A GENERALIZATION OF THE WEAK AMENABILITY OF SOME BANACH ALGEBRA

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Abstract. Let A be a Banach algebra and A^{**} be the second dual of it. We show that by some new conditions, A is weakly amenable whenever A^{**} is weakly amenable. We will study this problem under generalization, that is, if $(n+2)-th$ dual of A, $A^{(n+2)}$, is $T-S-$weakly amenable, then $A^{(n)}$ is $T-S-$weakly amenable where T and S are continuous linear mappings from $A^{(n)}$ into $A^{(n)}$.

1. Preliminaries and Introduction

Let A be a Banach algebra and A^*, A^{**}, respectively, are the first and second dual of A. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle$ - $\langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$. Arens [1] has shown that given any Banach algebra A, there exist two algebra multiplications on the second dual of A which extend multiplication on A. In the following, we introduce both multiplication which are given in [13]. The first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'ob''\rangle.$$

for all $a, b \in A$ and $a' \in A^*$.

We say that A is Arens regular if both multiplications are equal. Let a'' and b'' be elements of A^{**}. By *Goldstine's* Theorem [6, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - lim_\alpha a_\alpha$ and $b'' = weak^* - lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$lim_\alpha lim_\beta \langle a', a_\alpha b_\beta\rangle = \langle a''b'', a'\rangle$$

and

$$lim_\beta lim_\alpha \langle a', a_\alpha b_\beta\rangle = \langle a''ob'', a'\rangle.$$

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Thus A is Arens regular if and only if for every $a' \in A^*$, we have

$$lim_\alpha lim_\beta \langle a', a_\alpha b_\beta\rangle = lim_\beta lim_\alpha \langle a', a_\alpha b_\beta\rangle.$$

For more detail see [6, 13, 15].

Let X be a Banach $A-bimodule$. A derivation from A into X is a bounded linear mapping $D : A \to X$ such that

$$D(xy) = xD(y) + D(x)y \ for\ all\ x,\ y \in A.$$

The space of continuous derivations from A into X is denoted by $Z^1(A, X)$.

Easy example of derivations are the inner derivations, which are given for each $x \in X$ by

$$\delta_x(a) = ax - xa \ for\ all\ a \in A.$$

The space of inner derivations from A into X is denoted by $N^1(A, X)$. The Banach algebra A is said to be a amenable, when for every Banach $A-bimodule$ X, the inner derivations are only derivations existing from A into X^*. It is clear that A is amenable if and only if $H^1(A, X^*) = Z^1(A, X^*)/N^1(A, X^*) = \{0\}$.

A Banach algebra A is said to be a weakly amenable, if every derivation from A into A^* is inner. Similarly, A is weakly amenable if and only if $H^1(A, A^*) = Z^1(A, A^*)/N^1(A, A^*) = \{0\}$.

Suppose that A is a Banach algebra and X is a Banach $A-bimodule$. According to [5, pp.27 and 28], X^{**} is a Banach $A^{**}-bimodule$, where A^{**} is equipped with the first Arens product.

Let $A^{(n)}$ and $X^{(n)}$ be $n-th\ dual$ of A and X, respectively. By [19, page 4132-4134], if $n \geq 0$ is an even number, then $X^{(n)}$ is a Banach $A^{(n)}-bimodule$. Then for $n \geq 2$, we define $X^{(n)}X^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for all $x^{(n)} \in X^{(n)}$, $x^{(n-1)} \in X^{(n-1)}$ and $a^{(n-2)} \in A^{(n-2)}$ we define

$$\langle x^{(n)}x^{(n-1)}, a^{(n-2)}\rangle = \langle x^{(n)}, x^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, then for $n \geq 1$, we define $X^{(n)}X^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $x^{(n)} \in X^{(n)}$, $x^{(n-1)} \in X^{(n-1)}$ and $a^{(n-1)} \in A^{(n-1)}$ we define

$$\langle x^{(n)}x^{(n-1)}, a^{(n-1)}\rangle = \langle x^{(n)}, x^{(n-1)}a^{(n-1)}\rangle.$$

If $n = 0$, we take $A^{(0)} = A$ and $X^{(0)} = X$.

Now let X be a Banach $A-bimodule$ and $D : A \to X$ be a derivation. A problem which is of interest is under what conditions D'' is again a derivation. In [14, 5.9], this problem has been studied for the spacial case $X = A$, and they showed that D'' is a derivation if and only if $D''(A^{**})A^{**} \subseteq A^*$. We study this problem in the generality, that is, if $A^{(n+2)}$ is $T-S-$weakly amenable, then it follows that $A^{(n)}$ is $T-S-$ weakly amenable where T and S are continuous linear mapping from $A^{(n)}$ into $A^{(n)}$ and $n \geq 0$.

The main results of this paper can be summarized as follows:

a) Assume that A is a Banach algebra and $A^{(n+2)}$ has $T-w^*w$ property. If $A^{(n+2)}$ is weakly $T''-S''-$amenable, then $A^{(n)}$ is weakly $T-S-$amenable.

b) Let X be a Banach $A-bimodule$ and let $T,\ S : A^{(n)} \to A^{(n)}$ be continuous linear mappings. Let the mapping $a^{(n+2)} \to x^{(n+2)}T''(a^{(n+2)})$ be $weak^*$-to-$weak$ continuous for all $x^{(n+2)} \in X^{(n+2)}$. Then if $D : A^{(n)} \to X^{(n+1)}$ is a $T-S-derivation$, it follows that $D'' : A^{(n+2)} \to X^{(n+3)}$ is a $T''-S''-derivation$.

c) Let X be a Banach $A-bimodule$ and the mapping $a'' \to x''a''$ be $weak^*-to-weak$ continuous for all $x'' \in X^{**}$. If $D : A \to X^*$ is a derivation, then $D''(A^{**})X^{**} \subseteq A^*$.
d) Let X be a Banach $A - bimodule$ and $D : A \to X^*$ be a derivation. Suppose that $D'' : A^{**} \to X^{***}$ is surjective derivation. Then the mapping $a'' \to x''a''$ is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.
e) Suppose that X is a Banach $A - bimodule$ and A is Arens regular. Assume that $D : A \to X^*$ is a derivation and surjective. Then $D'' : A^{**} \to X^{***}$ is a derivation if and only if the mapping $a'' \to x''a''$ is $weak^*-to-weak$ continuous for all $x'' \in X^{**}$.
In every parts of this paper, $n \geq 0$ is even number.

2. **Weak amenability of Banach algebras**

Definition 2-1. Let X be a Banach $A - bimodule$ and T, S be continuous linear mappings from A into itself. We say that $D : A \to X$ is $T - S - derivation$, if

$$D(xy) = T(x)D(y) + D(x)S(y) \ for \ all \ x, \ y \in A.$$

Now let $x \in A$. Then we say that the linear mapping $\delta_x : A \to A$ is inner $T - S - derivation$, if for every $a \in A$ we have $\delta_x(a) = T(a)x - xS(a)$.
The Banach algebra A is said to be a $T - S-$amenable, when for every Banach $A-bimodule$ X, every $T-S-$derivations from A into X^* is inner $T-S-$derivations. The definition of weakly $T - S-$ amenable is similar.

Definition 2-2. Assume that A is a Banach algebra and $T : A \to A$ is a continuous linear mapping such that the mapping $b'' \to a''T''(b'') \ : \ A^{**} \to A^{**} \ is \ weak^* - to - weak$ continuous where $a'' \in A^{**}$. Then we say that $a'' \in A^{**}$ has $T - w^*w$ property. We say that $B \subseteq A^{**}$ has $T - w^*w$ property, if every $b \in B$ has $T - w^*w$ property.

Let A be a Banach algebra and A^{**} has $I - w^*w$ property whenever $I : A \to A$ is the identity mapping. Then, obviously that A is Arens regular. There are some non-reflexive Banach algebras which the second dual of them have $T - w^*w$ property. If A is Arens regular, then, in general, A^{**} has not $I - w^*w$ property. In the following we give some examples from Banach algebras that the second dual of them have $T - w^*w$ property or no.

(1) Let A be a non-reflexive Banach space and suppose that $\langle f, x\rangle = 1$ for some $f \in A^*$ and $x \in A$. We define the product on A as $ab = \langle f, a\rangle b$ for all $a, \ b \in A$. It is clear that A is a Banach algebra with this product, then A^{**} has $I - w^*w$ property whenever $I : A \to A$ is the identity mapping.
(2) Every reflexive Banach algebra has $T - w^*w$ property.
(3) Consider the algebra $c_0 = (c_0, .)$ is the collection of all sequences of scalars that convergence to 0, with the some vector space operations and norm as ℓ_∞. Then $c_0^{**} = \ell_\infty$ has $I - w^*w$ property whenever $I : c_0 \to c_0$ is the identity mapping.
(4) $L^1(G)^{**}$ and $M(G)^{**}$ have not $I - w^*w$ property whenever G is locally compact group, but when G is finite, $L^1(G)^{**}$ and $M(G)^{**}$ have $I - w^*w$ property.

Theorem 2-3. Assume that A is a Banach algebra and $A^{(n+2)}$ has $T-w^*w$ property. If $D : A^{(n)} \rightarrow A^{(n+1)}$ is a $T-S-derivation$, then $D'' : A^{(n+2)} \rightarrow A^{(n+3)}$ is a $T''-S''-derivation$.

Proof. Let $a^{(n+2)},\ b^{(n+2)} \in A^{(n+2)}$ and let $(a_\alpha^{(n)})_\alpha,\ (b_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \stackrel{w^*}{\rightarrow} a^{(n+2)}$ and $b_\beta^{(n)} \stackrel{w^*}{\rightarrow} b^{(n+2)}$. Due to $A^{(n+2)}$ has $T-w^*w$ property, we have $c^{(n+2)}T(a_\alpha^{(n)}) \stackrel{w}{\rightarrow} c^{(n+2)}T''(a^{(n+2)})$ for all $c^{(n+2)} \in A^{(n+2)}$. Using the $weak^*-to-weak^*$ continuity of D'', we obtain

$$lim_\alpha lim_\beta \langle T(a_\alpha^{(n)})D(b_\beta^{(n)}), c^{(n+2)}\rangle = lim_\alpha lim_\beta \langle D(b_\beta^{(n)}), c^{(n+2)}T(a_\alpha^{(n)})\rangle$$

$$= lim_\alpha \langle D''(b^{(n+2)}), c^{(n+2)}T(a_\alpha^{(n)})\rangle = \langle D''(b^{(n+2)}), c^{(n+2)}T''(a^{(n+2)})\rangle$$

$$= \langle T''(a^{(n+2)})D''(b^{(n+2)}), c^{(n+2)}\rangle.$$

Moreover, it is also clear that for every $c^{(n+2)} \in A^{(n+2)}$, we have

$$lim_\alpha lim_\beta \langle D(a_\alpha^{(n)})S(b_\beta^{(n)}), c^{(n+2)}\rangle = \langle D''(a^{(n+2)})S''(b^{(n+2)}), c^{(n+2)}\rangle.$$

Notice that in latest equalities, we didn't need $S-w^*w$ property for $A^{(n+2)}$. In the following, we take limit on the $weak^*$ topologies. Thus we have

$$D''(a^{(n+2)}b^{(n+2)}) = lim_\alpha lim_\beta D(a_\alpha^{(n)}b_\beta^{(n)}) = lim_\alpha lim_\beta T(a_\alpha^{(n)})D(b_\beta^{(n)}) +$$

$$lim_\alpha lim_\beta D(a_\alpha^{(n)})S(b_\beta^{(n)}) = T''(a^{(n+2)})D''(b^{(n+2)}) + D''(a^{(n+2)})S''(b^{(n+2)}).$$

□

Theorem 2-4. Assume that A is a Banach algebra and $A^{(n+2)}$ has $T-w^*w$ property. If $A^{(n+2)}$ is weakly $T''-S''-$amenable, then $A^{(n)}$ is weakly $T-S-$amenable.

Proof. Let $D : A^{(n)} \rightarrow A^{(n+1)}$ is a $T-S-derivation$, then by Theorem 2-3, $D'' : A^{(n+2)} \rightarrow A^{(n+3)}$ is a $T''-S''-derivation$. Since $A^{(n+2)}$ is weakly $T''-S''-$amenable, $D'' : A^{(n+2)} \rightarrow A^{(n+3)}$ is an inner $T''-S''-$ *derivation*. It follows that for every $a^{(n+2)} \in A^{(n+2)}$, we have

$$D''(a^{(n+2)}) = T''(a^{(n+2)})a^{(n+3)} - a^{(n+3)}S''(a^{(n+2)}).$$

for some $a^{(n+3)} \in A^{(n+3)}$. Take $a^{(n+1)} = a^{(n+3)} \mid_{A^{(n+1)}}$. Then for every $a^{(n)} \in A^{(n)}$, we have

$$D(a^{(n)}) = T(a^{(n)})a^{(n+1)} - a^{(n+1)}S(a^{(n)}).$$

It follows that D is inner $T-S-$derivation, and so proof is hold.

□

Corollary 2-5. Let A be a Banach algebra and $I : A \rightarrow A$ be identity mapping. If A^{**} has $I-w^*w$ property and A^{**} is weakly amenable, then A is weakly amenable.

Corollary 2-6. Let A be a Banach algebra. If $A^{***}A^{**} \subseteq A^*$ and A^{**} is weakly amenable, then A is weakly amenable.

Proof. We show that A^{**} has $I-w^*w$ property where $I : A \to A$ is identity mapping. Suppose that $a'',\ b'' \in A^{**}$ and $b''_\alpha \stackrel{w^*}{\to} b''$. Let $c''' \in A^{***}$. Since $c'''a'' \in A^*$, we have

$$\langle c''', a''b''_\alpha\rangle = \langle c'''a'', b''_\alpha\rangle = \langle b''_\alpha, c'''a''\rangle \to \langle b'', c'''a''\rangle = \langle c''', a''b''\rangle.$$

We conclude that $a''b''_\alpha \stackrel{w}{\to} a''b''$. So A^{**} has $I-w^*w$ property. By using Corollary 2-5, A is weakly amenable.

□

Example 2-7. c_0 is weakly amenable.

Proof. Since $\ell^\infty = c_0^{**}$ is weakly amenable and ℓ^∞ has $I-w^*w$ property by Corollary 2-5, proof is hold.

□

Theorem 2-8. Suppose that A is a Banach algebra and B is a closed subalgebra of $A^{(n+2)}$ that is consisting of $A^{(n)}$ where $n \in \mathbb{N} \cup \{0\}$. If B has $T-w^*w$ property and is weakly $T''-S''-$amenable, then $A^{(n)}$ is weakly $T-S-$amenable.

Proof. Suppose that $D : A^{(n)} \to A^{(n+1)}$ is a $T-S-derivation$ and $p : A^{(n+3)} \to B'$ is the restriction map, defined by $P(a^{(n+3)}) = a^{(n+3)} \mid_{B'}$ for every $a^{(n+3)} \in A^{(n+3)}$. Since B has $T-w^*w$ property, $\bar{D} = PoD'' \mid_B : B \to B'$ is a $T''-S''-derivation$. Since B is weakly $T''-S''-$amenable, there is $b' \in B'$ such that $\bar{D} = \delta_{b'}$. We take $a^{(n+1)} = b' \mid_{A^{(n+1)}}$, then $D = \bar{D}$ on $A^{(n+1)}$. Consequently, we have $D = \delta_{a^{(n+1)}}$.

□

Theorem 2-9. Let X be a Banach $A-bimodule$ and let $T,\ S : A^{(n)} \to A^{(n)}$ be continuous linear mappings. Let the mapping $a^{(n+2)} \to x^{(n+2)}T''(a^{(n+2)})$ be $weak^*$-to-$weak$ continuous for all $x^{(n+2)} \in X^{(n+2)}$. If $D : A^{(n)} \to X^{(n+1)}$ is a $T-S-derivation$, then $D'' : A^{(n+2)} \to X^{(n+3)}$ is a $T''-S''-derivation$.

Proof. Let $a^{(n+2)},\ b^{(n+2)} \in A^{(n+2)}$ and let $(a_\alpha^{(n)})_\alpha,\ (b_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $a_\alpha^{(n)} \stackrel{w^*}{\to} a^{(n+2)}$ and $b_\beta^{(n)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for all $x^{(n+2)} \in X^{(n+2)}$, we have $x^{(n+2)}T(a_\alpha^{(n)}) \stackrel{w}{\to} x^{(n+2)}a^{(n+2)}$. Consequently, we have

$$lim_\alpha lim_\beta \langle T(a_\alpha^{(n)})D(b_\beta^{(n)}), x^{(n+2)}\rangle = lim_\alpha lim_\beta \langle D(b_\beta^{(n)}), x^{(n+2)}T(a_\alpha^{(n)})\rangle$$
$$= lim_\alpha \langle D''(b^{(n+2)}), x^{(n+2)}T(a_\alpha^{(n)})\rangle = \langle D''(b^{(n+2)}), x^{(n+2)}T(a^{(n+2)})\rangle$$
$$= \langle T(a^{(n+2)})D''(b^{(n+2)}), x^{(n+2)}\rangle.$$

For every $x^{(n+2)} \in X^{(n+2)}$, we have also the following equalities

$$lim_\alpha lim_\beta \langle D(a_\alpha^{(n)})S(b_\beta^{(n)}), x^{(n+2)}\rangle = lim_\alpha lim_\beta \langle D(a_\alpha^{(n)}), S(b_\beta^{(n)})x^{(n+2)}\rangle$$
$$= lim_\alpha \langle D(a_\alpha^{(n)}), S(b^{(n+2)})x^{(n+2)}\rangle = \langle D''(a^{(n+2)}), S(b^{(n+2)})x^{(n+2)}\rangle$$
$$= \langle D''(a^{(n+2)})S(b^{(n+2)}), x^{(n+2)}\rangle.$$

In the following, we take limit on the $weak^*$ topologies. Using the $weak^*-to-weak^*$ continuity of D'', we obtain

$$D''(a^{(n+2)}b^{(n+2)}) = lim_\alpha lim_\beta D(a_\alpha^{(n)}b_\beta^{(n)}) = lim_\alpha lim_\beta T(a_\alpha^{(n)})D(b_\beta^{(n)}) +$$
$$lim_\alpha lim_\beta D(a_\alpha^{(n)})S(b_\beta^{(n)}) = T''(a^{(n+2)})D''(b^{(n+2)}) + D''(a^{(n+2)})S''(b^{(n+2)}).$$

Thus $D'' : A^{(n+2)} \to X^{(n+3)}$ is a $T'' - S'' - derivation$.

□

Corollary 2-10. Let X be a Banach $A - bimodule$ and the mapping $a'' \to x''a''$ be $weak^* - to - weak$ continuous for all $x'' \in X^{**}$. Then, if $H^1(A^{**}, X^{***}) = 0$, it follows that $H^1(A, X^*) = 0$.

Corollary 2-11. Let X be a Banach $A - bimodule$ and the mapping $a'' \to x''a''$ be $weak^* - to - weak$ continuous for all $x'' \in X^{**}$. If $D : A \to X^*$ is a derivation, then $D''(A^{**})X^{**} \subseteq A^*$.

Proof. By using Theorem 2-9 and [14, Corollary 4-3], proof is hold.

□

Theorem 2-12. Let X be a Banach $A - bimodule$ and $D : A \to X^*$ be a surjective derivation. Suppose that $D'' : A^{**} \to X^{***}$ is also a derivation. Then the mapping $a'' \to x''a''$ is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.

Proof. Let $a'' \in A^{**}$ such that $a''_\alpha \stackrel{w^*}{\to} a''$. We show that $x''a''_\alpha \stackrel{w}{\to} x''a''$ for all $x'' \in X^{**}$. Suppose that $x''' \in X^{***}$. Since $D''(A^{**}) = X^{***}$, by using [14, Corollary 4-3], we conclude that $X^{***}X^{**} = D''(A^{**})X^{**} \subseteq A^*$. Then $x'''x'' \in A^*$, and so we have the following equality

$$\langle x''', x''a''_\alpha \rangle = \langle x'''x'', a''_\alpha \rangle = \langle a''_\alpha, x'''x'' \rangle \to \langle a'', x'''x'' \rangle = \langle x''', x''a'' \rangle.$$

□

Corollary 2-13. Suppose that X is a Banach $A - bimodule$ and A is Arens regular. Assume that $D : A \to X^*$ is a surjective derivation. Then $D'' : A^{**} \to X^{***}$ is a derivation if and only if the mapping $a'' \to x''a''$ from A^{**} into X^{**} is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.

Proof. By using Corollary 2-11, Theorem 2-12 and [14, Corollary 4-3], proof is hold.

□

In the proceeding Corollary, if we omit the Arens regularity of A, then we have also the following conclusion.
Assume that $D : A \to X^*$ is a surjective derivation. Then, $D''(A^{**})X^{**} \subseteq A^*$ if and only if the mapping $a'' \to x''a''$ is $weak^* - to - weak$ continuous for all $x'' \in X^{**}$.

Corollary 2-14. Let A be a Banach algebra. Then we have the following results:

(1) Assume that A is Arens regular and $D : A \to A^*$ is a surjective derivation. Then $D'' : A^{**} \to A^{***}$ is a derivation if and only if A has $I - w^*w$ property whenever $I : A \to A$ is the identity mapping.

(2) Assume that $D : A \to A^*$ is a surjective derivation. Then, A has $I - w^*w$ property if and only if $D''(A^{**})A^{**} \subseteq A^*$. So it is clear that if $D : A \to A^*$ is a surjective derivation and $D''(A^{**})A^{**} \subseteq A^*$, then A is Arens regular.

Problem. Let S be a semigroup. Dose $C(S)^{**}$, $L^1(S)^{**}$ and $M(S)^{**}$ have $I - w^*w$ property? whenever I is the identity mapping.

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