\mathcal{U})$ is clearly a pseudo-$P_{\mathfrak{b}^+}$-point for every finite-to-one ϕ, $\mathcal{U}$ is not coherent to a Q-point by [3, Theorem 13.8.1]. Therefore $\mathrm{cf}^*(\mathrm{Sym}(\kappa)) \leq \mathfrak{b}(\mathcal{U})$. It suffices to apply the following result of Nyikos [15] (see [6, Proposition 5] or [3, Theorem 13.2.1, Corollary 10.3.2] for its proof): *If $\mathcal{L}$ is pseudo-$P_{\mathfrak{b}^+}$-point, then $\mathfrak{b}(\mathcal{L}) = \mathfrak{b}$.* □

Proof of Corollary 1.9. Let $\mathcal{U}$ be an ultrafilter generated by $\mathfrak{u}$ many subsets of ω. It is well-known that $\mathfrak{b}(\mathcal{U}) = \mathfrak{d}$ and $\mathcal{U}$ is coherent to any ultrafilter $\mathcal{F}$ such that $\mathfrak{b}(\mathcal{F}) > \mathfrak{u}$, see [3, Theorem 10.3.1] or [6, Theorem 12]. It suffices to apply Corollary 1.5 and the transitivity of the coherence relation. □

Lemma 2.8. *If $F \subset \omega^\omega$ is a finitely dominating family of strictly increasing functions, then $\bigcup_{f\in F} S_f$ generates $\mathrm{Sym}(\omega)$.*

Proof. Let $H = \langle \bigcup_{f\in F} S_f\rangle$ and $\pi \in \mathrm{Sym}(\omega)$ be such that all its orbits are finite, i.e. for every $n \in \omega$ the set $\{\pi^k(n) : k \in \omega\}$ is finite, where $\pi^1 = \pi$ and $\pi^{k+1} = \pi \circ \pi^k$. Let $\mathcal{A} = \{a_i : i \in \omega\}$ be the enumeration of orbits of π such that $\min a_i < \min a_{i+1}$ for all i. The following claim is obvious.

Claim 2.9. *There exist two increasing sequences $\langle n^0_i : i \in \omega\rangle$ and $\langle n^1_i : i \in \omega\rangle$ of natural numbers such that for every $a \in \mathcal{A}$ there exists a pair $\langle i, j\rangle \in \omega \times 2$ such that $a \subset [n^j_i, n^j_{i+1})$.*

Let $h \in \omega^\omega$ be an increasing function such that $h(n^j_i) \geq \max\{\pi(m), \pi^{-1}(m) : m \in [n^j_i, n^j_{i+1})\}$ for all i and j, and F_0 be a finite subset of F such that $h \leq^* \max F_0$. Fix any $a \in \mathcal{A}$ and find $\langle i, j\rangle \in \omega \times 2$ such that $a \subset [n^j_i, n^j_{i+1})$. Let $f \in F_0$ be such that $f(n^j_i) > h(n^j_i)$. By the definition of h the above implies $\pi(m), \pi^{-1}(m) \leq h(n^j_i) \leq f(n^j_i) \leq f(m)$ for every $m \in a$. Therefore for every

[2]This fact can be thought of as the analogue for topological groups of the following result proven in [1]: if for every sequence $\langle u_n : n \in \omega\rangle$ of open covers of an analytic space X there exists a sequence $\langle v_n : n \in \omega\rangle$ such that $v_n \in [u_n]^{<\omega}$ and $X = \bigcup_{n\in\omega} \cup v_n$, then X is σ-compact.

$a \in \mathcal{A}$ there exists $f_a \in F_0$ such that $\pi(m), \pi^{-1}(m) < f_a(m)$ for all $m \in a$. Set $\pi_f = \pi \upharpoonright \bigcup\{a \in \mathcal{A} : f_a = f\}$ and note that $\pi_f \in S_f$ and $\pi = \circ_{f\in F_0}\pi_f$ (the latter composition obviously does not depend on the order in which we take π_f's). Hence $\pi \in H$.

$\mathrm{Sym}(\omega)$ is easily seen to be a G_δ-subset of ω^ω. Therefore $\mathrm{Sym}(\omega)$ with the topology τ inherited from ω^ω is a Polish topological group. It is also easy to check that the set E of all permutations of ω with finite orbits is a dense G_δ of $(\mathrm{Sym}(\omega), \tau)$, and hence $E \circ E \supset \mathrm{Sym}(\omega)$ by the Baire Category Theorem. It suffices to note that $E \circ E \subset H$. □

Proof of Theorem 1.10. The first statement is a direct consequence of Lemma 2.8: Suppose that $\kappa = \mathrm{cf}^*(\mathrm{Sym}(\omega)) < \mathrm{cov}(\mathfrak{D}_{fin})$ and $\langle G_\alpha : \alpha < \kappa\rangle$ is an increasing sequence of proper subgroups of $\mathrm{Sym}(\omega)$ witnessing for that. Set $B_\alpha = \{f \in \omega^{\uparrow\omega} : S_f \subset G_\alpha\}$. By the definition of $\mathrm{cf}^*(\mathrm{Sym}(\omega))$, $\bigcup_{\alpha<\kappa} B_\alpha = \omega^{\uparrow\omega}$. Since $\kappa < \mathrm{cov}(\mathfrak{D}_{fin})$, there exists $\alpha < \kappa$ such that B_α is finitely dominating, which by Lemma 2.8 implies that $G_\alpha = \mathrm{Sym}(\omega)$ and hence contradicts the properness of G_α.

The second one follows from the fact that NCF implies that $\mathrm{cov}(\mathfrak{D}_{fin}) = \mathfrak{d}$. Indeed, suppose that NCF holds. Then $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$ for all ultrafilters $\mathcal{F}$, see e.g. [5, Theorem 16] or [3, 12.3.1]. In addition, every not finitely dominating subset of $\omega^{\uparrow\omega}$ is $\le_{\mathcal{F}}$-bounded for every ultrafilter $\mathcal{F}$. □

3. Appendix

Following the suggestion of the referee, we include here from [3] an essentially self-contained proof of the fact that there are no Q-points (in fact, rare ultrafilters) provided that $\mathfrak{r} < \mathfrak{s}$. This is a direct consequence of Corollary 3.3 and Proposition 3.4 below.

The easiest way to do this would be to simply copy relevant pieces of [3]. But since the book [3] is available online, this does not make much sense. Therefore we take another approach and present a simplified proof. The simplification comes mainly from the obvious equality $\mathcal{F} = \mathcal{F}^\perp$ which holds for all ultrafilters. However, this simplification seems to hide some ideas.

In what follows $\mathfrak{Fr}$ denotes the filter of cofinite subsets of ω. By a *semifilter* we mean a subset $\mathcal{S}$ of $[\omega]^\omega$ which is closed with respect to taking supersets of its elements and such that $S \cap A \in \mathcal{S}$ for all $S \in \mathcal{S}$ and $A \in \mathfrak{Fr}$. For a subset Ψ of $\omega \times \omega$ and $n \in \omega$ we set $\Psi(n) = \{m \in \omega : (n, m) \in \Psi\}$ and $\Psi^{-1}(n) = \{m \in \omega : (m, n) \in \Psi\}$. $\Psi \subset \omega\times\omega$ is called a *finite-to-finite multifunction*, if $\Psi(n), \Psi^{-1}(n)$ are finite and nonempty for all $n \in \omega$. The family of all finite-to-finite multifunction will usually be considered with the preorder $\subset^*$. A semifilter $\mathcal{S}_0$ is said to be *subcoherent* to a semifilter $\mathcal{S}_1$, if there exists a finite-to-finite multifunction Ψ such that $\Psi(\mathcal{S}_0) \subset \mathcal{S}_1$, where $\Psi(\mathcal{S}_0) = \{\Psi(S) : S \in \mathcal{S}_0\}$ and $\Psi(X) = \bigcup_{n\in X} \Psi(n)$ for all $X \subset \omega$. Semifilters $\mathcal{S}_0$ and $\mathcal{S}_1$ are called *coherent*, if each of them is subcoherent to the other one. A direct verification shows that the subcoherence relation is an equivalence relation. The equivalence class of a semifilter $\mathcal{S}$ will be denoted by $[\mathcal{S}]$. Each family $\mathcal{B}$ of infinite subsets of ω generates a semifilter, namely the smallest semifilter $\langle\mathcal{B}\rangle$ containing $\mathcal{B}$[3]. Given a semifilter $\mathcal{S}$, we denote by $\mathrm{non}[\mathcal{S}]$ the smallest size of a family $\mathcal{B} \subset [\omega]^\omega$ such that $\langle\mathcal{B}\rangle$ is not subcoherent to $\mathcal{S}$. For an ultrafilter

[3]Note that in this appendix the notation $\langle\cdot\rangle$ has a different meaning than in the main part of the paper.

$\mathcal{F}$ we denote by $\mathrm{cov}[\mathcal{F}]$ the minimal size of a family $\mathsf{S} \subset [\mathcal{F}]$ such that $\cap\mathsf{S} = \mathfrak{Fr}$. The increasing sequence of natural numbers whose range coincides with an infinite subset X of ω will be denoted by e_X. An ultrafilter $\mathcal{U}$ is called *rare* if the collection $\{e_F : F \in \mathcal{F}\}$ is dominating. It is clear that every Q-point is rare and the question whether the existence of a rare ultrafilter implies the existence of a Q-point is open.

The proof of the following statement is fairly simple and can be found in the introductory part of [3].

Proposition 3.1. (1) *For every finite-to-finite multifunction Ψ there exists an increasing sequence $\langle n_k : k \in \omega\rangle$ of natural numbers with $n_0 = 0$ such that $\Psi(n) \subset [n_{k-1}, n_{k+2})$ for all $n \in [n_k, n_{k+1})$. Therefore the cofinality of the family of all finite-to-finite multifunctions equals $\mathfrak{d}$ and any family of finite-to-finite multifunctions of size $< \mathfrak{b}$ has an upper bound.*

(2) $\mathrm{cov}[\mathcal{F}] \geq \mathfrak{b}$ *and* $\mathrm{non}[\mathcal{F}] \leq \mathfrak{d}$ *for all ultrafilters $\mathcal{F}$.*

(3) *Let $\mathcal{S}$ be a semifilter and $\mathcal{F}$ be a ultafilter. Then $\mathcal{S}$ is subcoherent (resp. coherent) to $\mathcal{F}$ if and only if there exists a monotone surjection $\psi : \omega \to \omega$ such that $\psi(\mathcal{S}) \subset \psi(\mathcal{F})$ (resp. $\psi(\mathcal{S}) = \psi(\mathcal{F})$).*

(4) *The restriction to ultrafilters of the coherence relation on the set of all semifilters coincides with the near coherence relation on ultrafilters (see the definition after Proposition 1.7.)*

The following statement is a special case of [3, Theorem 9.2.5].

Proposition 3.2. *Suppose that $\mathcal{F}$ is an ultrafilter, $\mathsf{C} \subset [\mathcal{F}]$, $|\mathsf{C}| < \mathrm{cov}[\mathcal{F}]$. Then for every family $\mathcal{B} \subset [\omega]^\omega$ of size less than $\mathrm{cov}[\mathcal{F}]$ there exists a monotone surjection $\psi : \omega \to \omega$ such that $\psi(\mathcal{B}) \subset \psi(\bigcap \mathsf{C})$.*

Proof. For every $B \in \mathcal{B}$ and $\mathcal{C} \in \mathsf{C}$ we denote by $\mathcal{C}_B$ the semifilter consisting of all infinite subsets X of ω such that

$$\exists C \in \mathcal{C}\, \forall a, b \in \omega\ (a, b \in \omega \setminus X \wedge [a, b) \cap C \neq \emptyset \to [a, b) \cap B \neq \emptyset).$$

Given an arbitrary $B \in \mathcal{B}$, consider the finite-to-finite multifunction $\Psi_B : \omega \Rightarrow \omega$ assigning to each $n \in \omega$ the interval $\Psi_B(n) = [n, \min(B \setminus [0, n))]$. Observe that $\Psi_B(\mathcal{C}) \subset \mathcal{C}_B$ for all $\mathcal{C} \in \mathsf{C}$. Indeed, suppose that $a, b \in \omega \setminus \Psi_B(C)$ for some $C \in \mathcal{C}$ and $[a, b) \cap C \neq \emptyset$. The inclusion $a \in \omega \setminus \Psi_B(C)$ means that $a \notin C$ and $a > \min(B \setminus [0, n))$ for all $n < a$ with $n \in C$. Similarly for b. Let $m \in C \cap [a, b)$. It follows from the above that $\min(B \setminus [0, m)) < b$, and hence $[a, b) \cap B \neq \emptyset$. Therefore $C \in \mathcal{C}$ is a witness for $\Psi_B(C)$ being an element of $\mathcal{C}_B$.

Observe that the semifilter $\langle \Psi_B(\mathcal{C})\rangle$ belongs to $[\mathcal{F}]$. Since $|\mathcal{B}|, |\mathsf{C}| < \mathrm{cov}[\mathcal{F}]$, the intersection $\bigcap\{\langle \Psi_B(\mathcal{C})\rangle : B \in \mathcal{B}, \mathcal{C} \in \mathsf{C}\}$ contains a co-infinite set X. Let $\langle n_k : k \in \omega\rangle$ be an increasing enumeration of $\omega \setminus X$ and $\psi^{-1}(k) = [n_k, n_{k+1})$. We claim that $\psi(\mathcal{B}) \subset \psi(\cap\mathsf{C})$. Indeed, let us fix $B \in \mathcal{B}$ and $\mathcal{C} \in \mathsf{C}$. Since $X \in \langle \Psi_B(\mathcal{C})\rangle \subset \mathcal{C}_B$, there exists $C \in C$ such that

$$\forall a, b \in \omega\ (a, b \in \omega \setminus X \wedge [a, b) \cap C \neq \emptyset \to [a, b) \cap B \neq \emptyset),$$

which means that $\psi(C) \subset \psi(B)$, and hence $\psi(B) \in \psi(\mathcal{C})$. Since B and $\mathcal{C}$ are arbitrary elements of $\mathcal{B}$ and C, respectively, our proof is completed. □

Corollary 3.3. *Let $\mathcal{F}$ be an ultrafilter. Then* $\mathrm{non}[\mathcal{F}] \geq \mathrm{cov}[\mathcal{F}]$.

The following proposition is a special case of [3, Theorem 13.8.1].

Proposition 3.4. *Let $\mathcal{F}$ be a rare ultrafilter. Then*

(1) $\mathrm{non}[\mathcal{F}] \le \mathfrak{r}$; *and*
(2) $\mathrm{cov}[\mathcal{F}] \ge \mathfrak{s}$.

Proof. 1. By the inequality $\mathrm{non}[\mathcal{F}] \le \mathfrak{d}$ we may assume $\mathfrak{r} < \mathfrak{d}$. Since $\mathcal{F}$ is rare, so is $\psi(\mathcal{F})$ for any monotone surjection $\psi : \omega \to \omega$. Applying Proposition 3.1(3) we conclude that no semifilter $\mathcal{S} \in [\mathcal{F}]$ can be generated by fewer than $\mathfrak{d}$ sets. Let $\mathcal{U}$ be an ultrafilter with $\mathcal{U} \subset \langle \mathcal{B} \rangle$ for some $\mathcal{B} \subset [\omega]^\omega$ with $|\mathcal{B}| = \mathfrak{r}$. It follows from the above that $\mathcal{U} \notin [\mathcal{F}]$, hence $\mathcal{U}$ is not subcoherent to $\mathcal{F}$, and consequently $\langle B \rangle$ is neither subcoherent to $\mathcal{F}$. This yields $\mathrm{non}[\mathcal{F}] \le |\mathcal{B}| = \mathfrak{r}$.

2. First we show that there exists a subfamily $\mathcal{B} \subset \mathcal{F}$ of size $|\mathcal{B}| = \mathfrak{b}$ without an infinite pseudointersection. Indeed, let $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ be a $\mathfrak{b}$-scale, i.e. an increasing and unbounded with respect to $\le^*$ sequence. Since $\mathcal{F}$ is rare, for every α there exists $F_\alpha \in \mathcal{F}$ such that $e_{F_\alpha} \ge^* f_\alpha$. If $X \in [\omega]^\omega$ is such that $X \subset^* F_\alpha$ and $F_\alpha \not\subset^* X$, then $e_X \ge^* f_\alpha$, and hence the existence of an infinite pseudointersection of $\langle F_\alpha : \alpha < \mathfrak{b} \rangle$ would contradict the unboundedness of $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$.

Thus for every semifilter $\mathcal{S} \in [\mathcal{F}]$ there exists a subfamily $\mathcal{S}' \in [\mathcal{S}]^{\mathfrak{b}}$ without an infinite pseudointersection.

Since $\mathrm{cov}[\mathcal{F}] \ge \mathfrak{b}$, we can assume that $\mathfrak{s} > \mathfrak{b}$. We proceed in the same way as in [3, Theorem 9.2.7(7)]. Set $\lambda = \mathrm{cov}[\mathcal{F}]$ and find a family $\mathsf{S} \subset [\mathcal{F}]$ such that $|\mathsf{S}| = \lambda$ and $\cap \mathsf{S} = \mathfrak{Fr}$. For every $\mathcal{S} \in \mathsf{S}$ find $\mathcal{B}_\mathcal{S} \subset \mathcal{S}$ of size $|\mathcal{B}_\mathcal{S}| = \mathfrak{b}$ such that $\mathcal{B}_\mathcal{S}$ has no infinite pseudointersection. It suffices to prove that $\bigcup\{\mathcal{B}_\mathcal{S} : \mathcal{S} \in \mathsf{S}\}$ is a splitting family. Indeed, let us fix $X \in [\omega]^\omega$. Since $\omega \setminus X \notin \mathfrak{Fr}$, there exists $\mathcal{S} \in \mathsf{S}$ such that $\omega \setminus X \notin \mathcal{S}$, and hence $B \not\subset^* \omega \setminus X$ for all $B \in \mathcal{B}_\mathcal{S}$. In other words, all elements of $\mathcal{B}_\mathcal{S}$ have infinite intersection with X. If none of the elements of $\mathcal{B}_\mathcal{S}$ splits X, we get that $X \subset^* B$ for all $B \in \mathcal{B}_\mathcal{S}$, which contradicts our choice of $\mathcal{B}_\mathcal{S}$. Therefore X is split by some element of $\mathcal{B}_\mathcal{S}$, and hence $\bigcup\{\mathcal{B}_\mathcal{S} : \mathcal{S} \in \mathsf{S}\}$ is a splitting family, which completes our proof. □

References

[1] Arkhangel'skiĭ, A., *Hurewicz spaces, analytic sets and fan tightness of function spaces,* Dokl. Akad. Nauk SSSR **287** (1986), 525–528. (In Russian)

[2] Banakh, T.; Repovš, D.; Zdomskyy, L., *o-Boundedness of free topological groups*, Topology Appl. **157** (2010), 466–481.

[3] Banakh, T.; Zdomskyy, L., *Coherence of Semifilters*, book in preparation. `http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/booksite.html`.

[4] Banakh, T.; Zdomskyy, L., *Selection principles and infinite games on multicovered spaces,* in: Selection Principles and Covering Properties in Topology (Lj. D.R. Koćinac, ed.), Quaderni di Matematica **18**, Dept. Math., Seconda Universita di Napoli, Caserta (2006), 1–51.

[5] Blass, A., *Near coherence of filters. I. Cofinal equivalence of models of arithmetic,* Notre Dame J. Form. Log. **27** (1986), 579–591.

[6] Blass, A.; Mildenberger, H., *On the cofinality of ultrapowers,* J. Symbolic Logic **64** (1999), 727–736.

[7] Blass, A.; Shelah, S.; *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed,* Ann. Pure Appl. Logic **33** (1987), 213–243.

[8] Blass, A.; Shelah, S., *Ultrafilters with small generating sets,* Israel J. Math. **65** (1989), 259–271.

[9] Canjar, R.M, *Countable ultraproducts without CH,* Ann. Pure Appl. Logic **37** (1988), 1–79.

[10] Canjar, R.M., *Cofinalities of countable ultraproducts: the existence theorem,* Notre Dame J. Form. Log. **30** (1989), 539–542.

[11] Guran, I., *Topological groups similar to Lindelöf groups*, Dokl. Akad. Nauk SSSR **256** (1981), 1305–1307. (In Russian)

[12] Laflamme, C.; Zhu, J.-P., *The Rudin-Blass ordering of ultrafilters,* J. Symbolic Logic **63** (1998), 584–592.

[13] Macpherson, H.D.; Neumann, P.M., *Subgroups of infinite symmetric groups*, J. Lond. Math. Soc. (2) **42** (1990), p. 64–84.

[14] Mildenberger, H.; Shelah, S.; Tsaban, B., *Covering the Baire space with meager sets,* Ann. Pure Appl. Logic **140** (2006), 60–71.

[15] Nyikos, P., *Special ultrafilters and cofinal subsets of* ω, preprint, 1984.

[16] Sakai, M., *Two properties of* $C_p(X)$ *weaker than the Fréchet Urysohn property,* Topology Appl. **153** (2006), 2795–2804.

[17] Shelah, S.; Tsaban, B., *Critical cardinalities and additivity properties of combinatorial notions of smallness,* J. Appl. Anal. **9** (2003), 149–162.

[18] Thomas, S., *Groupwise density and the cofinality of the infinite symmetric group*, Arch. Math. Logic **37** (1998), 483–493.

[19] Vaughan, J., *Small uncountable cardinals and topology*, in: Open problems in topology (eds. J. van Mill, G.M. Reed), Elsevier Sci. Publ., Amsterdam 1990, 197–216.

[20] Zdomskyy, L., *A semifilter approach to selection principles,* Comment. Math. Univ. Carolin. **46** (2005), 525–539.

[21] Zdomskyy, L., *o-Boundedness of free objects over a Tychonoff space*, Mat. Stud. **25** (2006), 10–28.

Department of Mathematics, Ivan Franko National University of Lviv, Ukraine; and, Instytut Matematyki, Uniwersytet Humanistyczno-Przyrodniczy Jana Kochanowskiego w Kielcach, Poland.

E-mail address: tbanakh@yahoo.com

URL: http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/bancv.html

Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P. O. Box 2964, Ljubljana, Slovenija 1001.

E-mail address: dusan.repovs@guest.arnes.si

URL: http://www.fmf.uni-lj.si/~repovs/index.htm

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Strasse 25, A-1090 Wien, Austria.

E-mail address: lzdomsky@gmail.com

URL: http://www.logic.univie.ac.at/~lzdomsky/