Interpolating Thin-Shell and Sharp Large-Deviation Estimates For Isotropic Log-Concave Measures

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Abstract

Given an isotropic random vector X with log-concave density in Euclidean space $\mathbb{R}^n$, we study the concentration properties of $|X|$. We show in particular that:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-cn^{\frac{1}{2}} \min(t^3, t)) \quad \forall t > 0 \ ,$$

for some universal constant $c > 0$. This improves the best known deviation results above the expectation on the thin-shell and mesoscopic scales due to Fleury and Klartag, respectively, and recovers the sharp large-deviation estimate of Paouris. Another new feature of our estimate is that it improves when X is ψ_α ($\alpha \in (1,2]$), in precise agreement with the sharp Paouris estimates. The upper bound on the thin-shell width $\sqrt{\mathbb{V}\mathrm{ar}(|X|)}$ we obtain is of the order of $n^{1/3}$, and improves down to $n^{1/4}$ when X is ψ_2. Our estimates thus continuously interpolate between a new best known thin-shell estimate and the sharp Paouris large-deviation one.

1 Introduction

Let a Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ be fixed. This work is dedicated to quantitative concentration properties of $|X|$, where X is an isotropic random vector in $\mathbb{R}^n$ with log-concave density. Recall that a random vector X in $\mathbb{R}^n$ (and its density) is called isotropic if $\mathbb{E}X = 0$ and $\mathbb{E}X \otimes X = Id$, i.e. its barycenter is at the origin and its covariance matrix is equal to the identity one. Taking traces, we observe that $\mathbb{E}|X|^2 = n$. Here and throughout we use $\mathbb{E}$ to denote expectation, $\mathbb{P}$ to denote probability, and $\mathbb{V}\mathrm{ar}$ to denote variance. A function $g : \mathbb{R}^n \to \mathbb{R}_+$ is called log-concave if $-\log g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex. Throughout this work, C,c,c_2,C', etc. denote universal positive numeric constants, independent of any other parameter and in particular the dimension n, whose value may change from one occurrence to the next.

It was conjectured by Anttila, Ball and Perissinaki [1] that $|X|$ is concentrated around its expectation significantly more than suggested by the trivial bound $\mathbb{V}\mathrm{ar}|X| \leq \mathbb{E}|X|^2 =$

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n. Namely, they conjectured that there exists a sequence $\{\varepsilon_n\}$ decreasing to 0 with the dimension n, so that X is concentrated within a "thin shell" of relative width $2\varepsilon_n$ around the (approximately) expected Euclidean norm of $\sqrt{n}$:

$$\mathbb{P}(\big||X| - \sqrt{n}\big| \geq \varepsilon_n \sqrt{n}) \leq \varepsilon_n \ . \tag{1.1}$$

Their conjecture was mainly motivated by the Central Limit Problem for log-concave measures, and as pointed out in [1], implies that most marginals of log-concave measures are approximately Gaussian.

A stronger version of this conjecture was put forth by Bobkov and Koldobsky [7]. It may be equivalently formulated as stating that the "thin-shell width" $\sqrt{\mathbb{V}\mathrm{ar}|X|}$ is bounded above by a universal constant C.

An even stronger conjecture is due to Kannan, Lovász and Simonovits [16]. In an equivalent form, it states that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathbb{V}\mathrm{ar}(f(X)) \leq C\mathbb{E}|\nabla f(X)|^2 \ .$$

Applied to the function $f(x) = |x|^p$ with $p = c\sqrt{n}$, the KLS conjecture implies (see [11] and Section 4) that:

$$\mathbb{P}(\big||X| - \sqrt{n}\big| \geq t\sqrt{n}) \leq C \exp(-c\sqrt{n}t) \quad \forall t \geq 0 \ . \tag{1.2}$$

It was shown by G. Paouris [29] that the predicted positive deviation estimate (1.2) indeed holds in the large:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-c\sqrt{n}t) \quad \forall t \geq C > 0 \ . \tag{1.3}$$

Moreover, Paouris showed that when X is ψ_α ($\alpha \in [1,2]$):

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-cn^{\frac{\alpha}{2}}t) \quad \forall t \geq C > 0 \ . \tag{1.4}$$

Recall that X (and its density) is said to be "ψ_α with constant $D > 0$" if:

$$(\mathbb{E}\,|\langle X, y\rangle|^p)^{1/p} \leq Dp^{1/\alpha} \left(\mathbb{E}\,|\langle X, y\rangle|^2\right)^{1/2} \quad \forall p \geq 2 \quad \forall y \in \mathbb{R}^n \ .$$

We will simply say that "X is ψ_α", if it is ψ_α with constant $D \leq C$, and not specify explicitly the dependence of the estimates on the parameter D. By Borell's Lemma [10] (see also [27, Appendix III]), it is well known that any X with log-concave density is ψ_1 with some universal constant, and so we only gain additional information when $\alpha > 1$.

The positive large-deviation estimate (1.4) is easily verified to be sharp (up to universal constants) for all $\alpha \in [1,2]$. However, this leaves open the concentration estimates in the bulk: positive deviation $\mathbb{P}(|X| \geq (1+t)\sqrt{n})$ when $t \in [0,C]$, and negative deviation $\mathbb{P}(|X| \leq (1-t)\sqrt{n})$ when $t \in [0,c]$ ($c \in (0,1)$); in particular, this gives no information on the thin-shell $\sqrt{\mathbb{V}\mathrm{ar}|X|}$. We remark that the small-ball estimates $\mathbb{P}(|X| \leq \varepsilon\sqrt{n})$ for $\varepsilon \in [0, 1-c]$ will mostly be disregarded in this work (see however Remark 1.4 below).

The first non-trivial estimate on the concentration of $|X|$ around its expectation was given by B. Klartag in [18], involving logarithmic improvements in n over the trivial bounds. This validated the conjectured thin-shell concentration (1.1), allowing Klartag to resolve the Central Limit Problem for log-concave measures. A different proof continuing Paouris' approach was given by Fleury, Guédon and Paouris in [13]. Klartag then improved in [19] his estimates from logarithmic to polynomial in n as follows (for any small $\varepsilon > 0$):

$$\mathbb{P}(\big||X| - \sqrt{n}\big| \geq t\sqrt{n}) \leq C_\varepsilon \exp(-c_\varepsilon n^{\frac{1}{3}-\varepsilon} t^{\frac{10}{3}-\varepsilon}) \quad \forall t \in [0,1] \; . \tag{1.5}$$

This implies in particular a thin-shell estimate of:

$$\sqrt{\mathbb{V}\mathrm{ar}|X|} \leq C_\varepsilon n^{\frac{1}{2}-\frac{1}{10}-\varepsilon} \; .$$

Note, however, that when $t = 1/2$, (1.5) does not recover the sharp positive large-deviation estimate of Paouris (1.3).

Recently in [12], B. Fleury improved Klartag's thin-shell estimate to:

$$\sqrt{\mathbb{V}\mathrm{ar}|X|} \leq C n^{\frac{1}{2}-\frac{1}{8}} \; ,$$

by obtaining the following deviation estimates:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq C \exp(-c n^{\frac{1}{4}} t^2) \quad \forall t \in [0,1] \; ;$$

$$\mathbb{P}(|X| \leq (1-t)\sqrt{n}) \leq C \exp(-c n^{\frac{1}{8}} t) \quad \forall t \in [0,1] \; .$$

Note, however, that when $t = 1/2$, Fleury's positive and negative large-deviation estimates are both inferior to those of Klartag, and so in the mesoscopic scale $t = n^{-\delta}$ ($\delta > 0$ small), Klartag's estimates still outperform Fleury's (and Paouris' ones are inapplicable). In addition, note that both Klartag and Fleury's estimates do not seem to improve under a ψ_α condition, contrary to the ones of Paouris. See also [8, 20, 11, 23] for further related results.

All of this suggests that one might hope for a concentration estimate which:

- Recovers the sharp positive large-deviation result of Paouris (1.4).
- Improves if X is ψ_α.
- Improves the best-known thin-shell estimate of Fleury.
- Improves the best-known positive mesoscopic-deviation estimate of Klartag.
- Interpolates continuously between all positive scales of t (bulk, mesoscopic, large-deviation).

The aim of this work is to provide precisely such an estimate.

1.1 The Results

Theorem 1.1. *Let X denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density, which is in addition ψ_α ($\alpha \in [1,2]$). Then:*

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-cn^{\frac{\alpha}{2}} \min(t^{2+\alpha}, t)) \quad \forall t > 0 \ , \tag{1.6}$$

and:

$$\mathbb{P}(|X| \leq (1-t)\sqrt{n}) \leq C\exp(-cn^{\frac{\alpha}{2(2+\alpha)}} t) \quad \forall t \in (0,1) \ . \tag{1.7}$$

In particular:

$$\sqrt{\mathbb{V}ar(|X|)} \leq Cn^{\frac{1}{2+\alpha}}. \tag{1.8}$$

Note that when $\alpha = 1$, as is the case for an arbitrary isotropic X with log-concave density, we obtain the following thin-shell estimate:

$$\sqrt{\mathbb{V}\mathrm{ar}(|X|)} \leq Cn^{\frac{1}{2}-\frac{1}{6}} \ .$$

Also note that we obtain $\mathbb{P}(|X| \geq (1+\varepsilon)\sqrt{n}) \leq \exp(-C_\varepsilon n^{\frac{\alpha}{2}})$ for *any* $\varepsilon > 0$, whereas Paouris' estimate (1.4) only ensures that this holds for $\varepsilon \geq C$ for some large enough $C > 0$.

Theorem 1.1 is a standard consequence of the following moment estimates, which are the main result of this work:

Theorem 1.2. *Let X denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density, which is in addition ψ_α ($\alpha \in [1,2]$). Then for any $2 \leq p \leq cn^{\alpha/2}$:*

$$(\mathbb{E}|X|^p)^{\frac{1}{p}} \leq \left(1 + C\left(\frac{p}{n^{\frac{\alpha}{2}}}\right)^{\frac{1}{\alpha+1}}\right)\left(\mathbb{E}|X|^2\right)^{\frac{1}{2}} \ . \tag{1.9}$$

Note that using $p = 4$ in (1.9) we obtain $\mathbb{V}\mathrm{ar}(|X|^2) \leq C_2 n^{2-\frac{\alpha}{2(\alpha+1)}}$, which only yields $\sqrt{\mathbb{V}\mathrm{ar}(|X|)} \leq C_3 n^{\frac{1}{2}-\frac{\alpha}{4(\alpha+1)}}$, an inferior estimate to (1.8). The reason for this discrepancy is due to the fact that our moment estimates for relatively small values of p may be a-posteriori improved, by integrating by parts and using the deviation estimates of Theorem 1.1 (see [11, Lemma 6]):

Corollary 1.3. *With the same assumptions as in Theorem 1.2, for any $2 \leq p \leq c_2 n^{\frac{\alpha}{2(\alpha+2)}}$:*

$$(\mathbb{E}|X|^p)^{\frac{1}{p}} \leq \left(1 + C_2\frac{p}{n^{\frac{\alpha}{\alpha+2}}}\right)\left(\mathbb{E}|X|^2\right)^{\frac{1}{2}} \ .$$

Remark 1.4. Note that our estimates for negative mesoscopic and large deviation are still inferior to those of Klartag. The best known negative large-deviation and small-ball estimates are due to Paouris [30], who showed that there exists a constant $C > 1$ so that:

$$\mathbb{P}(|X| \leq \varepsilon\sqrt{n}) \leq (C\varepsilon)^{cn^{\frac{\alpha}{2}}} \quad \forall \varepsilon \in (0, 1/C) \ . \tag{1.10}$$

It should be possible to extend our methods to handle negative moments p in Theorem 1.2, resulting in a continuous interpolation between (1.7) and (1.10), and thus improving over Klartag's negative mesoscopic deviation estimates; we leave this for another note.

1.2 The Approach

We let $G_{n,k}$ denote the Grassmann manifold of all k-dimensional linear subspaces of $\mathbb{R}^n$, and $SO(n)$ the group of rotations. Fixing a Euclidean structure on $\mathbb{R}^n$, and given a linear subspace F, we denote by $S(F)$ and $B_2(F)$ the unit-sphere and unit-ball in F, respectively. When $F = \mathbb{R}^n$, we simply write S^{n-1} and B_2^n. We denote by P_F the orthogonal projection onto F in $\mathbb{R}^n$, and given a random vector Y with density g, we denote by $\pi_F g$ the marginal density of g on F, i.e. the density of $P_F Y$. When $F = \mathrm{span}(\theta)$, $\theta \in S^{n-1}$, we denote the density $\pi_\theta g$ on $\mathbb{R}$ given by $\pi_\theta g(t) := \pi_F g(t\theta)$.

For the proof of Theorem 1.2, we use many of the ingredients developed previously by Klartag [19], and adapted to the language of moments by Fleury [11, 12]:

- It is enough to verify (1.9) with X replaced by $Y = (X + G_n)/\sqrt{2}$, where G_n denotes a standard Gaussian random vector in $\mathbb{R}^n$.

- It is useful to first project Y onto a lower-dimensional subspace $F \in G_{n,k}$. This idea also appears in essence in the work of Paouris [29]. Klartag and Paouris use V. Milman's approach to Dvoretzky's theorem [25, 27] for identifying lower-dimensional structures in most marginals $P_F Y$. Fleury, on the other hand, takes an average over the Haar measure on $G_{n,k}$, which is more efficient (see [12] or below):

$$\frac{(\mathbb{E}|Y|^p)^{1/p}}{(\mathbb{E}|Y|^2)^{1/2}} \leq \frac{(\mathbb{E}_{F,Y}|P_F Y|^p)^{1/p}}{(\mathbb{E}_{F,Y}|P_F Y|^2)^{1/2}} \ . \tag{1.11}$$

- Rewriting using the invariance of the Haar measure and polar coordinates:

$$\frac{(\mathbb{E}_{F,Y}|P_F Y|^p)^{1/p}}{(\mathbb{E}_{F,Y}|P_F Y|^2)^{1/2}} = \frac{(\mathbb{E}_U h_{k,p}(U))^{1/p}}{(\mathbb{E}_U h_{k,2}(U))^{1/2}} \ , \tag{1.12}$$

  where U is uniformly distributed over $SO(n)$, $E_0 \in G_{n,k}$, $\theta_0 \in S(E_0)$, g denotes the density of Y in $\mathbb{R}^n$, and $h_{k,q} : SO(n) \to \mathbb{R}_+$ is defined as:

$$h_{k,p}(u) := \mathrm{Vol}(S^{k-1}) \int_0^\infty t^{p+k-1} \pi_{u(E_0)} g(tu(\theta_0)) dt \ . \tag{1.13}$$

  To control the ratio in (1.12), a good bound on the log-Lipschitz constant $L_{k,q}$ of $h_{k,q}$ is required.

Our main technical result in this work is the following improvement over the log-Lipschitz bounds of Klartag from [19]:

Theorem 1.5. *Under the same assumptions as in Theorem 1.1,* $L_{k,p} \leq C \max(k,p)^{1/\alpha+1/2}$.

Contrary to Klartag's analytical approach for controlling the log-Lipschitz constant, ours is completely based on geometric convexity arguments, employing the convex bodies K_{k+q} introduced by K. Ball in [3], and a variation on the L_q-centroid bodies, which were introduced by E. Lutwak and G. Zhang in [22].

Fleury proceeds by employing three additional ingredients:

- As shown by Borell [9], for any log-concave function w on $\mathbb{R}_+$:

$$q \mapsto \log \frac{\int_0^\infty t^{q-1} w(t) dt}{\Gamma(q)} \text{ is concave on } \mathbb{R}_+ \ . \tag{1.14}$$

  Consequently, $q \mapsto \log(h_{k,q}(u)/\Gamma(k+q))$ is concave for any fixed $u \in SO(n)$. This ingredient was also used in [13].

- As follows e.g. from the work of Bakry and Émery [2] (see also [21]), for any Lipschitz function $f : SO(n) \to \mathbb{R}_+$, the following log-Sobolev inequality is satisfied (see Sections 2 and 3 for definitions):

$$\mathbb{E}\mathrm{nt}_U(f) \leq \frac{c}{n} \mathbb{E}_U(|\nabla f|^2 / f) \ . \tag{1.15}$$

- The latter log-Sobolev inequality implies via the Herbst argument, that for any log-Lipschitz function $f : SO(n) \to \mathbb{R}_+$ with log-Lipschitz constant bounded above by L, the following reverse Hölder inequality holds (see [12, (15)]):

$$(\mathbb{E}_U f^q)^{\frac{1}{q}} \leq \exp\left(C \frac{L^2}{n} (q - r) \right) (\mathbb{E}_U f^r)^{\frac{1}{r}} \quad \forall q > r > 0 \ . \tag{1.16}$$

We proceed by using these ingredients as our predecessors, but our proof corrects the slight inefficiency of Fleury's approach in the resulting large-deviation estimate (witnessed by the comparison to Klartag's estimate earlier). The improvement here comes from the fact that we take the derivative in p of (1.11), and optimize on the dimension k for each p separately, as opposed to optimizing on a single k directly in (1.11). However, this by itself would not yield the improvement in the thin-shell estimate - the latter is due to our improved log-Lipschitz estimate in Theorem 1.5. Only by combining this improved log-Lipschitz estimate with our variation on Fleury's method, are we able to recover the sharp large-deviation estimates of Paouris (1.4).

The rest of this work is organized as follows. In Section 2 we prove a more general version of Theorem 1.5. In Section 3 we provide a complete proof of Theorem 1.2. In Section 4, we derive for completeness Theorem 1.1 from Theorem 1.2. In Section 5, we provide some concluding remarks. In the Appendix, we provide a proof of Proposition 2.6 and other lemmas, whose purpose is to handle the case when X is not centrally-symmetric (non-even density).

2 An improved log-Lipschitz estimate

Let $M_{k,l}(\mathbb{R})$ denote the set of k by l matrices over $\mathbb{R}$. We equip

$$SO(n) = \left\{ U \in M_{n,n}(\mathbb{R}); U^t U = Id \ , \ det(U) = 1 \right\}$$

with its standard (left and right) invariant Riemannian metric g, which we specify for concreteness on $T_{Id}SO(n)$, the tangent space at the identity element $Id \in SO(n)$.

Fixing an orthonormal basis of $\mathbb{R}^n$ and taking the derivative of the relation $U^t U = Id$, we see that this tangent space may be identified with all anti-symmetric matrices $\left\{B \in M_{n,n}(\mathbb{R}); B^t + B = 0\right\}$. Given $B \in T_{Id}SO(n)$, we set $|B|^2 := g_{Id}(B,B) = \frac{1}{2}\|B\|_{HS}^2$, where recall the Hilbert-Schmidt norm of $A \in M_{k,l}(\mathbb{R})$ is given by $\|A\|_{HS}^2 := tr(A^t A) = \sum_{1\leq i\leq k, 1\leq j\leq l} A_{i,j}^2$. The factor of $\frac{1}{2}$ above is simply a convenience to ensure that a full 2π degree rotation in any two-plane leaving the orthogonal complement in place, has geodesic length 2π, and to prevent further appearances of factors like $\sqrt{2}$ later on. Up to this factor, this metric coincides with the one induced from the natural embedding $SO(n) \subset \mathbb{R}^{n^2}$.

2.1 Main Result

Throughout this section, let Y denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density g. Given an integer k between 1 and n, a linear subspace $E_0 \in G_{n,k}$ and $\theta_0 \in S(E_0)$, we recall the definition of the function $h_{k,p} : SO(n) \to \mathbb{R}_+$:

$$h_{k,p}(U) := \mathrm{Vol}(S^{k-1}) \int_0^\infty t^{p+k-1} \pi_{U(E_0)} g(tU(\theta_0)) dt \quad , \quad U \in SO(n) \; . \tag{2.1}$$

Note that $\pi_E g$ is log-concave for any $E \in G_{n,k}$ by the Prékopa–Leindler theorem (e.g. [14]).

When $Y = (X + G_n)/\sqrt{2}$, where (as throughout this work) X denotes an isotropic random vector in $\mathbb{R}^n$ with log-concave density, an upper bound on the log-Lipschitz constant (i.e. the Lipschitz constant of the logarithm) of:

$$U \mapsto \pi_{U(E_0)} g(tU(\theta_0))$$

was obtained by Klartag [19, Lemma 3.1], playing a crucial role in his polynomial estimates on the thin-shell of an isotropic log-concave measure. When $t \leq C\sqrt{k}$, Klartag's estimate is of the order of k^2. In [12], Fleury defined a truncated version of (2.1), where the integral ranges up to $C\sqrt{k}$. Klartag's estimate obviously implies the same bound on the log-Lipschitz constant of this truncated version of $h_{k,p}$.

Our main technical result in this work is the following improved estimate on the log-Lipschitz constant of $h_{k,p}$, which is completely based on geometric convexity arguments. Note that we do not need any truncation, nor do we need to assume that Y has been convolved with a Gaussian to obtain a meaningful estimate. However, the improvement over Klartag's k^2 bound appears after this convolution.

Theorem 2.1. *The log-Lipschitz constant $L_{k,p}$ of $U \mapsto h_{k,p}(U)$ is bounded above by $C \max(k,p) dist(Z^+_{\max(k,p)}(g), B_2^n)$.*

Here $Z_q^+(w) \subset \mathbb{R}^n$ ($q \geq 1$) denotes the *one-sided* L_q-centroid body of the density w (which may not have total mass one), defined via its support functional:

$$h_{Z_q^+(w)}(y) = \left(2 \int_{\mathbb{R}^n} \langle x, y\rangle_+^q \, w(x) dx\right)^{1/q} \quad ,$$

(here as usual $a_+ := \max(a,0)$). When w is even, this coincides with the more standard definition of the L_q-centroid body, introduced by E. Lutwak and G. Zhang in [22] (under a different normalization):

$$h_{Z_q(w)}(y) = \left(\int_{\mathbb{R}^n} |\langle x,y \rangle|^q \, w(x) dx \right)^{1/q} .$$

Clearly:

$$Z_q^+(w) \subset 2^{1/q} Z_q(w) .$$

In any case, when w is the characteristic function of a set K, we denote $Z_q^+(K) := Z_q^+(1_K)$, and similarly for $Z_q(K)$. Lastly, the *geometric distance* $\text{dist}(K,L)$ between two subsets $K, L \subset \mathbb{R}^n$ is defined as:

$$\text{dist}(K,L) := \inf \left\{ C_2/C_1 ; C_1 L \subset K \subset C_2 L \ , \ C_1, C_2 > 0 \right\} .$$

A very useful result for handling the non-even case is due to Grünbaum [15]:

Lemma 2.2. *Let X_1 denote a random variable on $\mathbb{R}$ with log-concave density and barycenter at the origin. Then $\frac{1}{e} \leq \mathbb{P}(X_1 \geq 0) \leq 1 - \frac{1}{e}$.*

Note that by definition Y (and its density g) is ψ_α ($\alpha \geq 1$) iff $Z_q(g) \subset C q^{1/\alpha} Z_2(g)$ for some fixed universal constant $C > 1$ and all $q \geq 2$. Also recall that by Borell's Lemma [10], a log-concave probability density g is always ψ_1, and that moreover:

$$1 \leq q_1 \leq q_2 \quad \Rightarrow \quad Z_{q_1}(g) \subset Z_{q_2}(g) \subset C \frac{q_2}{q_1} Z_{q_1}(g) . \tag{2.2}$$

If in addition the barycenter of g is at the origin, then repeating the argument leading to (2.2) and using Lemma 2.2, one verifies:

$$1 \leq q_1 \leq q_2 \quad \Rightarrow \quad \left(\frac{2}{e}\right)^{\frac{1}{q_1} - \frac{1}{q_2}} Z_{q_1}^+(g) \subset Z_{q_2}^+(g) \subset C \left(\frac{2e-2}{e}\right)^{\frac{1}{q_1} - \frac{1}{q_2}} \frac{q_2}{q_1} Z_{q_1}^+(g) . \tag{2.3}$$

Note that $Z_2(g) = B_2^n$ by definition of isotropicity, and one may similarly show (see Lemma A.5) that $c B_2^n \subset Z_2^+(g) \subset \sqrt{2} B_2^n$. It follows immediately from (2.3) that $\text{dist}(Z_k^+(g), B_2^n) \leq Ck$, and we see that Theorem 2.1 recovers Klartag's k^2 order of magnitude when $p \leq k$ (which is the case of interest in the subsequent analysis).

The improvement over Klartag's bound comes from the following elementary:

Lemma 2.3. *Let X_0 denote a random-vector in $\mathbb{R}^n$ with log-concave density and barycenter at the origin. Set $Y_0 = (X_0 + G_n)/\sqrt{2}$ and denote by g_0 its density. Then:*

1. *$Z_q^+(g_0) \supset c\sqrt{q} B_2^n$ for all $q \geq 2$.*
2. *If X_0 is ψ_α ($\alpha \in [1,2]$), then so is Y_0.*

Proof. Given $\theta \in S^{n-1}$, denote $Y_1 = \pi_\theta Y_0$, $X_1 = \pi_\theta X_0$ and $G_1 = \pi_\theta G_n$ (a one-dimensional standard Gaussian random variable). We have:

$$h^q_{Z^+_q(g_0)}(\theta) = 2\mathbb{E}(Y_1)^q_+ = \frac{2}{2^{q/2}}\mathbb{E}\left(X_1 + G_1\right)^q_+ \geq \frac{2}{2^{q/2}}\mathbb{E}(G_1)^q_+\mathbb{P}(X_1 \geq 0) \ .$$

When X_0 is centrally-symmetric $\mathbb{P}(X_1 \geq 0) = 1/2$. In the general case, since X_1 has log-concave density on $\mathbb{R}$ and barycenter at the origin, Lemma 2.2 implies that $\mathbb{P}(X_1 \geq 0) \geq 1/e$, and hence:

$$h^q_{Z^+_q(g_0)}(\theta) \geq \frac{1}{e2^{q/2}}\mathbb{E}|G_1|^q \ ,$$

by the symmetry of G_1. An elementary calculation shows that $c_1\sqrt{q} \leq (\mathbb{E}|G_1|^q)^{1/q} \leq c_2\sqrt{q}$ for all $q \geq 1$, concluding the first claim. The second claim follows similarly since:

$$h^q_{Z_q(g_0)}(\theta) = \mathbb{E}|Y_1|^q = \mathbb{E}\left|\frac{X_1+G_1}{\sqrt{2}}\right|^q \leq \frac{2^{q-1}}{2^{q/2}}\mathbb{E}(|X_1|^q + |G_1|^q) \ .$$

□

Corollary 2.4. *If X is ψ_α ($\alpha \in [1,2]$) and $Y = (X+G_n)/\sqrt{2}$, then $dist(Z^+_q(g), B^n_2) \leq Cq^{1/\alpha-1/2}$. Consequently, Theorem 2.1 implies that $L_{k,p} \leq C\max(k,p)^{1/\alpha+1/2}$.*

2.2 Proof of Theorem 2.1

For convenience, we assume that $2 \leq k \leq n/2$, although it will be clear from the proof that this is immaterial. By the symmetry and transitivity of $SO(n)$, and since $E_0 \in G_{n,k}$ was arbitrary, it is enough to bound $|\nabla_{U_0}\log h_{k,p}|$ at $U_0 = Id$. We complete θ_0 to an orthonormal basis $\left\{\theta_0, e^2, \ldots, e^k\right\}$ of E_0, and take $\left\{e^{k+1}, \ldots, e^n\right\}$ to be any completion to an orthonormal basis of $\mathbb{R}^n$. In this basis, the anti-symmetric matrix $M := \nabla_{Id}\log h_{k,p} \in T_{Id}SO(n)$ looks as follows:

$$M = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline -M_2 & 0 \end{array}\right) \ , \ M_1 = \left(\begin{array}{c|c} 0 & V_1 \\ \hline -V_1 & 0 \end{array}\right) \ , \ M_2 = \left(\begin{array}{c} V_2 \\ \hline V_3 \end{array}\right) \ , \tag{2.4}$$

where $M_1 \in M_{k,k}(\mathbb{R})$, $M_2 \in M_{k,n-k}(\mathbb{R})$, $V_1 \in M_{1,k-1}(\mathbb{R})$, $V_2 \in M_{1,n-k}(\mathbb{R})$ and $V_3 \in M_{k-1,n-k}(\mathbb{R})$. Indeed, the lower $n-k$ by $n-k$ block of M is clearly 0, since rotations in $E_0^\perp$, the orthogonal complement to E_0, leave $\pi_{U(E_0)}g$ and hence $h_{k,p}$ unaltered; and the lower $k-1$ by $k-1$ block of M_1 is 0 since rotations which fix θ_0 and act invariantly on E_0 preserve $h_{k,p}$ as well. Consequently $|\nabla_{Id}\log h_{k,p}|^2 = \|V_1\|^2_{HS} + \|V_2\|^2_{HS} + \|V_3\|^2_{HS}$. We will analyze the contributions of these three terms separately.

Denote by T_i ($i = 1, 2, 3$) the subspace of $T_{Id}SO(n)$ having the form (2.4) with $V_j = 0$ for $j \neq i$. Given $B \in T_i$, we call the geodesic $\mathbb{R} \ni s \mapsto U_s := \exp_{Id}(sB) \in SO(n)$ a *Type-i movement*. Clearly $\|V_i\|_{HS} = \sup_{0 \neq B \in T_i} \langle \nabla_{Id} \log h_{k,p}, B \rangle / |B|$, so our goal now is to obtain a uniform upper bound on the derivative of $\log h_{k,p}$ induced by a Type-i movement.

To this end, we recall the following crucial fact, due to K. Ball [3, Theorem 5] in the even case, and verified to still hold in the general one by Klartag [17, Theorem 2.2]:

Theorem. *Let w denote a log-concave function on $\mathbb{R}^m$ with $0 < \int w < \infty$ and $w(0) > 0$. Given $q \geq 1$, set:*

$$\|x\| = \|x\|_{K_q(w)} := \left(q \int_0^\infty t^{q-1} w(tx) dt \right)^{-\frac{1}{q}} \;\; , \;\; x \in \mathbb{R}^m \; .$$

Then for all $x, y \in \mathbb{R}^m$, $0 \leq \|x\| < \infty$, $\|x\| = 0$ iff $x = 0$, $\|\lambda x\| = \lambda \|x\|$ for all $\lambda \geq 0$, and $\|x + y\| \leq \|x\| + \|y\|$.

We will thus say that $\|\cdot\|_{K_q(w)}$ defines a norm, even though it may fail to be even, and denote by $K_q(w) := \{x \in \mathbb{R}^m; \|x\|_{K_q(w)} \leq 1\}$ its associated convex compact unit-ball. Note that the constant q in front of the integral above is simply a convenient normalization for later use. We also set $\|x\|_{\hat{K}_q(w)} := \max(\|x\|_{K_q(w)}, \|-x\|_{K_q(w)})$, having unit-ball $\hat{K}_q(w) = K_q(w) \cap -K_q(w)$. Note that the triangle inequality implies that:

$$\left| \|x\|_{K_q(w)} - \|y\|_{K_q(w)} \right| \leq \|x - y\|_{\hat{K}_q(w)} \; . \tag{2.5}$$

Finally, note that since B_2^m is centrally-symmetric, then $C_1 B_2^m \subset K \cap -K \subset K \subset C_2 B_2^m$ iff $C_1 B_2^m \subset K \subset C_2 B_2^m$, and hence:

$$\frac{\|x\|_{\hat{K}_q(w)}}{\|y\|_{K_q(w)}} \leq \mathrm{dist}(K_q(w), B_2^m) \frac{|x|}{|y|} \; . \tag{2.6}$$

2.2.1 Type-1 movement

Let $B \in T_1$ with $|B| = 1$ generate a Type-1 movement $\{U_s\}$, and denote $\xi_0 = \frac{d}{ds} U_s(\theta_0)|_{s=0} \in T_{\theta_0} S(\mathbb{R}^n)$. Using henceforth the natural embedding $T_\theta S(\mathbb{R}^n) \subset T_\theta \mathbb{R}^n \subset \mathbb{R}^n$, we see that ξ_0 lies in the orthogonal complement of θ_0 in E_0, and $|B| = 1$ ensures that $|\xi_0| = 1$. So U_s is a rotation in the $\{\theta_0, \xi_0\}$ plane, and $U_s(E_0) = E_0$. Recalling the definition of $h_{k,p}$, we see that:

$$|\langle \nabla_{Id} \log h_{k,p}, B \rangle| = \left| \frac{d}{ds} \log h_{k,p}(U_s) \Big|_{s=0} \right| = (k+p) \left| \frac{d}{ds} \log \|U_s(\theta_0)\|_{K_{k+p}(\pi_{E_0} g)} \Big|_{s=0} \right| \; .$$

Estimating the derivative of the norm using the triangle-inequality (2.5) and (2.6), we immediately obtain:

$$|\langle \nabla_{Id} \log h_{k,p}, B \rangle| \leq (k+p) \frac{\|\xi_0\|_{\hat{K}_{k+p}(\pi_{E_0} g)}}{\|\theta_0\|_{K_{k+p}(\pi_{E_0} g)}} \leq (k+p) \mathrm{dist}(K_{k+p}(\pi_{E_0} g)), B_2(E_0)) \; .$$

2.2.2 Type-2 movement

Let $B \in T_2$ with $|B| = 1$ generate a Type-2 movement $\{U_s\}$, and denote $\theta_s := U_s(\theta_0)$ and $\xi_s := \frac{d}{ds}\theta_s \in T_{\theta_s}S(\mathbb{R}^n)$. Now $\xi_0 \in S(E_0^\perp)$, and the rotation U_s in the $\{\theta_s, \xi_s\} = \{\theta_0, \xi_0\}$ plane rotates E_0 into $E_s := U_s(E_0)$. Denote $H := \text{span}(E_s, \xi_s) \in G_{n,k+1}$, and observe that:

$$h_{k,p}(U_s) = \text{Vol}(S^{k-1}) \int_0^\infty \int_{-\infty}^\infty t^{p+k-1} \pi_H g(t\theta_s + r\xi_s) dr dt \; .$$

Performing the change of variables $r = vt$, which is valid except at the negligible point $t = 0$, we obtain:

$$h_{k,p}(U_s) = \text{Vol}(S^{k-1}) \int_0^\infty \int_{-\infty}^\infty t^{p+k} \pi_H g(t(\theta_s + v\xi_s)) dv dt = c_{p,k} \int_{-\infty}^\infty \|\theta_s + v\xi_s\|_{K_{k+p+1}(\pi_H g)}^{-(k+p+1)} dv \; ,$$

where $c_{p,k} = \text{Vol}(S^{k-1})/(k+p+1)$ is totally immaterial. Using that $\frac{d}{ds}\xi_s = -\theta_s$ and the triangle inequality for $\|\cdot\|_{K_{k+p+1}(\pi_H g)}$, we obtain:

$$\begin{aligned} |\langle \nabla_{Id} \log h_{k,p}, B \rangle| &= \left| \frac{d}{ds} \log h_{k,p}(U_s) \Big|_{s=0} \right| \leq (k+p+1) \sup_{v \in \mathbb{R}} \frac{\|\xi_0 - v\theta_0\|_{\hat{K}_{k+p+1}(\pi_H g)}}{\|\theta_0 + v\xi_0\|_{K_{k+p+1}(\pi_H g)}} \\ &\leq (k+p+1) \text{dist}(K_{k+p+1}(\pi_H g), B_2(H)) \sup_{v \in \mathbb{R}} \frac{|\xi_0 - v\theta_0|}{|\theta_0 + v\xi_0|} \\ &= (k+p+1) \text{dist}(K_{k+p+1}(\pi_H g), B_2(H)) \; , \end{aligned}$$

where we have used the fact that θ_0 and ξ_0 are orthogonal unit vectors in the last equality.

2.2.3 Type-3 movement

Finally, we analyze the most important movement type, which is responsible for a subspace of movements of dimension $(k-1)(n-k)$ (out of the $\dim G_{n,k} + \dim S^{k-1} = k(n-k) + (k-1)$ dimensional subspace of non-trivial movements).

Let $0 \neq B \in T_3$ generate a Type-3 movement $\{U_s\}$, and note that $U_s(\theta_0) = \theta_0$. Set $e_s^j := U_s(e^j)$, $j = 2, \ldots, k$, and note that $f^j := \frac{d}{ds} e_s^j|_{s=0} \in E_0^\perp$. Denote $F_0 := \text{span}\left\{f^2, \ldots, f^k\right\}$, and note that by slightly perturbing B if necessary, we may assume that F_0 is $k-1$ dimensional. Finally, set $H = E_0 \oplus F_0 \in G_{n,2k-1}$, and notice that H is invariant under U_s (since U_s is an isometry acting as the identity on the orthogonal complement). Consequently, $H = E_s \oplus F_s$, where $F_s := U_s(F_0)$, and so:

$$h_{k,p}(U_s) = \text{Vol}(S^{k-1}) \int_0^\infty \int_{F_s} t^{p+k-1} \pi_H g(t\theta_0 + y) dy dt \; .$$

Using the change of variables $y = zt$, we obtain (with $c_{p,k} = \text{Vol}(S^{k-1})/(2k-1+p)$):

$$h_{k,p}(U_s) = \text{Vol}(S^{k-1}) \int_0^\infty \int_{F_s} t^{p+2k-2} \pi_H g(t(\theta_0 + z)) dz dt = c_{p,k} \int_{F_s} \|\theta_0 + z\|_{K_{2k-1+p}(\pi_H g)}^{-(2k-1+p)} dz \; ,$$

which we rewrite, since U_s is orthogonal, as:

$$h_{k,p}(U_s) = c_{p,k} \int_{F_0} \|\theta_0 + U_s(z)\|_{K_{2k-1+p}(\pi_H g)}^{-(2k-1+p)} dz \ .$$

As usual, the triangle inequality for $\|\cdot\|_{K_{2k-1+p}(\pi_H g)}$ implies that:

$$|\langle \nabla_{Id} \log h_{k,p}, B\rangle| = \left| \frac{d}{ds} \log h_{k,p}(U_s)\Big|_{s=0} \right| \leq (2k-1+p) \sup_{z \in F_0} \frac{\|Bz\|_{\hat{K}_{2k-1+p}(\pi_H g)}}{\|\theta_0 + z\|_{K_{2k-1+p}(\pi_H g)}} \ ,$$

and so:

$$\begin{aligned} \frac{|\langle \nabla_{Id} \log h_{k,p}, B\rangle|}{(2k-1+p)\mathrm{dist}(K_{2k-1+p}(\pi_H g), B_2(H))} &\leq \sup_{z \in F_0} \frac{|Bz|}{|\theta_0 + z|} \leq \|B\|_{op} \sup_{z\in F_0} \frac{|z|}{\sqrt{1+|z|^2}} \\ &\leq \frac{\|B\|_{HS}}{\sqrt{2}} = |B| \ , \end{aligned}$$

where we have used that θ_0 is perpendicular to F_0, and that $\|B\|_{op} \leq \|B\|_{HS}/\sqrt{2}$ for any anti-symmetric matrix B (here $\|B\|_{op}$ denotes its operator norm), as may be easily verified by using the Cauchy–Schwarz inequality.

2.3 Distance of K_{m+p} to Euclidean ball

To conclude the proof of Theorem 2.1, it remains to control the geometric distance of $K_{m+p}(\pi_H g)$ to a Euclidean ball, for $H \in G_{n,m}$ with m of the order of k. To this end, we compare $K_{m+p}(\pi_H g)$ to $Z_p(\pi_H g) = P_H Z_p(g)$. Our motivation comes from the work of Paouris [29], who noted that:

$$Z_p(\pi_H g) = Z_p(K_{m+p}(\pi_H g)) \ ,$$

and using the trivial $Z_p(K) \subset conv(K \cup -K)$ for any set K of volume 1, obtained an upper bound on $\mathrm{Vol}(Z_p(\pi_H g))$ by bounding above $\mathrm{Vol}(K_{m+p}(\pi_H g))$. In this work, on the other hand, we take the converse path, passing from K_{m+p} bodies to Z_p ones, and consequently need to introduce the Z_p^+ bodies to handle non-even densities. Moreover, we require bounds on $Z_p^+(K)$ both from above and from below, which turn out to be more laborious in the non-even case (when K is not centrally-symmetric).

Since the distance to the Euclidean ball cannot increase under orthogonal projections, and since $c_1 Z_k^+(g) \subset c_2 Z_m^+(g) \subset c_3 Z_{2k-1}^+(g) \subset c_4 Z_k^+(g)$ when $k \leq m \leq 2k-1$ by (2.3), it remains to establish the following:

Theorem 2.5. *Let w denote a log-concave function on $\mathbb{R}^m$ with $0 < \int w < \infty$ and barycenter at the origin. Then for any $p \geq 1$:*

$$dist(K_{m+p}(w), B_2^m) \leq C\, dist(Z^+_{\max(p,m)}(w), B_2^m) \ .$$

For the proof, we recall several useful properties of the bodies $K_q(w)$ and $Z_q^+(K)$. First, it is known (see [4, 3, 26] for the even case and [17, Lemmas 2.5,2.6] or [30, Lemma 3.2 and (3.12)] for the general one) that under the assumptions of Theorem 2.5:

$$1 \leq q_1 \leq q_2 \quad \Rightarrow \quad e^{-m(\frac{1}{q_1}-\frac{1}{q_2})} \frac{K_{q_1}(w)}{w(0)^{1/q_1}} \subset \frac{K_{q_2}(w)}{w(0)^{1/q_2}} \subset \frac{\Gamma(q_2+1)^{1/q_2}}{\Gamma(q_1+1)^{1/q_1}} \frac{K_{q_1}(w)}{w(0)^{1/q_1}} \ . \tag{2.7}$$

Second, integration in polar coordinates (cf. [29]) directly shows that:

$$Z_p^+(K_{m+p}(w)) = Z_p^+(w) \ . \tag{2.8}$$

Lastly, we require the following proposition, which is well-known in the even-case (e.g. [28, Lemma 4.1]), but requires more work in the general one (note for instance that the barycenter of $K_{m+p}(w)$ below need not be at the origin); its proof is postponed to the Appendix.

Proposition 2.6.

$$C_1 Z_p^+(K_{m+p}(w)) \subset \mathit{Vol}(K_{m+p}(w))^{1/p} K_{m+p}(w) \subset C_2 Z_p^+(K_{m+p}(w)) \left(\frac{\Gamma(m+p+1)}{\Gamma(m)\Gamma(p+1)}\right)^{1/p} \tag{2.9}$$

Proof of Theorem 2.5. Note that by Stirling's formula, (2.9) with (2.8) implies that:

$$\mathrm{dist}(K_{m+p}(w), B_2^m) \leq C \frac{p+m}{p} \mathrm{dist}(Z_p^+(w), B_2^m) \ ,$$

and so when $p \geq m$ the asserted claim follows. Otherwise, using (2.7), Stirling's formula, (2.9) and (2.8), we see that if $q \geq p$ then:

$$\mathrm{dist}(K_{m+p}(w), B_2^m) \leq C_1 \frac{m+q}{m+p} \mathrm{dist}(K_{m+q}(w), B_2^m) \leq C_2 \frac{m+q}{m+p} \frac{m+q}{q} \mathrm{dist}(Z_q^+(w), B_2^m) \ .$$

Setting $q = m$, the case $p < m$ is also resolved. □

The proof of Theorem 2.1 is now complete.

3 Moment Estimates

In this section we provide a complete proof of Theorem 1.2.

3.1 Reductions

Given X as in Theorem 1.2, set $Y := (X + G_n)/\sqrt{2}$, where G_n is an independent standard Gaussian random vector in $\mathbb{R}^n$. Note that Y is centrally-symmetric and isotropic, and by the Prékopa–Leindler Theorem, has log-concave density.

We repeat the argument of Fleury for reducing the moment estimation problem from X to Y and for passing from integration on $\mathbb{R}^n$ to $SO(n)$. By the symmetry and independence of G_n, convexity of $t \mapsto t^p$ and the Cauchy–Schwarz inequality, we have:

$$\mathbb{E}|Y|^{2p} = E\left(\frac{|X+G_n|^2}{2}\right)^p = \frac{1}{2}\mathbb{E}\left(\left(\frac{|X+G_n|^2}{2}\right)^p + \left(\frac{|X-G_n|^2}{2}\right)^p\right)$$
$$\geq \mathbb{E}\left(\frac{|X|^2+|G_n|^2}{2}\right)^p \geq \mathbb{E}|X|^p|G_n|^p = \mathbb{E}|X|^p\mathbb{E}|G_n|^p \geq \mathbb{E}|X|^p(\mathbb{E}|G_n|^2)^{p/2} = n^{p/2}\mathbb{E}|X|^p \ .$$

Since $\mathbb{E}|X|^2 = \mathbb{E}|Y|^2 = n$, we deduce:

$$\frac{(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|X|^2)^{1/2}} \leq \left(\frac{(\mathbb{E}|Y|^{2p})^{1/2p}}{(\mathbb{E}|Y|^2)^{1/2}}\right)^2 \ , \tag{3.1}$$

and it remains to obtain (1.9) with X replaced by Y, with an obvious modification of the constants.

Next, since $|x|^p = a_{n,k,p}\mathbb{E}_F|P_F x|^p$, where F is uniformly distributed on $G_{n,k}$ (according to its Haar probability measure), with k to be determined later on, we have:

$$\frac{\mathbb{E}|Y|^p}{\mathbb{E}|G_n|^p} = \frac{\mathbb{E}\mathbb{E}_F|P_F Y|^p}{\mathbb{E}\mathbb{E}_F|P_F G_n|^p} = \frac{\mathbb{E}\mathbb{E}_F|P_F Y|^p}{\mathbb{E}|G_k|^p} \ ,$$

where G_i denotes a standard Gaussian random vector on $\mathbb{R}^i$. A direct calculation shows that:

$$\mathbb{E}|G_i|^p = 2^{p/2-1}\frac{\Gamma((p+i)/2)}{\Gamma(i/2)} \ ,$$

and hence:

$$\mathbb{E}|Y|^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\mathbb{E}\mathbb{E}_F|P_F Y|^p \ .$$

Passing to polar coordinates on $F \in G_{n,k}$ and using the invariance of the Haar measures on $G_{n,k}$, $S(F)$ and $SO(n)$ under the action of $SO(n)$, we verify that:

$$\mathbb{E}\mathbb{E}_F|P_F Y|^p = \mathbb{E}_U h_{k,p}(U) \ ,$$

where U is uniformly distributed on $SO(n)$.

3.2 Controlling the derivative

We now deviate from Fleury's argument and proceed to estimate:

$$\frac{d}{dp}\log((\mathbb{E}|Y|^p)^{\frac{1}{p}}) = \frac{d}{dp}\log((\mathbb{E}_U h_{k,p}(U))^{\frac{1}{p}}) + \frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\right) \ . \tag{3.2}$$

Given $u \in SO(n)$, we introduce the (non-probability) measure μ_u on $\mathbb{R}_+$ having density $\mathrm{Vol}(S^{k-1})t^{k-1}\pi_{u(F_0)}g(tu(\theta_0))$, where g is the density of Y on $\mathbb{R}^n$. We define the (probability) measure $\mu_{k,p} := \mathbb{E}_U\mu_U$ on $\mathbb{R}_+$, and write:

$$h_{k,p}(u) = \mathbb{E}_{\mu_u}(t^p) \ , \ \mathbb{E}_U h_{k,p}(U) = \mathbb{E}_U\mathbb{E}_{\mu_U}(t^p) = \mathbb{E}_{\mu_{k,p}}(t^p) \ .$$

Here and in the sequel we use the following convention: given a measure space (Ω, μ), which does not necessarily have total mass 1, and a measurable $f : \Omega \to \mathbb{R}_+$, we set:

$$\mathbb{E}_\mu f = \mathbb{E}_\mu(f) = \int f d\mu \quad , \quad \mathrm{Ent}_\mu(f) = \mathbb{E}_\mu(f \log f) - \mathbb{E}_\mu(f) \log(\mathbb{E}_\mu(f)) \ .$$

A useful fact, easily verified by direct calculation, is that:

$$\frac{d}{dp} \log((\mathbb{E}_\mu f^p)^{\frac{1}{p}}) = \frac{1}{p^2} \frac{\mathrm{Ent}_\mu(f^p)}{\mathbb{E}_\mu(f^p)} \ .$$

We proceed with estimating (3.2). As explained:

$$\frac{d}{dp} \log((\mathbb{E}_U h_{k,p}(U))^{\frac{1}{p}}) = \frac{1}{p^2} \frac{\mathrm{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}_{\mu_{k,p}}(t^p)} = \frac{1}{p^2} \frac{\mathrm{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}_U h_{k,p}(U)} \ . \tag{3.3}$$

Our main idea here is to decompose the numerator as follows:

$$\mathrm{Ent}_{\mu_{k,p}}(t^p) = \mathbb{E}_U \mathrm{Ent}_{\mu_U}(t^p) + \mathrm{Ent}_U \mathbb{E}_{\mu_U}(t^p) = \mathbb{E}_U \mathrm{Ent}_{\mu_U}(t^p) + \mathrm{Ent}_U h_{k,p}(U) \ . \tag{3.4}$$

The contribution of the second term in (3.4) is controlled using the log-Sobolev inequality (1.15):

$$\frac{1}{p^2} \frac{\mathrm{Ent}_U h_{k,p}(U)}{\mathbb{E}_U h_{k,p}(U)} \leq \frac{c}{p^2 n} \frac{\mathbb{E}_U(|\nabla \log h_{k,p}|^2(U) h_{k,p}(U))}{\mathbb{E}_U h_{k,p}(U)} \leq \frac{c L_{k,p}^2}{p^2 n} \ , \tag{3.5}$$

where recall $L_{k,p}$ denotes the log-Lipschitz constant of $u \mapsto h_{k,p}(u)$. To control the contribution of the first term in (3.4), we first write given $u \in SO(n)$:

$$\frac{1}{p^2} \frac{\mathrm{Ent}_{\mu_u}(t^p)}{\mathbb{E}_{\mu_u}(t^p)} = \frac{d}{dp} \log((\mathbb{E}_{\mu_u} t^p)^{\frac{1}{p}}) = \frac{d}{dp} \frac{1}{p} \left(\log \frac{h_{k,p}(u)}{\Gamma(k+p)} - \log \frac{h_{k,0}(u)}{\Gamma(k)} + \log \frac{\Gamma(k+p)}{\Gamma(k)} + \log h_{k,0}(u) \right) \ .$$

By Borell's concavity result (1.14), we realize that:

$$\frac{d}{dp} \frac{1}{p} \left(\log \frac{h_{k,p}(u)}{\Gamma(k+p)} - \log \frac{h_{k,0}(u)}{\Gamma(k)} \right) \leq 0 \ ,$$

and hence:

$$\frac{1}{p^2} \frac{\mathrm{Ent}_{\mu_u}(t^p)}{E_{\mu_u}(t^p)} \leq \frac{d}{dp} \left(\frac{1}{p} \log \frac{\Gamma(k+p)}{\Gamma(k)} \right) - \frac{1}{p^2} \log h_{k,0}(u) \ .$$

Plugging this estimate back into (3.3) and (3.4), we obtain:

$$\frac{1}{p^2} \frac{\mathbb{E}_U \mathrm{Ent}_{\mu_U}(t^p)}{\mathbb{E}_U \mathbb{E}_{\mu_U}(t^p)} \leq \frac{d}{dp} \left(\frac{1}{p} \log \frac{\Gamma(k+p)}{\Gamma(k)} \right) + \frac{1}{p^2} \frac{\mathbb{E}_U \log(1/h_{k,0}(U)) h_{k,p}(U)}{\mathbb{E}_U h_{k,p}(U)} \ . \tag{3.6}$$

By using the Jensen and Cauchy–Schwarz inequalities, we bound the second term by:

$$\frac{\mathbb{E}_U \log(1/h_{k,0}(U)) h_{k,p}(U)}{\mathbb{E}_U h_{k,p}(U)} \leq \log \left(\frac{\mathbb{E}_U \frac{h_{k,p}(U)}{h_{k,0}(U)}}{\mathbb{E}_U h_{k,p}(U)} \right) \leq \log \left(\frac{(\mathbb{E}_U h_{k,p}(U)^2)^{1/2}}{\mathbb{E}_U h_{k,p}(U)} (\mathbb{E}_U h_{k,0}(U)^{-2})^{1/2} \right) \ .$$

We now use the reverse Hölder inequality (1.16) for comparing the various moments above. Denoting $\|f\|_q := (\mathbb{E}_U|f(U)|^q)^{1/q}$, we have:

$$\|h_{k,p}\|_2 \leq \exp\left(\frac{CL_{k,p}^2}{n}\right)\|h_{k,p}\|_1 \ ,$$

$$\left\|h_{k,0}^{-1}\right\|_2 \leq \exp\left(\frac{2CL_{k,0}^2}{n}\right)\left\|h_{k,0}^{-1}\right\|_0 = \exp\left(\frac{2CL_{k,0}^2}{n}\right)\frac{1}{\|h_{k,0}\|_0} \leq \exp\left(\frac{3CL_{k,0}^2}{n}\right)\frac{1}{\|h_{k,0}\|_1} \ ,$$

where $\|f\|_0$ is as usual interpreted as $\exp(\mathbb{E}_U \log|f(U)|)$. Since $\|h_{k,0}\|_1 = \mathbb{E}_U h_{k,0}(U) = \mathbb{E}_{\mu_{k,p}}(1) = 1$, we conclude that:

$$\frac{\mathbb{E}_U \log(1/h_{k,0}(U))h_{k,p}(U)}{\mathbb{E}_U h_{k,p}(U)} \leq \frac{C}{p^2 n}(L_{k,p}^2 + 3L_{k,0}^2) \ . \tag{3.7}$$

Now, plugging all the estimates (3.5), (3.6), (3.7) into (3.3) using the decomposition (3.4), and plugging the result into (3.2), we obtain:

$$\frac{d}{dp}\log((\mathbb{E}|Y|^p)^{\frac{1}{p}}) \leq \frac{c}{p^2 n}(2L_{k,p}^2+3L_{k,0}^2)+\frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma(k+p)}{\Gamma(k)}\right)+\frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\right) \ .$$

3.3 Optimizing on the dimension

As observed by Fleury, using that the function $\frac{d}{dp}\log\Gamma(p)$ is concave, the contribution of the last term above is easily verified to be non-positive and moreover insignificant relative to the second term, so we just bound it from above by 0. For the second term, we estimate using Jensen's inequality, for any $q > 0$:

$$\frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma(k+p)}{\Gamma(k)}\right) = \frac{1}{pq}\frac{\int_0^\infty \log(t^q)t^{p+k-1}\exp(-t)dt}{\Gamma(p+k)} - \frac{1}{p^2}\log\frac{\Gamma(k+p)}{\Gamma(k)}$$
$$\leq \frac{1}{pq}\log\frac{\Gamma(k+p+q)}{\Gamma(k+p)} - \frac{1}{p^2}\log\frac{\Gamma(k+p)}{\Gamma(k)} = \frac{1}{p}\log\left(\frac{\Gamma(k+p+q)^{1/q}}{\Gamma(k+p)^{1/q}}\frac{\Gamma(k)^{1/p}}{\Gamma(k+p)^{1/p}}\right) \ .$$

Applying Stirling's formula and setting $q = (p+k-1)\frac{p}{k-1}$, one verifies that:

$$\frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma(k+p)}{\Gamma(k)}\right) \leq \frac{C}{k} \ .$$

Plugging our estimates for $L_{k,q}$ obtained in Corollary 2.4, we conclude that if X is ψ_α ($\alpha \in [1,2]$), then:

$$\frac{d}{dp}\log((\mathbb{E}|Y|^p)^{\frac{1}{p}}) \leq C\left(\frac{k^{1+2/\alpha}}{p^2 n} + \frac{1}{k}\right) \quad \forall k \in [p,n] \ .$$

Optimizing on k in the above range, we set:

$$k = \lceil p^{1/\beta} n^{1/(2\beta)} \rceil \ , \ \beta := 1 + \frac{1}{\alpha} \ ,$$

which is guaranteed to satisfy $k \in [p, n]$ whenever $2 \leq p \leq n^{\alpha/2}$, and obtain for such p:

$$\frac{d}{dp} \log((E|Y|^p)^{\frac{1}{p}}) \leq \frac{C_2}{p^{1/\beta} n^{1/(2\beta)}} \ .$$

Integrating over p, we obtain for p in that range:

$$(\mathbb{E}|Y|^p)^{\frac{1}{p}} \leq \exp\left(C_3 \frac{p^{\frac{1}{\alpha+1}}}{n^{\frac{\alpha}{2(\alpha+1)}}} \right) (\mathbb{E}|Y|^2)^{\frac{1}{2}} \ ,$$

and together with the reduction (3.1) from X to Y, the conclusion of Theorem 1.2 follows.

4 Deviation Estimates

Obtaining the deviation estimates of Theorem 1.1 from the moment estimates of Theorem 1.2 is completely standard, exactly as in [12]. For completeness, we provide a brief description.

Proof of Theorem 1.1. Set:

$$\varepsilon_{n,\alpha} := \min\left(1, \frac{2^{\frac{\alpha+2}{\alpha+1}} C}{n^{\frac{\alpha}{2(\alpha+1)}}}\right) \ ,$$

and note that there exists a constant $t_0 \in (0, 1]$, so that:

$$\forall t \in [\varepsilon_{n,\alpha}, t_0] \quad \exists p \in [2, cn^{\alpha/2}] \quad \text{such that} \quad t = 2C \frac{p^{\frac{1}{\alpha+1}}}{n^{\frac{\alpha}{2(\alpha+1)}}} \ . \tag{4.1}$$

Here $c, C > 0$ are the two constants appearing in Theorem 1.2, which guarantee that for p in the above range:

$$(\mathbb{E}|X|^p)^{1/p} \leq \left(1 + \frac{t}{2}\right) \sqrt{n} \ .$$

Since $\frac{1+t}{1+t/2} \geq 1 + t/3$ for $t \in [0, 1]$, we obtain by the Markov–Chebyshev inequality:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \mathbb{P}(|X| \geq (1+t/3)(\mathbb{E}|X|^p)^{1/p}) \leq (1+t/3)^{-p} \leq \exp(-pt/4) \ .$$

Expressing p as a function of t, for t in the range specified in (4.1), and plugging this above, we obtain:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-c_1 n^{\alpha/2} t^{2+\alpha}) \quad \forall t \in [\varepsilon_{n,\alpha}, t_0] \ .$$

To extend this estimate to the entire interval $[0, t_0]$, note that:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq (1+t)^{-2} \leq \exp(-t/2) \quad \forall t \in [0, \varepsilon_{n,\alpha}] \ ,$$

and so adjusting the constants appearing above:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-c_2 n^{\alpha/2} t^{2+\alpha}) \quad \forall t \in [0, t_0] \ .$$

Finally, a standard application of Borell's lemma [10] (e.g. as in [29]), ensures that:

$$\mathbb{P}(|X| \geq (1+t)\sqrt{n}) \leq \exp(-c_3 n^{\alpha/2} t) \quad \forall t \geq t_0 \ ,$$

concluding the proof of the positive deviation estimate (1.6).

For the proof of the negative deviation estimate (1.7), observe that there exists a constant $c_4 > 0$, so that setting $p_0 := c_4 n^{\frac{\alpha}{2(\alpha+2)}}$, Theorem 1.2 implies that:

$$\mathbb{E}|X|^{2p_0} \leq \left(1 + C\frac{p_0^{\frac{1}{\alpha+1}}}{n^{\frac{\alpha}{2(\alpha+1)}}}\right)^{2p_0} (\mathbb{E}|X|^2)^{p_0} \leq \frac{17}{16}(\mathbb{E}|X|^2)^{p_0} \leq \frac{17}{16}(\mathbb{E}|X|^{p_0})^2 \ .$$

Consequently $\mathbb{V}\mathrm{ar}|X|^{p_0} \leq \frac{1}{16}(\mathbb{E}|X|^{p_0})^2$, and Chebyshev's inequality implies:

$$\begin{aligned} \frac{1}{4} \ &\geq \ \mathbb{P}\left(||X|^{p_0} - \mathbb{E}|X|^{p_0}| \geq \frac{1}{2}\mathbb{E}|X|^{p_0}\right) \\ &\geq \ \mathbb{P}\left(|X| \geq \frac{1}{2^{1/p_0}}(\mathbb{E}|X|^{p_0})^{1/p_0}\right) \geq \mathbb{P}\left(|X| \leq \left(1 - \frac{c_5}{p_0}\right)\sqrt{n}\right) \ . \end{aligned} \tag{4.2}$$

On the other hand, the positive deviation estimate (1.6) implies that:

$$\mathbb{P}\left(|X| \leq \left(1 + \frac{c_6}{p_0}\right)\sqrt{n}\right) \geq \frac{3}{4} \ . \tag{4.3}$$

Setting $t_1 := \frac{\max(c_5, c_6)}{p_0}$ and given $t \in (t_1, 1)$, we set $\lambda := \frac{2t_1}{t+t_1} \in (0,1)$ so that:

$$(1 - t_1) = \lambda(1-t) + (1-\lambda)(1+t_1) \ ,$$

and by the log-concavity of the function $\mathbb{R}_+ \ni s \mapsto \mathbb{P}(|X| \leq s)$ (a consequence of Prékopa–Leindler), it follows that:

$$\mathbb{P}\left(|X| \leq (1-t_1)\sqrt{n}\right) \geq \mathbb{P}\left(|X| \leq (1-t)\sqrt{n}\right)^{\lambda} \mathbb{P}\left(|X| \leq (1+t_1)\sqrt{n}\right)^{1-\lambda} \ .$$

Using the estimates (4.2) and (4.3), we deduce:

$$\mathbb{P}\left(|X| \leq (1-t)\sqrt{n}\right) \leq \left(\frac{1}{4}\right)^{1/\lambda} \left(\frac{4}{3}\right)^{1/\lambda - 1} \leq \frac{1}{4}\frac{1}{3^{1/\lambda}} \leq \exp(-c_7 p_0 t) \quad \forall t \in (t_1, 1) \ .$$

The negative deviation estimate (1.7) immediately follows.

Lastly, we observe that:

$$\sqrt{\mathbb{V}\mathrm{ar}|X|} \leq C\frac{n^{1/2}}{p_0} = C_2 n^{\frac{1}{2+\alpha}} \ ,$$

e.g. by integrating by parts and using the positive and negative deviation estimates (see e.g. [11, Lemma 6]). □

5 Concluding Remarks

Remark 5.1. Examining the proof, it is easy to verify that if the log-Lipschitz constant $L_{k,p}$ of $h_{k,p} : SO(n) \to \mathbb{R}_+$ satisfies:

$$p \le k \quad \Rightarrow \quad L_{k,p} \le C p^{\beta} k^{\gamma} \ , \ \beta, \gamma \in \mathbb{R} \ ,$$

then the sharp large-deviation estimate $\mathbb{P}(|X| \ge C\sqrt{n}) \le \exp(-\sqrt{n})$ is recovered if and only if $\beta + \gamma = 3/2$. Of course, since $p \le k$, it is better to have larger β, and this affects the resulting thin-shell estimate. Our estimates yield $\beta = 0$ and $\gamma = 3/2$.

Remark 5.2. Using a theorem of Bobkov [6], we improve the best-known bound on the Cheeger constant $D_{Che}(\mu)$ of a general log-concave isotropic measure μ in $\mathbb{R}^n$ to $D_{Che}(\mu) \ge c n^{-\frac{5}{12}}$, bringing us a little bit closer to the full KLS conjecture $D_{Che}(\mu) \ge c > 0$ (we refer to [6] for missing definitions and background). Note that the estimate improves all the way to $D_{Che}(\mu) \ge c n^{-\frac{3}{8}}$ when μ is ψ_2.

Appendix

In the Appendix, we prove several properties of the bodies $Z_p^+(K)$ which are needed for the results of Section 2.

Our main goal is to establish Proposition 2.6. Given $\theta \in S^{m-1}$, we denote $H_\theta^+ := \{x \in \mathbb{R}^m ; \langle x, \theta \rangle \ge 0\}$ and set $H_\theta^- := H_{-\theta}^+$. For the proof, we require several lemmas.

Lemma A.1. *Let K denote a convex body in $\mathbb{R}^m$, and given $\theta \in S^{m-1}$, denote $f_\theta = \pi_\theta \mathbf{1}_K$. Then:*

$$\left(\frac{f_\theta(0)}{\|f_\theta\|_\infty} \right)^{1/p} \left(\frac{\Gamma(m)\Gamma(p+1)}{\Gamma(m+p+1)} \right)^{1/p} h_K(\theta) \le \frac{h_{Z_p^+(K)}(\theta)}{(2\,Vol(K \cap H_\theta^+))^{1/p}} \le h_K(\theta) \ .$$

Proof. The right inequality is straightforward from the definitions. The left inequality is derived by following the proof of [28, Lemma 4.1], which uses the fact that the $1/(m-1)$ power of any one-dimensional marginal of K is a concave function. □

To control the left-most term in Lemma A.1, we have:

Lemma A.2. *Let $\mu = f(x)dx$ denote a log-concave probability measure on $\mathbb{R}$. Then for any $\varepsilon > 0$:*

$$\varepsilon \le \int_0^\infty f(x)dx \le 1 - \varepsilon \quad \Rightarrow f(0) \ge \varepsilon \, \|f\|_\infty \ .$$

Proof. Consider the function $\mathcal{I} : [0,1] \to \mathbb{R}_+$ given by $\mathcal{I}(v) = \min(f \circ F^{-1}(v), f \circ F^{-1}(1-v))$, where $F(x) = \int_{-\infty}^x f(t)dt$. By a result of Bobkov [5], $\mathcal{I}$ is the isoperimetric profile of the measure-metric space $(\mathbb{R}, |\cdot|, \mu)$ (see [5] for definitions), and furthermore, $\mathcal{I}$ is a concave and symmetric function on $[0,1]$. If $v = F(0)$ is such that $v \in [\varepsilon, 1-\varepsilon]$, it follows by the concavity of $\mathcal{I}$ that:

$$f(0) \geq \mathcal{I}(v) \geq 2\min(v, 1-v)\mathcal{I}(1/2) \geq 2\varepsilon f(m) \ , \tag{A.1}$$

where $m = F^{-1}(1/2)$ is the median of μ. But by [24, Lemma 2.7], $f(m) \geq \|f\|_\infty /2$, which together with (A.1) concludes the assertion. $\square$

This reduces our task to showing:

Lemma A.3. *If w is a log-concave function on $\mathbb{R}^m$ with barycenter at the origin, then:*

$$\forall \theta \in S^{m-1} \quad \frac{1}{C} \leq \left(\frac{Vol(K_{m+p}(w) \cap H_\theta^+)}{Vol(K_{m+p}(w) \cap H_\theta^-)} \right)^{1/p} \leq C \ .$$

Proof. Note that we may normalize and rescale so that $w(0) = 1$ and $\int_{\mathbb{R}^m} w(x)dx = 1$. Using polar-coordinates, we have for any convex (in fact, star-shaped) body K containing the origin:

$$\mathrm{Vol}(K \cap H_\theta^+) = \frac{1}{m} \int_{S^{m-1} \cap H_\theta^+} \|\xi\|_K^{-m} \, d\xi \ . \tag{A.2}$$

Using (2.7), we see that:

$$\forall \xi \in S^{m-1} \quad e^{-\frac{mp}{m+p}} \|\xi\|_{K_m(w)}^{-m} \leq \|\xi\|_{K_{m+p}(w)}^{-m} \leq \frac{\Gamma(m+p+1)^{\frac{m}{m+p}}}{\Gamma(m+1)} \|\xi\|_{K_m(w)}^{-m} \ .$$

Plugging this into (A.2) and using Stirling's formula, we verify that:

$$\forall \theta \in S^{m-1} \quad e^{-p} \leq \frac{\mathrm{Vol}(K_{m+p}(w) \cap H_\theta^+)}{\mathrm{Vol}(K_m(w) \cap H_\theta^+)} \leq C^p \ . \tag{A.3}$$

Using (A.2), the definition of $K_m(w)$ and polar-coordinates again, we see that $\mathrm{Vol}(K_m(w) \cap H_\theta^+) = \int_{H_\theta^+} w(x)dx = \mathbb{P}(W_1 \geq 0)$, where W_1 is the random variable on $span(\theta)$ having density $\pi_\theta w$. Since this density is log-concave by the Prékopa–Leindler theorem, and since the barycenter of W_1 is at the origin, Lemma 2.2 implies that:

$$\frac{1}{e-1} \leq \frac{\mathrm{Vol}(K_m(w) \cap H_\theta^+)}{\mathrm{Vol}(K_m(w) \cap H_\theta^-)} \leq e-1 \ .$$

Together with (A.3), this concludes the proof. $\square$

Corollary A.4. *With the same assumptions as in Lemma A.3:*

$$\forall \theta \in S^{m-1} \quad \frac{1}{C'} \leq \left(\frac{Vol(K_{m+p}(w) \cap H_\theta^+)}{Vol(K_{m+p}(w))} \right)^{1/p} \leq C' \ .$$

Proof of Proposition 2.6. Applying Lemma A.1 with $K = K_{m+p}(w)$ and using Corollary A.4, we obtain for all $\theta \in S^{m-1}$:

$$c \left(\frac{f_\theta(0)}{\|f_\theta\|_\infty} \right)^{1/p} \left(\frac{\Gamma(m)\Gamma(p+1)}{\Gamma(m+p+1)} \right)^{1/p} \leq \mathrm{Vol}(K_{m+p}(w))^{-1/p} \frac{h_{Z_p^+(K_{m+p}(w))}(\theta)}{h_{K_{m+p}(w)}(\theta)} \leq C \ .$$

Lemma A.2 together with Lemma A.3 imply that:

$$\forall \theta \in S^{m-1} \quad \left(\frac{f_\theta(0)}{\|f_\theta\|_\infty} \right)^{1/p} \geq c' > 0 \ ,$$

and hence:

$$c'' \left(\frac{\Gamma(m)\Gamma(p+1)}{\Gamma(m+p+1)} \right)^{1/p} K_{m+p}(w) \subset \mathrm{Vol}(K_{m+p}(w))^{-1/p} Z_p^+(K_{m+p}(w)) \subset C K_{m+p}(w) \ .$$

Rearranging terms, the assertion of Proposition 2.6 follows. □

Finally, we prove:

Lemma A.5. *If $g : \mathbb{R}^m \to \mathbb{R}_+$ is log-concave and isotropic then $Z_2^+(g) \supset c B_2^m$.*

Proof. Given $\theta \in S^{n-1}$, denote $g_0 := \pi_\theta g$; as usual, it is an isotropic log-concave probability density on $\mathbb{R}$. Comparing moments using (2.7) with $m = 1$, $q_1 = 1$ and $q_2 = 3$, we obtain:

$$3 \int_0^\infty t^2 g_0(t) dt \geq \frac{\left(\int_0^\infty g_0(t) dt \right)^3}{e^2 g_0(0)^2} \ .$$

Applying now the reverse comparison in both directions θ and $-\theta$ and summing, we obtain:

$$3 = 3 \int_{-\infty}^\infty t^2 g_0(t) dt \leq \frac{\Gamma(4)}{g_0(0)^2} \left(\left(\int_0^\infty g_0(t) dt \right)^3 + \left(\int_{-\infty}^0 g_0(t) dt \right)^3 \right) \ .$$

Combining these two estimates and using Lemma 2.2 to control $\int_0^\infty g_0(t)dt$, the assertion follows with e.g. $c = (3e^2(1 + (e-1)^3))^{-1/2}$. □

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