Multivariate Rogers-Szegö polynomials and flags in finite vector spaces

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Abstract

We give a recursion for the multivariate Rogers-Szegö polynomials, along with another recursive functional equation, and apply them to compute special values. We also consider the sum of all q-multinomial coefficients of some fixed degree and length, and give a recursion for this sum which follows from the recursion of the multivariate Rogers-Szegö polynomials, and generalizes the recursion for the Galois numbers. The sum of all q-multinomial coefficients of degree n and length m is the number of flags of length $m-1$ of subspaces of an n-dimensional vector space over a field with q elements. We give a combinatorial proof of the recursion for this sum of q-multinomial coefficients in terms of finite vector spaces.

1 Introduction

For a parameter $q \neq 1$, and a positive integer n, let $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$, and $(q)_0 = 1$. For non-negative integers n and k, with $n \geq k$, the *q-binomial coefficient* or *Gaussian polynomial*, denoted $\binom{n}{k}_q$, is defined as $\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$.

The *Rogers-Szegö polynomial* in a single variable, denoted $H_n(t)$, is defined as

$$H_n(t) = \sum_{k=0}^{n} \binom{n}{k}_q t^k.$$

The Rogers-Szegö polynomials first appeared in papers of Rogers [12, 13] which led up to the famous Rogers-Ramanujan identities, and later were independently studied by Szegö [15]. They are important in combinatorial number theory ([1, Ex. 3.3–3.9] and [4, Sec. 20]), symmetric function theory [16], and are key examples of orthogonal polynomials [2]. They also have applications in mathematical physics [8, 9].

The Rogers-Szegö polynomials satisfy the recursion (see [1, p. 49])

$$H_{n+1}(t) = (1+t)H_n(t) + t(q^n - 1)H_{n-1}(t). \tag{1.1}$$

Letting $t = 1$, we have $H_n(1) = \sum_{k=0}^{n} \binom{n}{k}_q$, which, when q is the power of a prime, is the total number of subspaces of an n-dimensional vector space over a field with q elements. The numbers $G_n = H_n(1)$ are the *Galois numbers*, and from (1.1), satisfy the recursion $G_{n+1} = 2G_n + (q^n - 1)G_{n-1}$. The Galois numbers were studied from the point of view of finite vector spaces by Goldman and Rota [5], and have been studied extensively, for example, in [11, 6].

For non-negative integers $k_1, k_2, \ldots, k_m$ such that $k_1 + \cdots + k_m = n$, we define the q-multinomial coefficient of length m as

$$\binom{n}{k_1, k_2, \ldots, k_m}_q = \frac{(q)_n}{(q)_{k_1}(q)_{k_2} \cdots (q)_{k_m}},$$

so that $\binom{n}{k}_q = \binom{n}{k, n-k}_q$. Define the homogeneous Rogers-Szegö polynomial in m variables for $m \geq 2$, denoted $\tilde{H}_n(t_1, t_2, \ldots, t_m)$, by

$$\tilde{H}_n(t_1, t_2, \ldots, t_m) = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m}_q t_1^{k_1} \cdots t_m^{k_m},$$

and define the Rogers-Szegö polynomial in $m - 1$ variables, denoted $H_n(t_1, \ldots, t_{m-1})$, by

$$H_n(t_1, \ldots t_{m-1}) = \tilde{H}(t_1, \ldots, t_{m-1}, 1).$$

The homogeneous multivariate Rogers-Szegö polynomials were first defined by Rogers [12] in terms of their generating function, and several of their properties are given by Fine [4, Section 21]. The definition of the multivariate Rogers-Szegö polynomial H_n is given by Andrews in [1, Chap. 3, Ex. 17], along with a generating function, although there is little other study of these polynomials elsewhere in the literature (however, there is a non-symmetric version of a bivariate Rogers-Szegö polynomial [3]). In Section 2, we give a recursion for the multivariate Rogers-Szegö polynomials which generalizes (1.1). The result, given in Theorem 2.1 below, follows quickly from the generating function for the multivariate Rogers-Szegö polynomials, although seems not to have been noted before. We also give a few other properties of the multivariate Rogers-Szegö polynomials in Section 2 which complement those given in [4, Section 21].

Finally, in Section 3, we concentrate on the value $H_n(1, 1, \ldots, 1)$, of the Rogers-Szegö polynomial $H_n(t_1, \ldots, t_{m-1})$ when we let $t_1 = \cdots = t_{m-1} = 1$. This is the sum of all q-multinomial coefficients of length m, which we denote by $G_n^{(m)}$, so

$$G_n^{(m)} = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m}_q.$$

These generalize the Galois numbers $G_n = G_n^{(2)}$, and in particular, $G_n^{(m)}$ is the total number of flags of subspaces of length m of an n-dimensional vector space over a field with q elements when q is the power of a prime. The main task of Section 3 is to study the numbers $G_n^{(m)}$ in the context of finite vector spaces, independent of the Rogers-Szegö polynomials, in the spirit of the study of Goldman and Rota. A recursion for the numbers $G_n^{(m)}$ is immediately obtained in Corollary 3.1 from the recursion of the multivariate Rogers-Szegö polynomials, for which we provide a combinatorial proof in terms of finite vector spaces. We prove the recursion for the generalized Galois numbers by obtaining a recursive formula for q-multinomial coefficients, which itself implies the recursion for the multivariate Rogers-Szegö polynomials.

2   Rogers-Szegö polynomials

For any $a, r \neq 1$, and $n \geq 1$, we define $(a; r)_n$ by

$$(a; r)_n = (1 - a)(1 - ar) \cdots (1 - ar^{n-1}),$$

and $(a; r)_0 = 1$. Define $(a; r)_\infty$ by the formal infinite product

$$(a; r)_\infty = \prod_{i=0}^{\infty} (1 - ar^i).$$

For our fixed parameter $q \neq 1$, and any $a \neq 1$, define $(a)_n = (a; q)_n$ and $(a)_\infty = (a; q)_\infty$, so $(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$ for $n \geq 1$ as in the introduction.

Then we have [1, Cor. 2.2] the following formal identity, originally due to Euler:

$$\sum_{n=0}^{\infty} \frac{x^n}{(r; r)_n} = (x; r)_\infty^{-1}. \tag{2.1}$$

We may use (2.1) to give a generating function for the multivariate Rogers-Szegö polynomials. The following is stated in [1, Ex. 3.17], but we include the proof here for the sake of self-containment.

Lemma 2.1. *The multivariate Rogers-Szegö polynomials have the following generating function:*

$$\sum_{n=0}^{\infty} \frac{H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n = (t_1 x)_\infty^{-1} \cdots (t_{m-1} x)_\infty^{-1} (x)_\infty^{-1}.$$

Proof. By (2.1), we have

$$
\begin{aligned}
(t_1 x)_\infty^{-1} \cdots (t_{m-1} x)_\infty^{-1} (x)_\infty^{-1} &= \left(\sum_{k_1=0}^{\infty} \frac{t_1^{k_1} x^{k_1}}{(q)_{k_1}} \right) \cdots \left(\sum_{k_{m-1}=0}^{\infty} \frac{t_{m-1}^{k_{m-1}} x^{k_{m-1}}}{(q)_{k_{m-1}}} \right) \left(\sum_{k_m=0}^{\infty} \frac{x^{k_m}}{(q)_{k_m}} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k_1 + \cdots + k_m = n} \frac{t_1^{k_1} \cdots t_{m-1}^{k_{m-1}}}{(q)_{k_1} \cdots (q)_{k_m}} \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m}_q t_1^{k_1} \cdots t_{m-1}^{k_{m-1}} \right) \frac{x^n}{(q)_n} \\
&= \sum_{n=0}^{\infty} \frac{H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n,
\end{aligned}
$$

as claimed. $\square$

For any finite set of variables X, we let $\mathbf{e}_i(X)$ denote the ith elementary symmetric polynomial in the variables X. We can now give the recursion of the multivariate Rogers-Szegö polynomials which generalizes (1.1).

Theorem 2.1. *The Rogers-Szegö polynomials in $m - 1$ variables satisfy the following recursion:*

$$H_{n+1}(t_1, \ldots, t_{m-1}) = \sum_{i=0}^{m-1} \mathbf{e}_{i+1}(t_1, \ldots, t_{m-1}, 1)(-1)^i \frac{(q)_n}{(q)_{n-i}} H_{n-i}(t_1, \ldots, t_{m-1}).$$

Proof. Let

$$F(x, t_1, \ldots, t_{m-1}) = \sum_{n=0}^{\infty} \frac{H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n = (t_1 x)_\infty^{-1} \cdots (t_{m-1} x)_\infty^{-1} (x)_\infty^{-1},$$

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by Lemma 2.1. Since $(1 - t_i x)(t_i x)_\infty^{-1} = (t_i xq)_\infty^{-1}$ and $(1-x)(x)_\infty^{-1} = (xq)_\infty^{-1}$, we have

$$(1 - t_1 x) \cdots (1 - t_{m-1} x)(1 - x) F(x, t_1, \ldots, t_{m-1}) = (t_1 xq)_\infty^{-1} \cdots (t_{m-1} xq)_\infty^{-1} (xq)_\infty^{-1}$$
$$= F(xq, t_1, \ldots, t_{m-1}). \quad (2.2)$$

By the definition of the elementary symmetric polynomials, we have

$$(1 - t_1 x) \cdots (1 - t_{m-1} x)(1 - x) = \sum_{i=0}^{m} (-1)^i \mathbf{e}_i(t_1, \ldots, t_{m-1}, 1) x^i.$$

From this and (2.2), we have

$$\left(\sum_{i=0}^{m} (-1)^i \mathbf{e}_i(t_1, \ldots, t_{m-1}, 1) x^i \right) \sum_{n=0}^{\infty} \frac{H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n = \sum_{n=0}^{\infty} \frac{q^n H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n,$$

which may be re-written as

$$\sum_{n=0}^{\infty} \frac{(1 - q^n) H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n = \left(\sum_{i=0}^{m-1} (-1)^i \mathbf{e}_{i+1}(t_1, \ldots, t_{m-1}, 1) x^{i+1} \right) \sum_{n=0}^{\infty} \frac{H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n.$$

Comparing the coefficients of x^{n+1} in both sides of the above expression, we obtain

$$\frac{H_{n+1}(t_1, \ldots, t_{m-1})}{(q)_n} = \sum_{i=0}^{m-1} \mathbf{e}_{i+1}(t_1, \ldots, t_{m-1}, 1) \frac{(-1)^i}{(q)_{n-i}} H_{n-i}(t_1, \ldots, t_{m-1}),$$

which yields the desired result. $\qquad \square$

Note that we have $(-1)^i (q)_n / (q)_{n-i} = (q^n - 1)(q^{n-1} - 1) \cdots (q^{n-i+1} - 1)$. For example, applying Theorem 2.1 to the Rogers-Szegö polynomials in two variables, we obtain

$$H_{n+1}(t_1, t_2) = (1 + t_1 + t_2) H_n(t_1, t_2) + (t_1 t_2 + t_1 + t_2)(q^n - 1) H_{n-1}(t_1, t_2)$$
$$+ t_1 t_2 (q^n - 1)(q^{n-1} - 1) H_{n-2}(t_1, t_2).$$

When the Rogers-Szegö polynomial $H_n(t)$ in a single variable is evaluated at $t = -1$, we get the following identity for the alternating sum of q-binomial coefficients originally due to Gauss:

$$H_n(-1) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k = \begin{cases} \frac{(q)_n}{(q^2; q^2)_{n/2}} = \prod_{j<n, j \text{ odd}} (1 - q^j) & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

As pointed out in [4, Sec. 21], this identity may be generalized by evaluating the multivariate Rogers-Szegö polynomials at roots of unity. That is, if $\omega = e^{2\pi i/m}$ is a primitive m-th root of unity, where $m \geq 2$, and let $n \geq 0$, then

$$H_n(\omega, \omega^2, \ldots, \omega^{m-1}) = \begin{cases} \frac{(q)_n}{(q^m; q^m)_{n/m}} = \prod_{j<n, m \nmid j} (1 - q^j) & \text{if } m | n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

This is calculated in [4] by applying the generating function in Lemma 2.1. We note that we may also compute it quickly from Theorem 2.1 as follows. We have $\mathbf{e}_i(\omega, \ldots, \omega^{m-1}, 1) = 0$ for $1 \leq i \leq m - 1$, while $\mathbf{e}_m(\omega, \ldots, \omega^{m-1}, 1) = (-1)^{m+1}$, since these roots of unity are the roots of $x^m - 1$. We have $H_0(t_1, \ldots, t_{m-1}) = 1$, and for $0 < n \leq m - 1$, we have $H_n(t_1, \ldots, t_{m-1})$

is a symmetric polynomial in $t_1, \ldots, t_{m-1}, 1$, of degree $n < m$, with zero constant term. We may thus write $H_n(t_1, \ldots, t_{m-1})$ as a polynomial in $\mathbf{e}_i(t_1, \ldots, t_{m-1}, 1)$, with $0 < i < n$. Now, $H_n(\omega, \ldots, \omega^{m-1})$ can be written as a polynomial in $\mathbf{e}_i(\omega, \ldots, \omega^{m-1}, 1)$, for $1 \leq i \leq m-1$, which are all 0. It follows that $H_n(\omega, \ldots, \omega^{m-1}) = 0$ for these n. By Theorem 2.1 and the values of the elementary symmetric polynomials, if $n \geq m$ then

$$H_n(\omega, \ldots, \omega^{m-1}) = \sum_{i=0}^{m-1} \mathbf{e}_{i+1}(\omega, \ldots, \omega^{m-1}, 1)(-1)^i \frac{(q)_{n-1}}{(q)_{n-1-i}} H_{n-1-i}(\omega, \ldots, \omega^{m-1})$$
$$= (1 - q^{n-1})(1 - q^{n-2}) \cdots (1 - q^{n-m+1}) H_{n-m}(\omega, \ldots, \omega^{m-1}).$$

The values (2.4) now follow by induction.

Another value of the Rogers-Szegö polynomial of a single variable is

$$H_n(q^{1/2}) = \frac{(q)_n}{(q^{1/2}; q^{1/2})_n} = (q^{1/2}; -q^{1/2})_n = \prod_{j=1}^{n}(1 + q^{j/2}).$$

This is generalized in [4] with the value

$$H_n(q^{1/m}, q^{2/m}, \ldots, q^{(m-1)/m}) = \frac{(q)_n}{(q^{1/m}; q^{1/m})_n} = \prod_{j=1}^{n}(1 + q^{j/m} + \cdots + q^{j(m-1)/m}). \qquad (2.5)$$

Fine also gives a generalization of both (2.4) and (2.5), and applies it to obtain a bi-basic identity [4, 21.4].

The Rogers-Szegö polynomial $H_n(t)$ also takes the value

$$H_n(-q) = \frac{(q)_n}{(q^2; q^2)_{\lfloor n/2 \rfloor}} = \prod_{j \leq n, j \text{ odd}}(1 - q^j), \qquad (2.6)$$

which is applied in finding identities involving Hall-Littlewood functions in [16], for example. A generalization of (2.6) for the multivariate Rogers-Szegö polynomials is not covered above, and so we obtain one now. Let $\omega = e^{2\pi i/m}$ be a primitive m-th root of unity, where $m \geq 2$, and let $n \geq 0$. Then

$$H_n(\omega q, \omega^2 q, \ldots, \omega^{m-1} q) = \frac{(q)_n}{(q^m; q^m)_{\lfloor n/m \rfloor}} = \prod_{j \leq n, m \nmid j}(1 - q^j), \qquad (2.7)$$

which we calculate as follows. By Lemma 2.1 and (2.1), we have

$$\sum_{n=0}^{\infty} \frac{H_n(\omega q, \ldots, \omega^{m-1} q)}{(q)_n} x^n = \frac{1}{(x)_\infty (x\omega q)_\infty \cdots (x\omega^{m-1} q)_\infty}$$

$$= \frac{1}{1-x} \prod_{j=1}^{\infty} \prod_{l=0}^{m-1} \frac{1}{1 - x\omega^l q^j} = \frac{1}{1-x} \prod_{j=1}^{\infty} \frac{1}{1 - x^m q^{mj}}$$

$$= \frac{1 + x + \cdots + x^{m-1}}{(1 - x^m)(1 - x^m q^m)(1 - x^m q^{2m}) \cdots} = \frac{1 + x + \cdots + x^{m-1}}{(x^m; q^m)_\infty}$$

$$= (1 + x + \cdots + x^{m-1}) \sum_{k=0}^{\infty} \frac{1}{(q^m; q^m)_k} x^{mk}.$$

Comparing the coefficients of x^n, we obtain (2.7).

The value (2.6) of $H_n(-q)$ could also be computed by using (2.3) along with the functional equation [4, 20.64b]

$$H_n(tq) = H_n(t) - t(1-q^n)H_{n-1}(t). \qquad (2.8)$$

The next result, which generalizes (2.8) to multivariate Rogers-Szegö polynomials, has a very similar form and proof to Theorem 2.1.

Theorem 2.2. *Let $m \geq 2$, and let $J \subseteq \{1, \ldots, m-1\}$, where $|J| \neq 0$. For $1 \leq i \leq m-1$, define $s_i = t_i q$ if $i \in J$, and $s_i = t_i$ otherwise. Let $\mathbf{e}_i(t_J)$ be the ith elementary symmetric polynomial in the set of variables $t_J = \{t_j \mid j \in J\}$. Then for $n \geq |J|$,*

$$H_n(s_1, \ldots, s_{m-1}) = \sum_{i=0}^{|J|} \mathbf{e}_i(t_J)(-1)^i \frac{(q)_n}{(q)_{n-i}} H_{n-i}(t_1, \ldots, t_{m-1}).$$

Proof. Let $F(x, t_1, \ldots, t_{m-1})$ be the generating function for $H_n(t_1, \ldots, t_{m-1})$ as in the proof of Theorem 2.1. Then we have

$$\left(\prod_{j \in J}(1 - t_j x) \right) F(x, t_1, \ldots, t_{m-1}) = \left(\prod_{j \in J}(t_j q x)_\infty^{-1} \right) \left(\prod_{i \notin J, 1 \leq i \leq m-1}(t_i x)_\infty^{-1} \right) (x)_\infty^{-1}$$

$$= F(x, s_1, \ldots, s_{m-1}). \qquad (2.9)$$

Since $\prod_{j \in J}(1 - t_j x) = \sum_{i=0}^{|J|}(-1)^i \mathbf{e}_i(t_J)x^i$, it follows from (2.9) that we have

$$\sum_{n=0}^\infty \frac{H_n(s_1, \ldots, s_{m-1})}{(q)_n} x^n = \left(\sum_{i=0}^{|J|}(-1)^i \mathbf{e}_i(t_J)x^i \right) \sum_{n=0}^\infty \frac{H_n(t_1, \ldots, t_{m-1})}{(q)_n} x^n.$$

Comparing the coefficient of x^n in both sides of the above gives the result. □

Now we may compute the value (2.7) by applying Theorem 2.2 in the following way. Note that since $\omega, \omega^2, \ldots, \omega^{m-1}$, are the roots of $x^m + x^{m-1} + \cdots + 1$, then $\mathbf{e}_i(\omega, \ldots, \omega^{m-1}) = (-1)^i$ for $0 \leq i \leq m-1$. If we are able to compute the values $H_i(\omega q, \ldots, \omega^{m-1}q)$ for $i < m$, then we use Theorem 2.2 with $J = \{1, \ldots, m-1\}$ and $t_i = \omega^i$ to obtain, when $n \geq m$,

$$H_n(\omega q, \ldots, \omega^{m-1}q) = \sum_{i=1}^{m-1} \frac{(q)_n}{(q)_{n-i}} H_{n-i}(\omega, \ldots, \omega^{m-1}).$$

The values (2.7) then follow for $n \geq m$ when plugging in the values (2.4). We can compute $H_n(\omega q, \ldots, \omega^{m-1}q)$ for $n < m$ using Theorem 2.2 as well. For $n = 1$, we begin by taking $J = \{1\}$ and $t_i = \omega^i$ to obtain $H_1(\omega q, \omega^2, \ldots, \omega^{m-1}) = \omega(q-1)$. Then take $J = \{2\}$, $t_1 = \omega q$, and $t_i = \omega^i$ for $i > 1$. Applying Theorem 2.2 then gives $H_1(\omega q, \omega^2 q, \omega^3, \ldots, \omega^{m-1}) = (\omega + \omega^2)(q - 1)$. Continuing in this way, we get

$$H_1(\omega q, \omega^2 q, \ldots, \omega^{m-1}q) = (\omega + \cdots + \omega^{m-1})(q - 1) = (1 - q).$$

The values for $1 < n < m$ may be computed similarly.

3 Flags in finite vector spaces

Now let q be the power of a prime, and let $\mathbb{F}_q$ denote a finite field with q elements. If V is an n-dimensional vector space over $\mathbb{F}_q$, then the q-binomial coefficient $\binom{n}{k}_q$ is the number of k-dimensional subspaces of V (see [7, Thm. 7.1] or [14, Prop. 1.3.18]). When we evaluate the Rogers-Szegö polynomial at $t = 1$, we obtain

$$H_n(1) = \sum_{k=0}^{n} \binom{n}{k}_q,$$

which is the total number of subspaces of an n-dimensional vector space over $\mathbb{F}_q$. We define the Galois numbers as $G_n = H_n(1)$. As mentioned in the introduction, the recursion for the Rogers-Szegö polynomials (1.1) gives the following recursion for the Galois numbers, which was studied by Goldman and Rota [5]:

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1}, \quad G_0 = 1, G_1 = 2. \tag{3.1}$$

The recursion (3.1) was proved bijectively by counting subspaces of finite vector spaces by Nijenhuis, Solow, and Wolf [11]. The proof in [11] is obtained by proving the following result bijectively, from which (3.1) follows.

Lemma 3.1. *For integers $n \geq k \geq 1$, we have*

$$\binom{n+1}{k}_q = \binom{n}{k}_q + \binom{n}{k-1}_q + (q^n - 1)\binom{n-1}{k-1}_q.$$

We now consider the meaning of a q-multinomial coefficient in terms of vector spaces over $\mathbb{F}_q$. It follows from the definition of a q-multinomial coefficient and the fact that $\binom{n}{k}_q = \binom{n}{n-k}_q$ that we have

$$\binom{n}{k_1, k_2, \ldots, k_m}_q = \binom{n}{k_1}_q \binom{n-k_1}{k_2}_q \cdots \binom{n - k_1 - \cdots - k_{m-2}}{k_{m-1}}_q$$
$$= \binom{n}{n-k_1}_q \binom{n-k_1}{n-k_1-k_2}_q \cdots \binom{n - k_1 - \cdots - k_{m-2}}{n - k_1 - \cdots - k_{m-2} - k_{m-1}}_q.$$

So, if V is an n-dimensional vector space over $\mathbb{F}_q$, the q-multinomial coefficient $\binom{n}{k_1, \ldots, k_m}_q$ is equal to the number of ways to choose an $(n - k_1)$-dimensional subspace W_1 of V, an $(n - k_1 - k_2)$-dimensional subspace W_2 of W_1, and so on, until finally we choose an $(n - k_1 - \cdots - k_{m-1})$-dimensional subspace W_{m-1} of some $(n - k_1 - \cdots - k_{m-2})$-dimensional subspace W_{m-2} (see also [10, Sec. 1.5]). That is,

$$W_{m-1} \subseteq W_{m-2} \subseteq \cdots \subseteq W_2 \subseteq W_1$$

is a *flag* of subspaces of V of length $m - 1$, where $\dim W_i = n - \sum_{j=1}^{i} k_j$.

If we evaluate the Rogers-Szegö polynomial in $m - 1$ variables at $t_1 = t_2 = \cdots = t_{m-1} = 1$, we obtain

$$H_n(1, 1, \ldots, 1) = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m}_q,$$

which, by the discussion above, counts the total number of flags of subspaces of length $m - 1$ in an n-dimensional $\mathbb{F}_q$-vector space. We denote this quantity by $G_n^{(m)}$, so that the Galois number $G_n = G_n^{(2)}$. We may apply Theorem 2.1 to obtain a recursion for the numbers $G_n^{(m)}$, generalizing the recursion in (3.1), by noticing that the number of terms in the elementary symmetric polynomial $\mathbf{e}_{i+1}(t_1, \ldots, t_{m-1}, 1)$ is $\binom{m}{i+1}$.

Corollary 3.1. *The numbers $G_n^{(m)}$ satisfy the following recursion, for $n \geq m - 1$:*

$$G_{n+1}^{(m)} = \sum_{i=0}^{m-1} \binom{m}{i+1}(-1)^i \frac{(q)_n}{(q)_{n-i}} G_{n-i}^{(m)}.$$

In this section, we prove Corollary 3.1 combinatorially in terms of finite vector spaces, by proving an analog of Lemma 3.1.

We need some notation. Let $\underline{k}$ denote the m-tuple $(k_1, \ldots, k_m)$, and write the corresponding q-multinomial coefficient as

$$\binom{n}{k_1, \ldots, k_m}_q = \binom{n}{\underline{k}}_q.$$

For a subset $J \subseteq \{1, \ldots, m\}$, let $\underline{e}_J$ denote the m-tuple $(e_1, \ldots, e_m)$, where

$$e_i = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

For example, if $m = 3$, $J = \{1, 3\}$, and $\underline{k} = (k_1, k_2, k_3)$, then

$$\binom{n}{\underline{k} - \underline{e}_J}_q = \binom{n}{k_1 - 1, k_2, k_3 - 1}_q.$$

The following is our generalization of Lemma 3.1.

Lemma 3.2. *For $m \geq 2$, and any $k_1, \ldots, k_m > 0$ such that $k_1 + \cdots + k_m = n + 1$, we have*

$$\binom{n+1}{k_1, \ldots, k_m}_q = \sum_{J \subseteq \{1, \ldots, m\}, |J| > 0} (-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k} - \underline{e}_J}_q$$

Before proving Lemma 3.2, we explain why it implies Corollary 3.1. First note that we may get a version of Lemma 3.2 which allows any of the $k_i = 0$ as follows. If we want l of the k_i's to be 0, we start with applying Lemma 3.2 to a q-multinomial coefficient of length $m - l$, using the $m - l$ nonzero k_i's, and note that the equation in Lemma 3.2 is not affected by inserting 0's into the appropriate positions of all the q-multinomial coefficients in both sides. That is, if some $k_i = 0$, we may still apply Lemma 3.2, while ignoring these k_i, or equivalently, we may apply Lemma 3.2 to the q-multinomial coefficient obtained by removing the k_i's which are 0, and re-inserting these 0's in all q-multinomial coefficients in the sum the end.

Now consider the sum of all q-multinomial coefficients of the form $\binom{n+1}{k_1, \ldots, k_m}_q$, while applying the more general version of Lemma 3.2 just discussed. For any i, $0 \leq i \leq m - 1$, each term $\binom{n+1}{k_1, \ldots, k_m}_q$ in $G_{n+1}^{(m)}$ may be obtained by adding 1 to $i + 1$ of the l_i's in terms in $G_{n-i}^{(m)}$ of the form $\binom{n-i}{l_1, \ldots, l_m}_q$ in exactly $\binom{m}{i+1}$ ways. By Lemma 3.2, these terms contribute exactly what we need to conclude Corollary 3.1. By a similar argument, we may see that in fact the recursion for the multinomial Rogers-Szegö polynomials in Theorem 2.1 also follows from Lemma 3.2.

Proof of Lemma 3.2. We will prove this by induction on m, where the base case $m = 2$ is given by Lemma 3.1. Fix V to be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$. We know $\binom{n+1}{k_1, \ldots, k_m}_q$ is the number of flags of subspaces of V, $W_{m-1} \subset \cdots \subset W_2 \subset W_1$, where $\dim W_i = n+1-\sum_{j=1}^{i} k_j$. We must show that the right-hand side of the claimed identity in Lemma 3.2 also counts these

flags. We have, by choosing first the subspace W_1 and then the rest of the flag, and applying Lemma 3.1,

$$\binom{n+1}{k_1,\ldots,k_m}_q = \binom{n+1}{n+1-k_1}_q \binom{n+1-k_1}{k_2,\ldots,k_m}_q$$

$$= \left(\binom{n}{n+1-k_1}_q + \binom{n}{n-k_1}_q + (q^n-1)\binom{n-1}{n-k_1}_q\right)\binom{n+1-k_1}{k_2,\ldots,k_m}_q. \quad (3.2)$$

We have $\binom{n}{n+1-k_1}_q \binom{n+1-k_1}{k_2,\ldots,k_m}_q = \binom{n}{k_1-1,k_2,\ldots,k_m}_q$, which is the term corresponding to the subset $J = \{1\} \subset \{1,\ldots,m\}$. This may be thought of as the total number of ways of choosing our flag so that W_1 is contained in some fixed n-dimensional subspace of V. By our induction hypothesis, the number of remaining flags is given by

$$\left(\binom{n}{n-k_1}_q + (q^n-1)\binom{n-1}{n-k_1}_q\right) \sum_{I\subseteq\{1,\ldots,m-1\},|I|>0} (-1)^{|I|-1}\frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}}\binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q, \quad (3.3)$$

where $\underline{k}' = (k_2,\ldots,k_m)$. We have

$$\binom{n}{n-k_1}_q \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} = \frac{(q)_n}{(q)_{n-|I|+1}}\binom{n+1-|I|}{n-k_1-|I|+1}_q,$$

and

$$(q^n-1)\binom{n-1}{n-k_1}_q \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} = (-1)\frac{(q)_n}{(q)_{n-|I|}}\binom{n-|I|}{n-k_1-|I|+1}_q.$$

Given $I \subset \{1,\ldots,m-1\}$, $|I| > 0$, write $I+1 = \{i+1 \mid i \in I\} \subseteq \{2,\ldots,m\}$. If we let $\underline{k} = (k_1,\ldots,k_m)$, we now have

$$\binom{n}{n-k_1}_q (-1)^{|I|-1}\frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}}\binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q = (-1)^{|J|-1}\frac{(q)_n}{(q)_{n-|J|+1}}\binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q, \quad (3.4)$$

where $J = I+1$, $|J| = |I|$, and

$$(q^n-1)\binom{n-1}{n-k_1}_q (-1)^{|I|-1}\frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}}\binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q$$

$$= (-1)^{|J|-1}\frac{(q)_n}{(q)_{n-|J|+1}}\binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q, \quad (3.5)$$

where $J = \{1\} \cup (I+1)$, $|J| = |I|+1$. As I ranges over nonempty subsets of $\{1,\ldots,m-1\}$, $I+1$ and $\{1\} \cup (I+1)$ range over all nonempty subsets of $\{1,\ldots,m\}$ other than $\{1\}$. Finally, we substitute (3.4) and (3.5) into (3.3), and we see that the sum of all of these terms, along with $\binom{n+1}{k_1-1,k_2,\ldots,k_m}$ corresponding to $J = \{1\}$, gives the desired result. $\qquad\square$

While the inductive proof of Lemma 3.2 above works nicely, it somewhat disguises the way we count our flags to see the result bijectively. We conclude with an explanation of this count. First, we need to understand the combinatorial proof of Lemma 3.1 appearing in [11], which may be summarized as follows. Fix V to be an $(n+1)$-dimensional $\mathbb{F}_q$-vector space as before. There are $\binom{n+1}{k}_q$ ways to choose a k-dimensional subspace W of V. Fix a basis $\{v_1, v_2, \ldots, v_{n+1}\}$ of V. Any k-dimensional subspace W can be written as $\mathrm{span}(W', v)$ where W' is a $(k-1)$-dimensional

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subspace of $V' = \text{span}(v_1, \ldots, v_n)$. We may choose W in three distinct ways. If $v \in V'$, then W is a subspace of V', for which there are $\binom{n}{k}_q$ choices. Call this a *Type 1* subspace of V. If we take v to be a scalar multiple of v_{n+1}, then W is determined by W', for which there are $\binom{n}{k-1}_q$ choices. We call this a *Type 2* subspace of V. Finally, if v is neither in V' nor a scalar multiple of v_{n+1}, then we call W a *Type 3* subspace of V, and it can be shown that there are $(q^n - 1)\binom{n-1}{k-1}_q$ choices for W, giving Lemma 3.1.

We now fix a basis of *every* subspace U of V, so that we may speak of subspaces of Type 1, 2, or 3 of U. Consider a flag of subspaces of $V = W_0, W_{m-1} \subset \cdots \subset W_2 \subset W_1$, such that if we define k_i for $1 \leq i \leq m$ by $\sum_{j=1}^{i} k_j = n + 1 - \dim W_i$, then each $k_i > 0$. The total number of such flags is $\binom{n+1}{k_1, \ldots, k_m}_q$, and these flags may also be counted in the following way. We may choose W_1 to be a Type 1 subspace of V, or we may choose every W_i to be a Type 2 or Type 3 subspace of W_{i-1} for $i \leq m-1$, or we may choose W_i to be a Type 2 or Type 3 subspace of W_{i-1} for $i \leq r-1$ for some $r < m$ and W_r a Type 1 subspace of W_{r-1}. These cases account for all possibilities for such a flag of V. For a nonempty $J \subseteq \{1, \ldots, m\}$, let r be the maximum element of J. Then a closer look at the proof of Lemma 3.2 reveals that

$$(-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k} - \underline{e}_J}_q$$

is the number of ways to choose our flag such that W_j is a Type 3 subspace of W_{j-1} for $j \in J$ and $j < r$, W_i is a Type 2 subspace of W_{i-1} for $i \notin J$ and $i < r$, and W_r is a Type 1 subspace of W_{r-1} if $r < m$. These account for all $2^m - 1$ terms in the right-side of the equation in Lemma 3.2, and all possible ways to choose our flag.

References

[1] G. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, Mass.-London-Amsterdam, 1976.

[2] R. Askey and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* **54**, no. 319, 1985.

[3] W. Y. C. Chen, H. L. Saad, and L. H. Sun, The bivariate Rogers-Szegö polynomials, *J. Phys. A* **40** (2007), no. 23, 6071–6084.

[4] N. J. Fine, Basic Hypergeometric Series and Applications, Mathematical Surveys and Monographs 27, American Mathematical Society, Providence, RI, 1988.

[5] J. Goldman and G.-C. Rota, The number of subspaces of a vector space, in *Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968)*, Academic Press, New York, 1969, pp. 75–83.

[6] S. Hitzemann and W. Hochstättler, On the combinatorics of Galois numbers, *Discrete Math.* **310** (2010), no. 24, 3551–3557.

[7] V. Kac and P. Cheung, Quantum Calculus, Universitext, Springer-Verlag, New York, 2002.

[8] H. Karabulut, Distributed Gaussian polynomials as q-oscillator eigenfunctions, *J. Math. Phys.* **47** (2006), no. 1, 013508, 13 pp.

[9] A. J. Macfarlane, On q-analogues of the quantum harmonic oscillator and the quantum group SU(2)$_q$, *J. Phys. A* **22** (1989), no. 21, 4581–4588.

[10] K. Morrison, Integer sequences and matrices over finite fields, *J. Integer Seq.* **9** (2006), no. 2, Article 06.2.1, 28 pp.

[11] A. Nijenhuis, A. E. Solow, and H. S. Wilf, Bijective methods in the theory of finite vector spaces, *J. Combin. Theory Ser. A*, **37** (1984), no. 1, 80–84.

[12] L. J. Rogers, On a three-fold symmetry in the elements of Heine's series, *Proc. London Math. Soc.*, **24** (1893), 171–179.

[13] L. J. Rogers, On the expansion of some infinite products, *Proc. London Math. Soc.*, **24** (1893), 337–352.

[14] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Studies in Advanced Mathematics, 49, Cambridge University Press, Cambridge, 1997.

[15] G. Szegö, Ein Beitrag zur Theorie der Thetafunktionen, *S. B. Preuss. Akad. Wiss. Phys.-Math. Kl.*, (1926), 242–252.

[16] S. O. Warnaar, Rogers-Szegö polynomials and Hall-Littlewood symmetric functions, *J. Algebra* **303** (2006), no. 2, 810–830.

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