A NON-WEAKLY AMENABLE BANACH ALGEBRA WHOSE BIDUAL IS WEAKLY AMENABLE

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Abstract. We answer, in the negative, the question: *Does a Banach algebra inherit weak amenability from its bidual?*, which was open since 1996.

A Banach algebra A is said to be $n-$weakly amenable, $n \in \mathbb{N}$, if every (bounded) derivation from A into the $n^{\text{th}}-$dual Banach $A-$module $A^{(n)}$ is inner. This notion was first introduced and intensively studied in [3]. Trivially, 1-weak amenability is nothing else than weak amenability, which was first introduced in [1] for the commutative case and then in [7] for the general case.

We equip the bidual A^{**} of A with the first Arens product $\Box$ defined as follows. For $m, n \in A^{**}, a, b \in A, f \in A^*$;

$$\langle m\Box n, f\rangle = \langle m, n \cdot f\rangle \text{ where } \langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle \text{ and } \langle f \cdot a, b\rangle = \langle f, ab\rangle.$$

Equipped with this multiplication, A^{**} is a Banach algebra. For all these notions and facts, we refer the reader to [8].

The study of weak amenability of the bidual A^{**} of a Banach algebra A was initiated in [6]. In [2], S. Barootkoob and the author have investigated the question: *Does A inherit $n-$weak amenability from A^{**}?* And they showed that the answer is positive for an arbitrary Banach algebra, in the case where $n \geq 2$. To the best of my knowledge, however, no example was yet known whether this fails if one considers the case $n = 1$ instead. Indeed, in the case $n = 1$, the question has a completely different feature from that of $n \geq 2$. More explicitly, in this case we have actually the long-standing open problem of whether weak amenability of A^{**} forces A to be weakly amenable, which first posed in [6]. Some positive results, under certain conditions, can be found in [6, 4, 5, 2].

Here we answer the question in the negative. For this we are looking for a counterexample amongst the class of module extension Banach algebras, which has already known as a source of (counter-)examples for various purposes in functional analysis. For instance, Y. Zhang [9] has

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served this type of Banach algebras to provide several counterexamples in the theory of $n-$weak amenability.

Let $\mathfrak{A}$ be a Banach algebra and let X_0 be a nonzero Banach $\mathfrak{A}-$module with trivial right module action. Then the ℓ_1-direct sum $\mathfrak{A}\oplus X_0$ is a Banach algebra equipped with the multiplication

$$(a,x)(b,y)=(ab,ay)\qquad (a,b\in\mathfrak{A},\ x,y\in X_0).$$

One can simply identify $(\mathfrak{A}\oplus X_0)^*$ with the $\ell_\infty-$direct sum $\mathfrak{A}^*\oplus {X_0}^*$ and similarly $(\mathfrak{A}\oplus X_0)^{**}$ with the ℓ_1-direct sum $\mathfrak{A}^{**}\oplus X_0^{**}$. Moreover a direct verification reveals that

$$(m,F)\Box(n,G)=(m\Box n,mG)\qquad (m,n\in\mathfrak{A}^{**},\ F,G\in X_0^{**});$$

where mG is the usual left $(\mathfrak{A}^{**},\Box)-$module action of X_0^{**}.

The weak amenability of $\mathfrak{A}\oplus X_0$ is investigated in [9, Corollary 6.2]. Indeed, it is shown that: If $\mathfrak{A}$ is weakly amenable with a bounded approximate identity then $\mathfrak{A}\oplus X_0$ is weakly amenable if and only $\mathfrak{A}X_0$ is dense in X_0. We use this criterion to provide our main result.

Theorem 1. *Let $\mathfrak{A}$ be an infinite dimensional C^*-algebra which is a right ideal in $\mathfrak{A}^{**}$. Then the Banach algebra $A=\mathfrak{A}\oplus(\mathfrak{A}^{**})_0$ is not weakly amenable while A^{**} is weakly amenable.*

Proof. As $\mathfrak{A}(\mathfrak{A}^{**})_0$ is not dense in $(\mathfrak{A}^{**})_0$ (indeed, $\mathfrak{A}(\mathfrak{A}^{**})_0\subseteq\mathfrak{A}$ and $\mathfrak{A}$ is infinite-dimensional) and $\mathfrak{A}$ is weakly amenable with a bounded approximate identity, [9, Corollary 6.2] implies that $A=\mathfrak{A}\oplus(\mathfrak{A}^{**})_0$ is not weakly amenable. But the C^*-algebra $\mathfrak{A}^{**}$ is weakly amenable and $\mathfrak{A}^{**}{(\mathfrak{A}^{**})_0}^{**}={(\mathfrak{A}^{**})_0}^{**}$ (note that $\mathfrak{A}^{**}$ has an identity and the $\mathfrak{A}^{**}-$module action on ${(\mathfrak{A}^{**})_0}^{**}$ coincides with the first Arens product induced from $(\mathfrak{A}^{**})^{**}$). Therefore $A^{**}=\mathfrak{A}^{**}\oplus{(\mathfrak{A}^{**})_0}^{**}$ is weakly amenable again by [9, Corollary 6.2]. □

A very familiar candidate satisfying circumstances of the above theorem is $\mathfrak{A}=\mathcal{K}(H)$ (=the C^*-algebra of all compact operators on an infinite-dimensional Hilbert space H) whose bidual $\mathcal{K}(H)^{**}$ (as a Banach algebra) can be identified with $\mathcal{B}(H)$ (see [8, p. 103]). Therefore $A=\mathcal{K}(H)\oplus\mathcal{B}(H)_0$ is not weakly amenable while A^{**} is weakly amenable.

One can even have a commutative candidate, namely $\mathfrak{A}=c_0$, satisfying the above theorem. Therefore $A=c_0\oplus{\ell^\infty}_0$ is not weakly amenable while A^{**} is weakly amenable.

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