Constructive diagonalization of an element X of the Jordan algebra $\mathfrak{J}$ by the exceptional group F_4

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I. We know that any element X of the exceptional Jordan algebra $\mathfrak{J}$ is transformed to a diagonal form by the compact exceptional Lie group F_4. However, its proof is used the method which is reduced a contradiction. In this paper, we give a direct and constructive proof.

Let $\boldsymbol{H}$ be the field of quaternions and $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4, {e_4}^2 = -1$ the division Cayley algebra. For $K = \boldsymbol{H}, \mathfrak{C}$, let $\mathfrak{J}(3, K) = \{X \in M(3, K) \,|\, X^* = X\}$ with the Jordan multiplicaton $X \circ Y$, the inner product (X, Y) and the Freudenthal multiplication $X \times Y$ espectively by

$$X \circ Y = \frac{1}{2}(XY + YX), \quad (X, Y) = \mathrm{tr}(X \circ Y),$$
$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E)$$

(where E is the 3×3 unit matrix).

The simply connected compact Lie group F_4 is defined by

$$\begin{aligned} F_4 &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}(3, \mathfrak{C})) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}(3, \mathfrak{C})) \,|\, \alpha(X \times Y) = \alpha X \times \alpha Y\}. \end{aligned}$$

Then, we have the following Theorem ([1],[2]).

Theorem 1. *Any element X of $\mathfrak{J}(3, \mathfrak{C})$ can be transformed to a diagonal form by some element $\alpha \in F_4$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_i \in \boldsymbol{R}.$$

To give a constructive proof of this theorem 1, we will prepare some elements of F_4.

(1) Let $Sp(3) = \{A \in M(3, \boldsymbol{H}) \,|\, A^*A = E\}$. We shall show that the group F_4 contains $Sp(3)$ as subgroup : $Sp(3) \subset F_4$. An element $X = \mathfrak{J}(3, \mathfrak{C})$ is expressed by

$$X = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 & \overline{m}_2 \\ \overline{m}_3 & \xi_2 & m_1 \\ m_2 & \overline{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3e_4 & -a_2e_4 \\ -a_3e_4 & 0 & a_1e_4 \\ a_2e_4 & -a_1e_4 & 0 \end{pmatrix},$$

where $x_i = m_i + a_i e_4 \in \boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C}$. To such X, we associate an element

$$\begin{pmatrix} \xi_1 & m_3 & \overline{m}_2 \\ \overline{m}_3 & \xi_2 & m_1 \\ m_2 & \overline{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3)$$

of $\mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3$. In $\mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3$, we define a multiplication $\times$ by

$$(M + \boldsymbol{a}) \times (N + \boldsymbol{b}) = \Big(M \times N - \frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b} + \boldsymbol{b}^*\boldsymbol{a})\Big) - \frac{1}{2}(\boldsymbol{a}N + \boldsymbol{b}M).$$

Since this multilication corresponds to the Freudenthal multiplication in $\mathfrak{J}(3, \mathfrak{C})$, hereafter, we identify $\mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3$ with $\mathfrak{J}(3, \mathfrak{C})$.

Now, we define a map $\varphi : Sp(3) \to F_4$ by

$$\varphi(A)(M + \boldsymbol{a}) = AMA^* + \boldsymbol{a}A^*, \quad M + \boldsymbol{a} \in \mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3 = \mathfrak{J}(3, \mathfrak{C}).$$

It is not difficult to see that φ is well-defined : $\varphi(A) \in F_4$. Since φ is injective, we identify $Sp(3)$ with $\varphi(Sp(3))$: $Sp(3) \subset F_4$ ([3]).

(2) Let $G_2 = \{\alpha \in \mathrm{Iso}_R(\mathfrak{C}) \,|\, \alpha(xy) = (\alpha x)(\alpha y)\}$. The group F_4 contains G_2 as subgroup by the following way. We define a map $\phi : G_2 \to F_4$ by

$$\phi(\alpha) \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix}.$$

It is not difficult to see that ϕ is well-defined : $\phi(\alpha) \in F_4$. Since ϕ is injective, we identify G_2 with $\phi(G_2)$: $G_2 \subset F_4$.

(3) For $D = \mathrm{diag}(a, \overline{a}, 1), a \in \mathfrak{C}, |a| = 1$, we define an $\boldsymbol{R}$-linear transformation δ_a of $\mathfrak{J}(3, \mathfrak{C})$ by

$$\delta_a X = D_a X \overline{D}_a = \begin{pmatrix} \xi_1 & a x_3 a & a\overline{x}_2 \\ \overline{a}\overline{x}_3\overline{a} & \xi_2 & \overline{a}x_1 \\ x_2\overline{a} & \overline{x}_1 a & \xi_3 \end{pmatrix}.$$

Then, $\delta_a \in F_4$.

(4) For $T \in O(3) = \{T \in M(3, \boldsymbol{R}) \,|\, {}^tTT = E\}$, we define a transformation $\delta(T)$ of $\mathfrak{J}(3, \mathfrak{C})$ by

$$\delta(T)X = TXT^{-1}, \quad X \in \mathfrak{J}(3, \mathfrak{C}),$$

then, $\delta(T) \in F_4$.

Using (1) - (4), we will give a constructive proof of Theorem 1. Let $X = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}$ be a given element of $\mathfrak{J}(3, \mathfrak{C})$.

(i) Assume $x_1 \neq 0$ and let $a = x_1/|x_1|$. Applying δ_a on X, then, the x_1-part of $X' = \delta_a X$ becomes real.

(ii) Applying some $T = \begin{pmatrix} 1 & 0 \\ 0 & T' \end{pmatrix}, T' \in O(2) = \{T' \in M(2, \boldsymbol{R}) \,|\, {}^tT'T' = E\}$, then X' can be transformed to the form $X'' = \begin{pmatrix} \xi_1{}' & x_3{}'' & \overline{x}_2{}'' \\ \overline{x}_3{}'' & \xi_2{}'' & 0 \\ x_2{}'' & 0 & \xi_3{}'' \end{pmatrix}$.

(iii) Assume $x_2{}'' \neq 0$ and let $a = x_2{}''/|x_2{}''|$. Applying δ_a on X, then, the x_2-part of $X^{(3)} = \delta_a X''$ becomes real. That is, $X^{(3)}$ is of the form $X^{(3)} = \begin{pmatrix} \xi_1{}' & x_3{}^{(3)} & r_2 \\ \overline{x}_3{}^{(3)} & \xi_2{}'' & 0 \\ r_2 & 0 & \xi_3{}'' \end{pmatrix}$, r_2 is real.

(iv) Let $\boldsymbol{C} = \{x + ye_1 \,|\, x, y \in \boldsymbol{R}\}$ be the field of the complex numbers contained in $\boldsymbol{H}$: $\boldsymbol{C} \subset \boldsymbol{H} \subset \mathfrak{C}$. Since the group G_2 acts transitively on $S_r{}^6 = \{u \in \mathfrak{C} \,|\, u = -\overline{u}, |u| = r\}$, any element $x = x_0 + u \in \mathfrak{C}, x_0 \in \boldsymbol{R}, u \in \mathfrak{C}, u = -\overline{u}$ can be deformed to $c_3 = x_0 + x_1 e_1 \in \boldsymbol{C}$ by some $\alpha \in G_2$ ([2]). Applying this α on $X^{(3)}$, then $\alpha X^{(3)}$ is of the form $X^{(4)} = \begin{pmatrix} \xi_1{}' & c_3 & r_2 \\ \overline{c}_3 & \xi_2{}'' & 0 \\ r_2 & 0 & \xi_3{}'' \end{pmatrix} \in \mathfrak{J}(4, \boldsymbol{C}) \subset \mathfrak{J}(3, \boldsymbol{H})$.

(v) $X^{(4)}$ can be transformed to diagonal form by some $A \in Sp(3)$, that is, $AX^{(4)}A^*$ is of the form $\begin{pmatrix} \xi_1{}'' & 0 & 0 \\ 0 & \xi_2{}^{(3)} & 0 \\ 0 & 0 & \xi_3{}^{(3)} \end{pmatrix}$.

Note that all process of (i) - (v) are constructive. Thus, Theorem 1 is proved.

II. Let $\mathfrak{C}' = H \oplus \boldsymbol{H}e_4{}', e_4{}'^{\,2} = 1$ be the split Cayley algebra and $\mathfrak{J}(3, \mathfrak{C}') = \{X \in M(n, \mathfrak{C}') \,|\, X^* = X\}$ with the Jordan multiplicaton $X \circ Y = \dfrac{1}{2}(XY + YX)$. Then, the connected non-compact Lie group $F_{4(4)}$ is defined by

$$F_{4(4)} = \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}(3, \mathfrak{C}')) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\}.$$

Then, we have the following Theorem.

Theorem 2. *Any element X of $\mathfrak{J}'$ can not necessarily be transformed to a dinagonal form by element $\alpha \in F_{4(4)}$.*

Proof. We will give a counter example. Assume that the element

$$X_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_4{}' \\ 0 & -e_4{}' & 0 \end{pmatrix} \in \mathfrak{J}'$$

can be transformed to a diagonal form by some $\alpha \in F_{4(4)}$:

$$\alpha X_0 = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_i \in \boldsymbol{R}.$$

If we define an inner product (X, Y) in $\mathfrak{J}(3, \mathfrak{C}')$ by $(X, Y) = \mathrm{tr}(X \circ Y)$, then we know that any element $\alpha \in F_{4(4)}$ leaves the inner product invariant : $(\alpha X, \alpha Y) = (X, Y)$.

Now, we have

$$(\alpha X_0, \alpha X_0) = (X_0, X_0) = 2{e_4}'(-{e_4}') = -2.$$

On the other hand,

$$(\alpha X_0, \alpha X_0) = \Big(\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \Big) = {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2 \geq 0,$$

which contradicts the above. Therefore, X_0 can not be trasformed to a diagonal form.

References

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