THE STRUCTURE OF THE C^*-ALGEBRA OF A LOCALLY INJECTIVE SURJECTION

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1. Introduction

The classical connection between dynamical systems and C^*-algebras is the crossed product construction which associates a C^*-algebra to a homeomorphism of a compact metric space. This construction has been generalized stepwise by J. Renault ([Re]), V. Deaconu ([De1]) and C. Anantharaman-Delaroche ([An]) to local homeomorphisms and recently also to locally injective surjections by the second named author in [Th1]. The main motivation for the last generalisation was the wish to include the Matsumoto-type C^*-algebra of a subshift which was introduced by the first named author in [Ca].

In this paper we continue the investigation of the structure of the C^*-algebra of a locally injective surjection which was begun in [Th1]. The main goal here is to give necessary and sufficient conditions for the algebras, or at least any simple quotient of them, to be purely infinite; a property they are known to have in many cases. Recall that a simple C^*-algebra is said to be purely infinite when all its non-zero hereditary C^*-subalgebras contain an infinite projection. Our main result is that a simple quotient of the C^*-algebra arising from a locally injective surjection on a compact metric space of finite covering dimension, as in Section 4 of [Th1], is one of the following kinds:

1) a full matrix algebra $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, or
2) the crossed product $C(K) \times_f \mathbb{Z}$ corresponding to a minimal homeomorphism f of a compact metric space K of finite covering dimension, or
3) a unital purely infinite simple C^*-algebra.

In particular, when the algebra itself is simple it must be one of the three types, and in fact purely infinite unless the underlying map is a homeomorphism. Hence the problem of finding necessary and sufficient conditions for the C^*-algebra of a locally injective surjection on a compact metric space of finite covering dimension to be both simple and purely infinite has a strikingly straightforward solution: If the algebra is simple (and [Th1] gives necessary and sufficient conditions for this to happen) then it is automatically purely infinite unless the map in question is a homeomorphism. A corollary of this result is that if the C^*-algebra of a one-sided subshift is simple, then it is also purely infinite.

On the way to the proof of the main result we study the ideal structure. We find first the gauge invariant ideals, obtaining an insight which combined with methods and results of Katsura ([Ka]) leads to a list of the primitive ideals. We then identify the maximal ideals among the primitive ones and obtain in this way a description of the simple quotients which we use to obtain the conclusions described above. A fundamental tool all the way is the canonical locally homeomorphic extension

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discovered in [Th2] which allows us to replace the given locally injective map with a local homeomorphism. It means, however, that much of the structure we investigate gets described in terms of the canonical locally homeomorphic extension, and this is unfortunate since it may not be easy to obtain a satisfactory understanding of it for a given locally injective surjection. Still, it allows us to obtain qualitative conclusions of the type mentioned above.

Besides the C^*-algebras of subshifts our results cover of course also the C^*-algebras associated to a local homeomorphism by the construction of Renault, Deaconu and Anantharaman-Delaroche, provided the map is surjective and the space is of finite covering dimension. This means that the results have bearing on many classes of C^*-algebras which have been associated to various structures, for example the λ-graph systems of Matsumoto ([Ma]) and the continuous graphs of Deaconu ([De2]).

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2. The C^*-algebra of a locally injective surjection

Let X be a compact metric space and $\varphi : X \to X$ a locally injective surjection. Set

$$\Gamma_\varphi = \{(x,k,y) \in X \times \mathbb{Z} \times X : \ \exists n,m \in \mathbb{N}, \ k = n-m, \ \varphi^n(x) = \varphi^m(y)\} .$$

This is a groupoid with the set of composable pairs being

$$\Gamma_\varphi^{(2)} \ = \ \{((x,k,y),(x',k',y')) \in \Gamma_\varphi \times \Gamma_\varphi : \ y = x'\} .$$

The multiplication and inversion are given by

$$(x,k,y)(y,k',y') = (x,k+k',y') \text{ and } (x,k,y)^{-1} = (y,-k,x).$$

Note that the unit space of Γ_φ can be identified with X via the map $x \mapsto (x,0,x)$. To turn Γ_φ into a locally compact topological groupoid, fix $k \in \mathbb{Z}$. For each $n \in \mathbb{N}$ such that $n+k \geq 0$, set

$$\Gamma_\varphi(k,n) = \left\{(x,l,y) \in X \times \mathbb{Z} \times X : \ l = k, \ \varphi^{k+i}(x) = \varphi^i(y), \ i \geq n\right\} .$$

This is a closed subset of the topological product $X \times \mathbb{Z} \times X$ and hence a locally compact Hausdorff space in the relative topology. Since φ is locally injective $\Gamma_\varphi(k,n)$ is an open subset of $\Gamma_\varphi(k,n+1)$ and hence the union

$$\Gamma_\varphi(k) = \bigcup_{n \geq -k} \Gamma_\varphi(k,n)$$

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

$$\Gamma_\varphi = \bigcup_{k \in \mathbb{Z}} \Gamma_\varphi(k)$$

is then a locally compact Hausdorff space in the topology where each $\Gamma_\varphi(k)$ is an open and closed set. In fact, as is easily verified, Γ_φ is a locally compact groupoid in the sense of [Re] and a semi étale groupoid in the sense of [Th1]. The paper [Th1] contains a construction of a C^*-algebra from any semi étale groupoid, but we give here only a description of the construction for Γ_φ.

Consider the space $B_c(\Gamma_\varphi)$ of compactly supported bounded functions on Γ_φ. They form a $*$-algebra with respect to the convolution-like product

$$f \star g(x,k,y) = \sum_{z,n+m=k} f(x,n,z)g(z,m,y)$$

and the involution

$$f^*(x,k,y) = \overline{f(y,-k,x)}.$$

To turn it into a C^*-algebra, let $x \in X$ and consider the Hilbert space H_x of square summable functions on $\{(x',k,y') \in \Gamma_\varphi : \ y' = x\}$ which carries a representation π_x of the $*$-algebra $B_c(\Gamma_\varphi)$ defined such that

$$(\pi_x(f)\psi)(x',k,x) = \sum_{z,n+m=k} f(x',n,z)\psi(z,m,x) \tag{2.1}$$

when $\psi \in H_x$. One can then define a C^*-algebra $B_r^*(\Gamma_\varphi)$ as the completion of $B_c(\Gamma_\varphi)$ with respect to the norm

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\|.$$

The space $C_c(\Gamma_\varphi)$ of continuous and compactly supported functions on Γ_φ generate a $*$-subalgebra $\operatorname{alg}^* \Gamma_\varphi$ of $B_r^*(\Gamma_\varphi)$ which completed with respect to the above norm becomes the C^*-algebra $C_r^*(\Gamma_\varphi)$ which is our object of study. When φ is open, and hence a local homeomorphism, $C_c(\Gamma_\varphi)$ is a $*$-subalgebra of $B_c(\Gamma_\varphi)$ so that $\operatorname{alg}^* \Gamma_\varphi = C_c(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi)$ is then the completion of $C_c(\Gamma_\varphi)$. In this case $C_r^*(\Gamma_\varphi)$ is the algebra studied by Renault in [Re], by Deaconu in [De1], and by Anantharaman-Delaroche in [An].

The algebra $C_r^*(\Gamma_\varphi)$ contains several canonical C^*-subalgebras which we shall need in our study of its structure. One is the C^*-algebra of the open sub-groupoid

$$R_\varphi = \Gamma_\varphi(0)$$

which is a semi étale groupoid (equivalence relation, in fact) in itself. The corresponding C^*-algebra $C_r^*(R_\varphi)$ is the C^*-subalgebra of $C_r^*(\Gamma_\varphi)$ generated by the continuous and compactly supported functions on R_φ. Equally important are two canonical abelian C^*-subalgebras, D_{Γ_φ} and D_{R_φ}. They arise from the fact that the C^*-algebra $B(X)$ of bounded functions on X sits canonically inside $B_r^*(\Gamma_\varphi)$, cf. p. 765 of [Th1], and are then defined as

$$D_{\Gamma_\varphi} = C_r^*(\Gamma_\varphi) \cap B(X)$$

and

$$D_{R_\varphi} = C_r^*(R_\varphi) \cap B(X),$$

respectively. There are faithful conditional expectations $P_{\Gamma_\varphi} : C_r^*(\Gamma_\varphi) \to D_{\Gamma_\varphi}$ and $P_{R_\varphi} : C_r^*(R_\varphi) \to D_{R_\varphi}$, obtained as extensions of the restriction map $\operatorname{alg}^* \Gamma_\varphi \to B(X)$ to $C_r^*(\Gamma_\varphi)$ and $C_r^*(R_\varphi)$, respectively. When φ is open and hence a local homeomorphism, the two algebras D_{Γ_φ} and D_{R_φ} are identical and equal to $C(X)$, but in general the inclusion $D_{R_\varphi} \subseteq D_{\Gamma_\varphi}$ is strict.

Our approach to the study of $C_r^*(\Gamma_\varphi)$ hinges on a construction introduced in [Th2] of a compact Hausdorff space Y and a local homeomorphic surjection $\phi : Y \to Y$ such that (X,φ) is a factor of (Y,ϕ) and

$$C_r^*(\Gamma_\varphi) \simeq C_r^*(\Gamma_\phi). \tag{2.2}$$

Everything we can say about ideals and simple quotients of $C_r^*(\Gamma_\phi)$ will have bearing on $C_r^*(\Gamma_\varphi)$, but while the isomorphism (2.2) is equivariant with respect to the canonical gauge actions (see Section 4), it will not in general take $C_r^*(R_\varphi)$ onto $C_r^*(R_\phi)$. This is one reason why we will work with $C_r^*(\Gamma_\varphi)$ whenever possible, instead of using (2.2) as a valid excuse for working with local homeomorphisms only. Another is that it is generally not so easy to get a workable description of (Y,ϕ). As in [Th2] we will refer to (Y,ϕ) as the *canonical locally homeomorphic extension* of (X,φ). The space Y is the Gelfand spectrum of D_{Γ_φ} so when φ is already a local homeomorphism itself, the extension is redundant and $(Y,\phi)=(X,\varphi)$.

3. Ideals in $C_r^*(R_\varphi)$

Recall from [Th1] that there is a semi étale equivalence relation

$$R(\varphi^n)=\{(x,y)\in X\times X:\varphi^n(x)=\varphi^n(y)\}$$

for each $n\in\mathbb{N}$. They will be considered as open sub-equivalence relations of R_φ via the embedding $(x,y)\mapsto(x,0,y)\in\Gamma_\varphi(0)$. In this way we get embeddings $C_r^*(R(\varphi^n))\subseteq C_r^*(R(\varphi^{n+1}))\subseteq C_r^*(R_\varphi)$ by Lemma 2.10 of [Th1], and then

$$C_r^*(R_\varphi)=\overline{\bigcup_n C_r^*(R(\varphi^n))}, \tag{3.1}$$

cf. (4.2) of [Th1]. This inductive limit decomposition of $C_r^*(R_\varphi)$ defines in a natural way a similar inductive limit decomposition of D_{R_φ}. Set

$$D_{R(\varphi^n)}=C_r^*(R(\varphi^n))\cap B(X).$$

Lemma 3.1. $D_{R_\varphi}=\overline{\bigcup_{n=1}^\infty D_{R(\varphi^n)}}$.

Proof. Since $C_r^*(R(\varphi^n))\subseteq C_r^*(R_\varphi)$, it follows that

$$D_{R(\varphi^n)}=C_r^*(R(\varphi^n))\cap B(X)\subseteq C_r^*(R_\varphi)\cap B(X)=D_{R_\varphi}.$$

Hence

$$\overline{\bigcup_{n=1}^\infty D_{R(\varphi^n)}}\subseteq D_{R_\varphi}. \tag{3.2}$$

Let $x\in D_{R_\varphi}$ and let $\epsilon>0$. It follows from (3.1) that there is an $n\in\mathbb{N}$ and an element $y\in\operatorname{alg}^* R(\varphi^n)$ such that

$$\left\|x-P_{R_\varphi}(y)\right\|\leq\epsilon.$$

On $\operatorname{alg}^* R_\varphi$ the conditional expectation P_{R_φ} is just the map which restricts functions to X and the same is true for the conditional expectation $P_{R(\varphi^n)}$ on $\operatorname{alg}^* R(\varphi^n)$, where $P_{R(\varphi^n)}$ is the conditional expectation of Lemma 2.8 in [Th1] obtained by considering $R(\varphi^n)$ as a semi étale groupoid in itself. Hence $P_{R_\varphi}(y)=P_{R(\varphi^n)}(y)\in D_{R(\varphi^n)}$. It follows that we have equality in (3.2). ☐

In the following, by an ideal of a C^*-algebra we will always mean a closed and two-sided ideal. The next lemma is well known and crucial for the sequel.

Lemma 3.2. *Let Y be a compact Hausdorff space, M_n the C^*-algebra of n-by-n matrices for some natural number $n\in\mathbb{N}$ and p a projection in $C(Y,M_n)$. Set $A=pC(Y,M_n)p$ and let C_A be the center of A.*

For every ideal I in A there is an approximate unit for I in $I\cap C_A$.

Lemma 3.3. *Let $I, J \subseteq C_r^*(R_\varphi)$ be two ideals. Then*

$$I \cap D_{R_\varphi} \subseteq J \cap D_{R_\varphi} \;\Rightarrow\; I \subseteq J.$$

Proof. If $I \cap D_{R_\varphi} \subseteq J \cap D_{R_\varphi}$ it follows that $I \cap D_{R(\varphi^n)} \subseteq J \cap D_{R(\varphi^n)}$ for all n. Note that the center of $C_r^*(R(\varphi^n))$ is contained in $D_{R(\varphi^n)}$ since $D_{R(\varphi^n)}$ is maximal abelian in $C_r^*(R(\varphi^n))$ by Lemma 2.19 of [Th1]. By using Corollary 3.3 of [Th1] it follows therefore from Lemma 3.2 that there is a sequence $\{x_n\}$ in $I \cap D_{R(\varphi^n)}$ such that $\lim_{n\to\infty} x_n a = a$ for all $a \in I \cap C_r^*(R(\varphi^n))$. Since $x_n \in J \cap D_{R(\varphi^n)}$ this implies that $I \cap C_r^*(R(\varphi^n)) \subseteq J \cap C_r^*(R(\varphi^n))$ for all n. Combining with (3.1) we find that

$$I = \overline{\bigcup_n I \cap C_r^*(R(\varphi^n))} \subseteq \overline{\bigcup_n J \cap C_r^*(R(\varphi^n))} = J.$$

□

Recall from [Th1] that an ideal J in D_{R_φ} is said to be *R_φ-invariant* when $n^*Jn \subseteq J$ for all $n \in \operatorname{alg}^* R_\varphi$ supported in a bisection of R_φ. For every R_φ-invariant ideal J in D_{R_φ}, set

$$\widehat{J} = \left\{a \in C_r^*(R_\varphi) : \; P_{R_\varphi}(a^*a) \in J\right\}.$$

Theorem 3.4. *The map $J \mapsto \widehat{J}$ is a bijection between the R_φ-invariant ideals in D_{R_φ} and the ideals in $C_r^*(R_\varphi)$. The inverse is given by the map $I \mapsto I \cap D_{R_\varphi}$*

Proof. It follows from Lemma 2.13 of [Th1] that $\widehat{J} \cap D_{R_\varphi} = J$ for any R_φ-invariant ideal in D_{R_φ}. It suffices therefore to show that every ideal in $C_r^*(R_\varphi)$ is of the form $\widehat{J}$ for some R_φ-invariant ideal J in D_{R_φ}. Let I be an ideal in $C_r^*(R_\varphi)$. Set $J = I \cap D_{R_\varphi}$, which is clearly a R_φ-invariant ideal in D_{R_φ}. Since $\widehat{J} \cap D_{R_\varphi} = J = I \cap D_{R_\varphi}$ by Lemma 2.13 of [Th1], we conclude from Lemma 3.3 that $\widehat{J} = I$.

□

A subset $A \subseteq Y$ is said to be *ϕ-saturated* when $\phi^{-k}\left(\phi^k(A)\right) = A$ for all $k \in \mathbb{N}$.

Corollary 3.5. *(Cf. Proposition II.4.6 of* [Re]*) The map*

$$L \mapsto I_L = \left\{a \in C_r^*(R_\phi) : \; P_{R_\phi}(a^*a)(x) = 0 \;\forall x \in L\right\}$$

is a bijection from the non-empty closed ϕ-saturated subsets L of Y onto the set of proper ideals in $C_r^*(R_\phi)$.

Proof. Since ϕ is a local homeomorphism, we have that $D_{R_\phi} = C(Y)$ so the corollary follows from Theorem 3.4 by use of the well-known bijection between ideals in $C(Y)$ and closed subsets of Y. The only thing to show is that an open subset U of Y is ϕ-saturated if and only if the ideal $C_0(U)$ of $C(Y)$ is R_ϕ-invariant which is straightforward, cf. the proof of Corollary 2.18 in [Th1]. □

The next issue will be to determine which closed ϕ-saturated subsets of Y correspond to primitive ideals. For a point $x \in Y$ we define the *ϕ-saturation* of x to be the set

$$H(x) = \bigcup_{n=1}^{\infty} \{y \in Y : \; \phi^n(y) = \phi^n(x)\}.$$

The closure $\overline{H(x)}$ of $H(x)$ will be referred to as the *closed ϕ-saturation* of x. Observe that both $H(x)$ and $\overline{H(x)}$ are ϕ-saturated.

Proposition 3.6. *Let $L \subseteq Y$ be a non-empty closed ϕ-saturated subset. The ideal I_L is primitive if and only L is the closed ϕ-saturation of a point in Y.*

Proof. Since $C_r^*(R_\phi)$ is separable an ideal is primitive if and only if it is prime, cf. [Pe]. We show that I_L is prime if and only if $L = \overline{H(x)}$ for some $x \in Y$.

Assume first that $L = \overline{H(x)}$ and consider two ideals, I_1 and I_2, in $C_r^*(R_\phi)$ such that $I_1 I_2 \subseteq I_{\overline{H(x)}}$. By Corollary 3.5 there are closed ϕ-saturated subsets, L_1 and L_2, in Y such that $I_j = I_{L_j}$, $j = 1, 2$. It follows from Corollary 3.5 that $\overline{H(x)} \subseteq L_1 \cup L_2$. At least one of the L_j's must contain x, say $x \in L_1$. Since L_1 is ϕ-saturated and closed it follows that $\overline{H(x)} \subseteq L_1$, and hence that $I_1 \subseteq I_{\overline{H(x)}}$. Thus $I_{\overline{H(x)}}$ is prime.

Assume next that I_L is prime. Let $\{U_k\}_{k=0}^{\infty}$ be a base for the topology of L consisting of non-empty sets. We will construct sequences $\{B_k\}_{k=0}^{\infty}$ of compact non-empty neighbourhoods in L and non-negative integers $\{n_k\}_{k=0}^{\infty}$ such that

i) $B_k \subseteq B_{k-1}$ for $k \geq 1$, and
ii) $\phi^{n_k}(B_k) \subseteq \phi^{n_k}(U_k)$ for $k \geq 0$.

We start the induction by letting B_0 be any compact non-empty neighbourhood in U_0 and $n_0 = 0$. Assume then that $B_0, B_1, B_2, \dots, B_m$ and $n_0, n_1, \dots, n_m$ have been constructed. Choose a non-empty open subset $V_{m+1} \subseteq B_m$. Note that both of

$$L \backslash \bigcup_l \phi^{-l}\left(\phi^l(V_{m+1})\right)$$

and

$$L \backslash \bigcup_l \phi^{-l}\left(\phi^l(U_{m+1})\right)$$

are closed ϕ-saturated subsets of L, and hence of Y, and none of them is all of L. It follows from Corollary 3.5 and primeness of I_L that L is not contained in their union, which in turn implies that

$$\phi^{-n_{m+1}}\left(\phi^{n_{m+1}}(V_{m+1})\right) \cap \phi^{-n_{m+1}}\left(\phi^{n_{m+1}}(U_{m+1})\right)$$

is non-empty for some $n_{m+1} \in \mathbb{N}$. There is therefore a point $z \in V_{m+1}$ such that $\phi^{n_{m+1}}(z) \in \phi^{n_{m+1}}(U_{m+1})$, and therefore also a compact non-empty neighbourhood $B_{m+1} \subseteq V_{m+1}$ of z such that $\phi^{n_{m+1}}(B_{m+1}) \subseteq \phi^{n_{m+1}}(U_{m+1})$. This completes the induction. Let $x \in \bigcap_m B_m$. By construction every U_k contains an element from $H(x)$. It follows that $\overline{H(x)} = L$. □

4. On the ideals of $C_r^*(\Gamma_\varphi)$

The C^*-algebra $C_r^*(\Gamma_\varphi)$ carries a canonical circle action β, called the *gauge action*, defined such that

$$\beta_\lambda(f)(x, k, y) = \lambda^k f(x, k, y)$$

when $f \in C_c(\Gamma_\varphi)$ and $\lambda \in \mathbb{T}$, cf. [Th1]. As the next step we describe in this section the gauge-invariant ideals in $C_r^*(\Gamma_\varphi)$.

Consider first the function $m : X \to \mathbb{N}$ defined such that

$$m(x) = \#\{y \in X : \ \varphi(y) = \varphi(x)\}. \tag{4.1}$$

As shown in [Th1], $m \in D_{R(\varphi)} \subseteq D_{R_\varphi}$. Define a function $V_\varphi : \Gamma_\varphi \to \mathbb{C}$ such that

$$V_\varphi(x, k, y) = \begin{cases} m(x)^{-\frac{1}{2}} & \text{when } k = 1 \text{ and } y = \varphi(x) \\ 0 & \text{otherwise.} \end{cases}$$

Then V_φ is the product $V_\varphi = m^{-\frac{1}{2}} 1_{\Gamma_\varphi(1,0)}$ in $C_r^*(\Gamma_\varphi)$ and in fact an isometry which induces an endomorphism $\widehat{\varphi}$ of $C_r^*(R_\varphi)$, viz.

$$\widehat{\varphi}(a) = V_\varphi a V_\varphi^*$$

Together with $C_r^*(R_\varphi)$ the isometry V_φ generates $C_r^*(\Gamma_\varphi)$ which in this way becomes a crossed product $C_r^*(R_\varphi) \times_{\widehat{\varphi}} \mathbb{N}$ in the sense of Stacey, cf. [St] and [Th1]; in particular Theorem 4.6 in [Th1].

4.1. **Gauge invariant ideals.** Let $C_r^*(\Gamma_\varphi)^{\mathbb{T}}$ denote the fixed point algebra of the gauge action.

Lemma 4.1. *For each $k \in \mathbb{N}$ we have that $V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k$ is a C^*-subalgebra of $C_r^*(\Gamma_\varphi)^{\mathbb{T}}$,*

$$V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k \subseteq V_\varphi^{*k+1} C_r^*(R_\varphi) V_\varphi^{k+1}, \tag{4.2}$$

and

$$C_r^*(\Gamma_\varphi)^{\mathbb{T}} = \overline{\bigcup_{k=0}^{\infty} V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k}. \tag{4.3}$$

Proof. Since $V_\varphi^k V_\varphi^{*k} \in C_r^*(R_\varphi)$, it is easy to check that $V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k$ is a $*$-algebra. To see that $V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k$ is closed let $\{a_n\}$ be a sequence in $C_r^*(R_\varphi)$ such that $\left\{V_\varphi^{*k} a_n V_\varphi^k\right\}$ converges in $C_r^*(\Gamma_\varphi)$, say $\lim_{n\to\infty} V_\varphi^{*k} a_n V_\varphi^k = b$. It follows that

$$\left\{V_\varphi^k V_\varphi^{*k} a_n V_\varphi^k V_\varphi^{*k}\right\}$$

is Cauchy in $C_r^*(R_\varphi)$ and hence convergent, say to $a \in C_r^*(R_\varphi)$. But then $b = \lim_{n\to\infty} V_\varphi^{*k} a_n V_\varphi^k = \lim_{n\to\infty} V_\varphi^{*k} V_\varphi^k V_\varphi^{*k} a_n V_\varphi^k V_\varphi^{*k} V_\varphi^k = V_\varphi^{*k} a V_\varphi^k$. It follows that

$$V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k$$

is closed and hence a C^*-subalgebra. The inclusion (4.2) follows from the observation that $V_\varphi^k = V_\varphi^* V_\varphi^{k+1}$ and $V_\varphi C_r^*(R_\varphi) V_\varphi^* \subseteq C_r^*(R_\varphi)$.

It is straightforward to check that $\beta_\lambda(V_\varphi) = \lambda V_\varphi$ and that $C_r^*(R_\varphi) \subseteq C_r^*(\Gamma_\varphi)^{\mathbb{T}}$. The inclusion $\supseteq$ in (4.3) follows from this. To obtain the other, let $x \in C_r^*(\Gamma_\varphi)^{\mathbb{T}}$ and let $\epsilon > 0$. It follows from Theorem 4.6 of [Th1] and Lemma 1.1. of [BoKR] that there is an $n \in \mathbb{N}$ and an element

$$y \in \operatorname{Span} \bigcup_{i,j \leq n} V_\varphi^{*i} C_r^*(R_\varphi) V_\varphi^j$$

such that $\|x - y\| \leq \epsilon$. Then $\left\|x - \int_{\mathbb{T}} \beta_\lambda(y)\ d\lambda\right\| \leq \epsilon$ and since

$$\int_{\mathbb{T}} \beta_\lambda(y)\ d\lambda \in V_\varphi^{*n} C_r^*(R_\varphi) V_\varphi^n,$$

we see that (4.3) holds. $\square$

Lemma 4.2. *Let I be a gauge invariant ideal in $C_r^*(\Gamma_\varphi)$. It follows that*

$$I = \left\{ a \in C_r^*(\Gamma_\varphi) : \int_{\mathbb{T}} \beta_\lambda(a^*a)\ d\lambda \in I \cap C_r^*(\Gamma_\varphi)^{\mathbb{T}} \right\}.$$

Proof. Set $B = C_r^*(\Gamma_\varphi)/I$. Since I is gauge-invariant there is an action $\hat{\beta}$ of $\mathbb{T}$ on B such that $q \circ \beta = \hat{\beta} \circ q$, where $q : C_r^*(\Gamma_\varphi) \to B$ is the quotient map. Thus, if

$$y \in \left\{ a \in C_r^*(\Gamma_\varphi) : \int_{\mathbb{T}} \beta_\lambda(a^*a)\, d\lambda \in I \cap C_r^*(\Gamma_\varphi)^{\mathbb{T}} \right\},$$

we find that

$$\int_{\mathbb{T}} \hat{\beta}_\lambda(q(y^*y))\, d\lambda = q\left(\int_{\mathbb{T}} \beta_\lambda(y^*y)\, d\lambda \right) = 0.$$

Since $\int_{\mathbb{T}} \hat{\beta}_\lambda(\cdot)\, d\lambda$ is faithful we conclude that $q(y) = 0$, i.e. $y \in I$. This establishes the non-trivial part of the asserted identity. □

Lemma 4.3. *Let I, I' be gauge invariant ideals in $C_r^*(\Gamma_\varphi)$. Then*

$$I \cap D_{R_\varphi} \subseteq I' \cap D_{R_\varphi} \ \Rightarrow\ I \subseteq I'.$$

Proof. Assume that $I \cap D_{R_\varphi} \subseteq I' \cap D_{R_\varphi}$. It follows from Lemma 3.3 that $I \cap C_r^*(R_\varphi) \subseteq I' \cap C_r^*(R_\varphi)$. Then

$$\begin{aligned} I \cap V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k &= V_\varphi^{*k} \left(I \cap C_r^*(R_\varphi)\right) V_\varphi^k \\ &\subseteq V_\varphi^{*k} \left(I' \cap C_r^*(R_\varphi)\right) V_\varphi^k = I' \cap V_\varphi^{*k} C_r^*(R_\varphi) V_\varphi^k \end{aligned}$$

for all $k \in \mathbb{N}$. Hence Lemma 4.1 implies that $I \cap C_r^*(\Gamma_\varphi)^{\mathbb{T}} \subseteq I' \cap C_r^*(\Gamma_\varphi)^{\mathbb{T}}$. It follows then from Lemma 4.2 that $I \subseteq I'$.

□

Proposition 4.4. *The map $J \mapsto \widehat{J}$, where*

$$\widehat{J} = \left\{ a \in C_r^*(\Gamma_\varphi) : P_{\Gamma_\varphi}(a^*a) \in J \right\},$$

is a bijection from the Γ_φ-invariant ideals of D_{Γ_φ} onto the gauge invariant ideals of $C_r^*(\Gamma_\varphi)$. Its inverse is the map $I \mapsto I \cap D_{\Gamma_\varphi}$.

Proof. Since $P_{\Gamma_\varphi} \circ \beta = P_{\Gamma_\varphi}$ the ideal $\widehat{J}$ is gauge invariant. It follows from Lemma 2.13 of [Th1] that $\widehat{J} \cap D_{\Gamma_\varphi} = J$ so it suffices to show that

$$\widehat{I \cap D_{\Gamma_\varphi}} = I \tag{4.4}$$

when I is a gauge invariant ideal in $C_r^*(\Gamma_\varphi)$. It follows from Lemma 2.13 of [Th1] that $\widehat{I \cap D_{\Gamma_\varphi}} \cap D_{\Gamma_\varphi} = I \cap D_{\Gamma_\varphi}$. Since $D_{R_\varphi} \subseteq D_{\Gamma_\varphi}$ this implies that $\widehat{I \cap D_{\Gamma_\varphi}} \cap D_{R_\varphi} = I \cap D_{R_\varphi}$. Then (4.4) follows from Lemma 4.3.

□

To simplify notation, set $D = D_{\Gamma_\phi} = C(Y)$. Every ideal I in $C_r^*(\Gamma_\phi)$ determines a closed subset $\rho(I)$ of Y defined such that

$$\rho(I) = \{ y \in Y : f(y) = 0 \ \forall f \in I \cap D \}. \tag{4.5}$$

We say that a subset $F \subseteq Y$ is *totally ϕ-invariant* when $\phi^{-1}(F) = F$.

Lemma 4.5. *$\rho(I)$ is totally ϕ-invariant for every ideal I in $C_r^*(\Gamma_\phi)$.*

Proof. It suffices to show that $Y \setminus \rho(I)$ is totally ϕ-invariant, which is what we will do. Assume first that $x \in Y \setminus \rho(I)$. Then there is an $f \in I \cap D$ such that $f(x) \neq 0$. Choose an open bisection $W \subseteq \Gamma_\phi$ such that $(x, 1, \phi(x)) \in W$. Choose then $\eta \in C_c(\Gamma_\phi)$ such that $\eta((x, 1, \phi(x)) = 1$ and $\operatorname{supp} \eta \subseteq W$. It is not difficult to check that $\eta^* f \eta \in D$ and that $\eta^* f \eta(\phi(x)) = f(x) \neq 0$, and since $\eta^* f \eta \in I$, it follows that $\phi(x) \in Y \setminus \rho(I)$. Assume then that $\phi(x) \in Y \setminus \rho(I)$. Then there is a $g \in I \cap D$ such that $g(\phi(x)) \neq 0$. Choose an open bisection $W \subseteq \Gamma_\phi$ such that $(x, 1, \phi(x)) \in W$ and $\eta \in C_c(\Gamma_\phi)$ such that $\eta((x, 1, \phi(x)) = 1$ and $\operatorname{supp} \eta \subseteq W$. Then $\eta g \eta^* \in D$ and $\eta g \eta^*(\phi(x)) = g(x) \neq 0$, and since $\eta g \eta^* \in I$, this shows that $x \in Y \backslash \rho(I)$, proving that $\phi^{-1}(\rho(I)) = \rho(I)$. □

Thus every ideal in $C_r^*(\Gamma_\phi)$ gives rise to a closed totally ϕ-invariant subset of Y. To go in the other direction, let F be a closed totally ϕ-invariant subset of Y. Then $Y \backslash F$ is open and totally ϕ-invariant so that the reduction $\Gamma_\phi|_{Y\backslash F}$ is an étale groupoid in its own right, cf. [An]. In fact, ϕ restricts to local homeomorphic surjections $\phi : Y\backslash F \to Y \backslash F$ and $\phi : F \to F$, and

$$\Gamma_\phi|_{Y\backslash F} = \Gamma_{\phi|_{Y\backslash F}}.$$

Note that $C_r^*\left(\Gamma_\phi|_{Y\backslash F}\right) = C_r^*\left(\Gamma_{\phi|_{Y\backslash F}}\right)$ is an ideal in $C_r^*(\Gamma_\phi)$ because $Y\backslash F$ is totally ϕ-invariant.

Proposition 4.6. *(Cf. Proposition II.4.5 of* [Re]*.) Let F be a non-empty, closed and totally ϕ-invariant subset of Y. There is then a surjective $*$-homomorphism $\pi_F : C_r^*(\Gamma_\phi) \to C_r^*(\Gamma_{\phi|_F})$ which extends the restriction map $C_c(\Gamma_\phi) \to C_c\left(\Gamma_{\phi|_F}\right)$ and has the property that* $\ker \pi_F = C_r^*\left(\Gamma_{\phi|_{Y\backslash F}}\right)$*, i.e.*

$$0 \longrightarrow C_r^*\left(\Gamma_{\phi|_{Y\backslash F}}\right) \longrightarrow C_r^*(\Gamma_\phi) \xrightarrow{\pi_F} C_r^*(\Gamma_{\phi|_F}) \longrightarrow 0$$

is exact. Furthermore,

$$\rho(\ker \pi_F) = F. \tag{4.6}$$

Proof. Let $\dot{\pi_F} : C_c(\Gamma_\phi) \to C_c\left(\Gamma_{\phi|_F}\right)$ denote the restriction map which is surjective by Tietze's theorem. By using that F is totally ϕ-invariant, it follows straightforwardly that $\dot{\pi_F}$ is a $*$-homomorphism. Since $\pi_x \circ \dot{\pi_F} = \pi_x$ when $x \in F$, it follows that $\dot{\pi_F}$ extends by continuity to a $*$-homomorphism $\pi_F : C_r^*(\Gamma_\phi) \to C_r^*(\Gamma_{\phi|_F})$ which is surjective because $\dot{\pi_F}$ is. To complete the proof observe that

$$\ker \pi_F \cap D = C_0(Y\backslash F) = C_r^*\left(\Gamma_{\phi|_{Y\backslash F}}\right) \cap D.$$

The first identity shows that (4.6) holds, and since $\ker \pi_F$ and $C_r^*\left(\Gamma_{\phi|_{Y\backslash F}}\right)$ are both gauge-invariant ideals the second that they are identical by Lemma 4.3. □

By combining Proposition 4.4, Lemma 4.5 and Proposition 4.6 we obtain the following.

Theorem 4.7. *The map ρ is a bijection from the gauge-invariant ideals in $C_r^*(\Gamma_\phi)$ onto the set of closed totally ϕ-invariant subsets of Y. The inverse is the map which sends a closed totally ϕ-invariant subset $F \subseteq Y$ to the ideal*

$$\ker \pi_F = \left\{a \in C_r^*(\Gamma_\phi) : \ P_{\Gamma_\phi}(a^*a)(x) = 0 \ \forall x \in F\right\}.$$

We remark that since the isomorphism (2.2) is equivariant with respect to the gauge actions, Theorem 4.7 gives also a description of the gauge invariant ideals in $C_r^*(\Gamma_\varphi)$, as a complement to the one of Proposition 4.4.

4.2. **The primitive ideals.** We are now in position to obtain a complete description of the primitive ideals of $C^*_r(\Gamma_\phi)$. Much of what we do is merely a translation of Katsuras description of the primitive ideals in the more general C^*-algebras considered by him in [Ka]. Recall that because we only deal with separable C^*-algebras the primitive ideals are the same as the prime ideals, cf. 3.13.10 and 4.3.6 in [Pe].

Lemma 4.8. *Let I be an ideal in $C^*_r(\Gamma_\phi)$ and let A be a closed totally ϕ-invariant subset of Y. If $\rho(I) \subseteq A$, then $\ker \pi_A \subseteq I$.*

Proof. Since $\rho(I) \subseteq A$ it follows from the Stone-Weierstrass theorem that $C_0(Y \setminus A) \subseteq I \cap C(Y)$. Let $\{i_n\}$ be an approximate unit in $C_0(Y \setminus A)$. It follows from Proposition 4.6 that $\{i_n\}$ is also an approximate unit in $\ker \pi_A$. Since $\{i_n\} \subseteq I$ it follows that $\ker \pi_A \subseteq I$. $\square$

We say that a closed totally ϕ-invariant subset A of Y is *prime* when it has the property that if B and C also are closed and totally ϕ-invariant subsets of Y and $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$.

Let $\mathcal{M} := \{A \subseteq Y : A \text{ is non-empty, closed, totally } \phi\text{-invariant and prime}\}$. For $x \in Y$ let

$$\operatorname{Orb}(x) = \{y \in Y : \exists m, n \in \mathbb{N} : \phi^n(x) = \phi^m(y)\}.$$

We call $\operatorname{Orb}(x)$ the *total ϕ-orbit of x*.

Proposition 4.9. *(Cf. Proposition 4.13 and 4.4 of* [Ka]*.)*

$$\mathcal{M} = \{\overline{\operatorname{Orb}(x)} : x \in Y\}.$$

Proof. It is clear that $\overline{\operatorname{Orb}(x)} \in \mathcal{M}$ for every $x \in Y$. Assume that $A \in \mathcal{M}$ and let $\{U_k\}_{k=1}^\infty$ be a basis for A. We will by induction show that we can choose compact neighbourhoods $\{C_k\}_{k=0}^\infty$ and $\{C'_k\}_{k=0}^\infty$ in A and positive integers $(n_k)_{k=0}^\infty$ and $(n'_k)_{k=0}^\infty$ such that $C_k \subseteq U_k$ and $C'_k \subseteq \phi^{n_{k-1}}(C_{k-1}) \cap \phi^{n'_{k-1}}(C'_{k-1})$ for $k \geq 1$. For this set $C_0 = C'_0 = A$. Assume then that $n \geq 1$ and that $C_1, \dots, C_n, C'_1, \dots, C'_n, n_0, \dots, n_{n-1}$ and $n'_0, \dots, n'_{n-1}$ satisfying the conditions above have been chosen. Choose non-empty open subsets $V_n \subseteq C_n$ and $V'_n \subseteq C'_n$. We then have that

$$\bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V_n)) \text{ and } \bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V'_n))$$

are non-empty open and totally ϕ-invariant subsets of A, and thus that

$$A \setminus \bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V_n)) \text{ and } A \setminus \bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V'_n)) \tag{4.7}$$

are closed, totally ϕ-invariant subsets of Y. Since A is prime and is not contained in either of the sets from (4.7), it follows that A is not contained in

$$\left(A \setminus \bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V_n))\right) \bigcup \left(A \setminus \bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V'_n))\right),$$

whence

$$\left(\bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V_n))\right) \bigcap \left(\bigcup_{l,m=0}^{\infty} \phi^{-l}(\phi^m(V'_n))\right) \neq \emptyset.$$

It follows that there are positive integers n_n and n'_n such that $\phi^{n_n}(V_n) \cap \phi^{n'_n}(V'_n)$ is non-empty. Thus we can choose a compact neighbourhood $C_{n+1} \subseteq U_{n+1}$ and a compact neighbourhood $C'_{n+1} \subseteq \phi^{n_n}(V_n) \cap \phi^{n'_n}(V'_n) \subseteq \phi^{n_n}(C_n) \cap \phi^{n'_n}(C'_n)$ which is what is required for the induction step.

It is easy to check that

$$C'_0 \cap \phi^{-n'_0}(C'_1) \cap \dots \dots \cap \phi^{-n'_0 - \dots - n'_k}(C'_{k+1}), \ k = 0, 1, \dots$$

is a decreasing sequence of non-empty compact sets. It follows that there is an

$$x \in \bigcap_{k=0}^{\infty} \phi^{-n'_0 - \dots \dots - n'_k}(C'_{k+1}) \cap C'_0.$$

We have for every $k \in \mathbb{N}$ that $\phi^{n'_0 + \dots + n'_k}(x) \in C'_{k+1} \subseteq \phi^{n_k}(C_k) \subseteq \phi^{n_k}(U_k)$, and it follows that $\mathrm{Orb}(x)$ is dense in A, and thus that $A = \overline{\mathrm{Orb}(x)}$. $\square$

Proposition 4.10. *(Cf. Proposition 9.3 of* [Ka]*.) Assume that I is a prime ideal in $C^*_r(\Gamma_\phi)$. It follows that $\rho(I) \in \mathcal{M}$.*

Proof. It follows from Lemma 4.5 that $\rho(I)$ is closed and totally ϕ-invariant. To show that $\rho(I)$ is also prime, assume that B and C are closed totally ϕ-invariant subsets such that $\rho(I) \subseteq B \cup C$. It follows then from Lemma 4.8 that $\ker(\pi_{B\cup C}) \subseteq I$. Since $\ker \pi_B \cap \ker \pi_C \cap D = C_0(Y \backslash B) \cap C_0(Y \backslash C) = C_0\left(Y \backslash (B \cup C)\right) = \ker \pi_{B \cup C} \cap D$ it follows from Lemma 4.3 that $\ker \pi_B \cap \ker \pi_C = \ker \pi_{B\cup C}$. Therefore $\ker(\pi_B) \subseteq I$ or $\ker(\pi_C) \subseteq I$ since I is prime. Hence $\rho(I) \subseteq B$ or $\rho(I) \subseteq C$, thanks to (4.6). $\square$

We say that a point $x \in Y$ is ϕ-periodic if $\phi^n(x) = x$ for some $n > 0$. Let Per denote the set of ϕ-periodic points $x \in Y$ which are isolated in $\mathrm{Orb}(x)$ and let

$$\mathcal{M}_{\mathrm{Per}} = \{\overline{\mathrm{Orb}(x)} : x \in \mathrm{Per}\}$$

and

$$\mathcal{M}_{\mathrm{Aper}} = \mathcal{M} \setminus \mathcal{M}_{\mathrm{Per}}.$$

Let $A \subseteq Y$ be a closed totally ϕ-invariant subset. We say that $\phi|_A$ is *topologically free* if the set of ϕ-periodic points in A has empty interior in A.

Proposition 4.11. *(Cf. Proposition 11.3 of* [Ka]*.) Let $A \in \mathcal{M}$. Then $\phi|_A$ is topologically free if and only if $A \in \mathcal{M}_{\mathrm{Aper}}$.*

Proof. We will show that $\phi|_A$ is not topologically free if and only if $A \in \mathcal{M}_{\mathrm{Per}}$. If $x \in \mathrm{Per}$ and $A = \overline{\mathrm{Orb}(x)}$, then $\phi|_A$ is not topologically free because x is periodic and isolated in $\mathrm{Orb}(x)$ and thus in A. Assume then that $\phi|_A$ is not topologically free. There is then a non-empty open subset $U \subseteq A$ such that every element of U is ϕ-periodic. Choose $x \in A$ such that $A = \overline{\mathrm{Orb}(x)}$. Then $U \cap \mathrm{Orb}(x) \neq \emptyset$. Let $y \in U \cap \mathrm{Orb}(x)$. Then y is ϕ-periodic and $\overline{\mathrm{Orb}(y)} = \overline{\mathrm{Orb}(x)} = A$, so if we can show that y is isolated in $\mathrm{Orb}(y)$, then we have that $A \in \mathcal{M}_{\mathrm{Per}}$. Since y is ϕ-periodic there is an $n \geq 1$ such that $\phi^n(y) = y$. We claim that $U \subseteq \{y, \phi(y), \dots, \phi^{n-1}(y)\}$. It will then follow that y is isolated in A and thus in $\mathrm{Orb}(y)$.

Assume that $U \setminus \{y, \phi(y), \dots, \phi^{n-1}(y)\}$ is non-empty. Since it is also open it follows that $\mathrm{Orb}(y) \cap U \setminus \{y, \phi(y), \dots, \phi^{n-1}(y)\}$ is non-empty. Let $z \in \mathrm{Orb}(y) \cap U \setminus \{y, \phi(y), \dots, \phi^{n-1}(y)\}$. Since $z \in U$ there is an $m \geq 1$ so that $\phi^m(z) = z$, and since $z \in \mathrm{Orb}(y)$ there are $k, l \in \mathbb{N}$ such that $\phi^k(z) = \phi^l(y)$. But then $z = \phi^{mk}(z) = \phi^{(m-1)k+l}(y) \in \{y, \phi(y), \dots, \phi^{n-1}(y)\}$ and we have a contradiction. It follows that $U \subseteq \{y, \phi(y), \dots, \phi^{n-1}(y)\}$. $\square$

In particular, it follows from Proposition 4.11 that the elements of $\mathcal{M}_{\text{Aper}}$ are infinite sets.

Proposition 4.12. *(Cf. Proposition 11.5 of* [Ka]*.) Let $A \in \mathcal{M}_{\text{Aper}}$. Then $\ker \pi_A$ is the unique ideal I in $C_r^*(\Gamma_\phi)$ with $\rho(I) = A$.*

Proof. We have already in Proposition 4.6 seen that $\rho(\ker \pi_A) = A$. Assume that I is an ideal in $C_r^*(\Gamma_\phi)$ with $\rho(I) = A$. It follows then from Lemma 4.8 that $\ker \pi_A \subseteq I$. Thus it sufficies to show that $\pi_A(I) = \{0\}$. Note that $\pi_A(I)$ is an ideal in $C_r^*(\Gamma_{\phi|_A})$ with $\rho(\pi_A(I)) = A$. It follows that $\pi_A(I) \cap C(A) = \{0\}$. To conclude from this that $\pi_A(I) = \{0\}$ we will show that the points of A whose isotropy group in $\Gamma_{\phi|_A}$ is trivial are dense in A. It will then follow from Lemma 2.15 of [Th1] that $\pi_A(I) = \{0\}$ because $\pi_A(I) \cap C(A) = \{0\}$. That the points of A with trivial isotropy in $\Gamma_{\phi|_A}$ are dense in A is established as follows: The points in A with non-trivial isotropy in $\Gamma_{\phi|A}$ are the pre-periodic points in A. Let $\operatorname{Per}_n A$ denote the set of points in A with minimal period n under ϕ and note that $\operatorname{Per}_n A$ is closed and has empty interior since $\phi|_A$ is topologically free by Proposition 4.11. It follows that $A \backslash \phi^{-k}(\operatorname{Per}_n A)$ is open and dense in A for all k, n. By the Baire category theorem it follows that

$$A \backslash \left(\bigcup_{k,n} \phi^{-k}(\operatorname{Per}_n A) \right) = \bigcap_{k,n} A \backslash \phi^{-k}(\operatorname{Per}_n A)$$

is dense in A, proving the claim. □

Lemma 4.13. *Let $A \in \mathcal{M}_{\text{Aper}}$. Then $\ker \pi_A$ is a primitive ideal.*

Proof. Let $A = \overline{\operatorname{Orb}(x)}$. To show that $\ker \pi_A$ is primitive it suffices to show that it is prime, cf. Proposition 4.3.6 of [Pe]. Equivalently, it suffices to show that $C_r^*\left(\Gamma_{\phi|_A}\right)$ is a prime C^*-algebra. Consider therefore two ideals $I_j \subseteq C_r^*\left(\Gamma_{\phi|_A}\right)$, $j = 1, 2$, such that $I_1 I_2 = \{0\}$. Then

$$\{y \in A : \ f(y) = 0 \ \forall f \in I_1 \cap C(A)\} \cup \{y \in A : \ f(y) = 0 \ \forall f \in I_2 \cap C(A)\} = A.$$

In particular, x must be in $\{y \in A : \ f(y) = 0 \ \forall f \in I_j \cap C(A)\}$, either for $j = 1$ or $j = 2$. It follows then from Lemma 4.5, applied to $\phi|_A$, that

$$A = \{y \in A : \ f(y) = 0 \ \forall f \in I_j \cap C(A)\}.$$

Hence $I_j = \{0\}$ by Proposition 4.12 applied to $\phi|_A$. □

Let $A \in \mathcal{M}_{\text{Per}}$. Choose $x \in \operatorname{Per}$ such that $\overline{\operatorname{Orb}(x)} = A$, and let n be the minimal period of x. Then x is isolated in A. It follows that the characteristic functions $1_{(x,0,x)}$ and $1_{(x,n,x)}$ belong to $C_r^*(\Gamma_{\phi|_A})$. Let $p_x = 1_{(x,0,x)}$ and $u_x = 1_{(x,n,x)}$. For $w \in \mathbb{T}$ let $\dot{P}_{x,w}$ denote the ideal in $C_r^*(\Gamma_{\phi|_A})$ generated by $u_x - w p_x$.

Lemma 4.14. *Suppose that $x, y \in \operatorname{Per}$ and that $\overline{\operatorname{Orb}(x)} = \overline{\operatorname{Orb}(y)}$ and let $w \in \mathbb{T}$. Then $\dot{P}_{x,w} = \dot{P}_{y,w}$.*

Proof. By symmetry, it is enough to show that $\dot{P}_{y,w} \subseteq \dot{P}_{x,w}$. Since y is isolated in $\operatorname{Orb}(y)$, it is isolated in $\overline{\operatorname{Orb}(y)} = \overline{\operatorname{Orb}(x)}$. Thus y must belong to $\operatorname{Orb}(x)$. This means that there are $k, l \in \mathbb{N}$ such that $\phi^k(x) = \phi^l(y)$. Since y is ϕ-periodic, it follows that there is an $i \in \mathbb{N}$ such that $y = \phi^i(x)$. Let $A = \overline{\operatorname{Orb}(y)} = \overline{\operatorname{Orb}(x)}$. Since x and y are isolated in A we have that $1_{(x,i,y)} \in C_r^*(\Gamma_{\phi|_A})$. Let $v = 1_{(x,i,y)}$. It is easy to check that $v^* p_x v = p_y$ and that $v^* u_x v = u_y$. Thus $u_y - w p_y = v^*(u_x - w p_x) v \in \dot{P}_{x,w}$ and it follows that $\dot{P}_{y,w} \subseteq \dot{P}_{x,w}$. □

Let $A \in \mathcal{M}_{\mathrm{Per}}$ and let $w \in \mathbb{T}$. It follows from Lemma 4.14 that the ideal $\dot{P}_{x,w}$ does not depend of the particular choice of $x \in A \cap \mathrm{Per}$, as long as $\overline{\mathrm{Orb}(x)} = A$. We will therefore simply write $\dot{P}_{A,w}$ for $\dot{P}_{x,w}$. We then define $P_{A,w}$ to be the ideal $\pi_A^{-1}(\dot{P}_{A,w})$ in $C_r^*(\Gamma_\phi)$.

Proposition 4.15. *(Cf. Proposition 11.13 of* [Ka]*.) Let $A \in \mathcal{M}_{\mathrm{Per}}$. Then*

$$w \mapsto P_{A,w}$$

is a bijection between $\mathbb{T}$ and the set of primitive ideals P in $C_r^*(\Gamma_\phi)$ with $\rho(P) = A$.

Proof. The map $P \mapsto \pi_A(P)$ gives a bijection between the primitive ideals in $C_r^*(\Gamma_\phi)$ with $\ker \pi_A \subseteq P$ and the primitive ideals in $C_r^*(\Gamma_{\phi|_A})$, cf. Theorem 4.1.11 (ii) in [Pe]. The inverse of this bijection is the map $Q \mapsto \pi_A^{-1}(Q)$. If P is a primitive ideal in $C_r^*(\Gamma_\phi)$ with $\rho(P) = A$, it follows from Lemma 4.8 that $\ker \pi_A \subseteq P$. In addition $\rho(\pi_A(P)) = A$. If on the other hand Q is a primitive ideal in $C_r^*(\Gamma_{\phi|_A})$ with $\rho(Q) = A$, then $\pi_A^{-1}(Q)$ is a primitive ideal in $C_r^*(\Gamma_\phi)$ and $\rho(\pi_A^{-1}(Q)) = A$. Thus $P \mapsto \pi_A(P)$ gives a bijection between the primitive ideals in $C_r^*(\Gamma_\phi)$ with $\rho(P) = A$ and the primitive ideals Q in $C_r^*(\Gamma_{\phi|_A})$ with $\rho(Q) = A$.

Choose $x \in \mathrm{Per}$ such that $\overline{\mathrm{Orb}(x)} = A$. Let $\langle p_x \rangle$ be the ideal in $C_r^*(\Gamma_{\phi|_A})$ generated by p_x. Observe that $\dot{P}_{A,w} \subseteq \langle p_x \rangle$ for all $w \in \mathbb{T}$ since $p_x\,(u_x - w p_x) = u_x - w p_x$. The map $Q \mapsto Q \cap \langle p_x \rangle$ gives a bijection between the primitive ideals in $C_r^*(\Gamma_{\phi|_A})$ with $\langle p_x \rangle \nsubseteq Q$ and the primitive ideals in $\langle p_x \rangle$, cf. Theorem 4.1.11 (ii) in [Pe]. We claim that $\langle p_x \rangle \nsubseteq Q$ if and only if $\rho(Q) = A$. To see this, let Q be an ideal in $C_r^*(\Gamma_{\phi|_A})$. If $p_x \in Q$, then $x \notin \rho(Q)$ and $\rho(Q) \neq A$. If on the other hand $\rho(Q) \neq A$, then $x \notin \rho(Q)$ because $\rho(Q)$ is closed and totally ϕ-invariant and $\overline{\mathrm{Orb}(x)} = A$. It follows that there is an $f \in Q \cap C(A)$ such that $f(x) \neq 0$, whence $p_x \in Q$. This proves the claim and it follows that $Q \mapsto Q \cap \langle p_x \rangle$ gives a bijection between the primitive ideals in $C_r^*(\Gamma_{\phi|_A})$ with $\rho(Q) = A$ and the primitive ideals in $\langle p_x \rangle$.

The C^*-algebra $\langle p_x \rangle$ is Morita equivalent to $p_x C_r^*(\Gamma_{\phi|_A}) p_x$ via the $p_x C_r^*(\Gamma_{\phi|_A}) p_x$-$\langle p_x \rangle$ imprimitivity bimodule $p_x C_r^*(\Gamma_{\phi|_A})$, and therefore $T \mapsto p_x T p_x$ gives a bijection between the primitive ideals T in $\langle p_x \rangle$ and the primitive ideals in $p_x C_r^*(\Gamma_{\phi|A}) p_x$, cf. Proposition 3.24 and Corollary 3.33 in [RW]. Now note that

$$\{(x', n', y') \in \Gamma_{\phi|_A} : x' = y' = x\} = \{(x, kn, x) : k \in \mathbb{Z}\}$$

where n is the smallest positive integer such that $\phi^n(x) = x$. It follows that $p_x C_r^*(\Gamma_{\phi|A}) p_x$ is isomorphic to $C(\mathbb{T})$ under an isomorphism taking the canonical unitary generator of $C(\mathbb{T})$ to u_x. In this way we conclude that the primitive ideals of $p_x C_r^*(\Gamma_{\phi|A}) p_x$ are in one-to-one correspondance with $\mathbb{T}$ under the map

$$\mathbb{T} \ni w \mapsto p_x \overline{C_r^*(\Gamma_{\phi|A})\,(u_x - w p_x)\, C_r^*(\Gamma_{\phi|A})} p_x = p_x \dot{P}_{A,w} p_x.$$

This completes the proof. □

By combining Proposition 4.10, 4.12 and 4.15 we get the following theorem.

Theorem 4.16. *The set of primitive ideals in $C_r^*(\Gamma_\phi)$ is the disjoint union of $\{\ker \pi_A : A \in \mathcal{M}_{\mathrm{Aper}}\}$ and $\{P_{A,w} : A \in \mathcal{M}_{\mathrm{Per}},\ w \in \mathbb{T}\}$.*

4.3. **The maximal ideals.** The next step is to identify the maximal ideals among the primitive ones.

Lemma 4.17. *Assume that not all points of Y are pre-periodic and that $C_r^*(\Gamma_\phi)$ contains a non-trivial ideal. It follows that there is a non-trivial gauge-invariant ideal J in $C_r^*(\Gamma_\phi)$ such that $J \cap C(Y) \neq \{0\}$.*

Proof. Let I be a non-trivial ideal in $C_r^*(\Gamma_\phi)$. Assume first that $I \cap C(Y) = \{0\}$. Since we assume that not all points of Y are pre-periodic we can apply Lemma 2.16 of [Th1] to conclude that $J_0 = \overline{P_{\Gamma_\phi}(I)}$ is a non-trivial Γ_ϕ-invariant ideal in $C(Y)$. Then

$$J = \left\{a \in C_r^*(\Gamma_\phi) : \ P_{\Gamma_\phi}(a^*a) \in J_0\right\}$$

is a non-trivial gauge-invariant ideal by Theorem 4.4, and $J \cap C(Y) = J_0 \neq \{0\}$. Note that J contains I in this case. If $I \cap C(X) \neq \{0\}$ we set

$$J = \left\{a \in C_r^*(\Gamma_\phi) : \ P_{\Gamma_\phi}(a^*a) \in I \cap C(Y)\right\}$$

which is a non-trivial ideal in $C_r^*(\Gamma_\phi)$ such that $J \cap C(Y) = I \cap C(Y)$ by Lemma 2.13 of [Th1]. Since J is gauge-invariant, this completes the proof. □

Lemma 4.18. *Let $F \subseteq Y$ be a minimal closed non-empty totally ϕ-invariant subset. Then either*

1) *$F \in \mathcal{M}_{Aper}$ and $\ker \pi_F$ is a maximal ideal, or*
2) *$F = \mathrm{Orb}(x) = \{\phi^n(x) : n \in \mathbb{N}\}$, where $x \in \mathrm{Per}$.*

Proof. It follows from the minimality of F that $\overline{\mathrm{Orb}(x)} = F$ for all $x \in F$. We will show that 1) holds when F does not contain an element of Per, and that 2) holds when it does. Assume first that F does not contain any elements of Per. Then $F \in \mathcal{M}_{Aper}$. If there is a proper ideal I in $C_r^*(\Gamma_\phi)$ such that $\ker \pi_F \subsetneq I$, then $\pi_F(I)$ is a non-trivial ideal in $C_r^*\left(\Gamma_{\phi|_F}\right)$, and then it follows from Lemma 4.17 that there is a non-trivial gauge-invariant ideal J in $C_r^*\left(\Gamma_{\phi|_F}\right)$. By Theorem 4.7 $\rho(\pi_F^{-1}(J))$ is then a non-trivial closed totally ϕ-invariant subset of F, contradicting the minimality of F. Thus 1) holds when F does not contain an element from Per.

Assume instead that there is an $x \in F \cap \mathrm{Per}$. Then x is isolated in $\mathrm{Orb}(x)$, and thus in F. It follows that $F = \mathrm{Orb}(x)$, because if $y \in F \setminus \mathrm{Orb}(x)$ we would have that $x \notin \overline{\mathrm{Orb}(y)} = F$, which is absurd. Since F is compact, $\mathrm{Orb}(x)$ must be finite. Since ϕ is surjective we must then have that $\mathrm{Orb}(x) = \{\phi^n(x) : n \in \mathbb{N}\}$. Thus 2) holds if F contains an element from Per. □

Lemma 4.19. *Let I be a maximal ideal in $C_r^*(\Gamma_\phi)$. Then either $I = \ker \pi_F$ for some minimal closed totally ϕ-invariant subset $F \in \mathcal{M}_{Aper}$, or $I = P_{\mathrm{Orb}(x),w}$ for some $w \in \mathbb{T}$ and some $x \in \mathrm{Per}$ such that $\mathrm{Orb}(x) = \{\phi^n(x) : \ n \in \mathbb{N}\}$.*

Proof. Since I is also primitive we know from Theorem 4.16 that $I = \ker \pi_A$ for some $A \in \mathcal{M}_{Aper}$ or $I = P_{A,w}$ for some $A \in \mathcal{M}_{\mathrm{Per}}$ and some $w \in \mathbb{T}$. In the first case it follows that A must be a minimal closed totally ϕ-invariant subset since I is a maximal ideal. Assume then that $I = P_{A,w}$ for some $A \in \mathcal{M}_{\mathrm{Per}}$ and some $w \in \mathbb{T}$. In the notation from the proof of Proposition 4.15, observe that $\dot{P}_{A,w} \subseteq \langle p_x \rangle$ since $p_x(u_x - wp_x) = u_x - wp_x$. Note that $\dot{P}_{A,w} \neq \langle p_x \rangle$ because the latter of these ideals is gauge-invariant and the first is not. By maximality of I this implies that $\langle p_x \rangle = C_r^*\left(\Gamma_{\phi|_A}\right)$. On the other hand, $\mathrm{Orb}(x)$ is an open totally ϕ-invariant subset of A and $p_x \in C_r^*\left(\Gamma_{\phi|_{\mathrm{Orb}(x)}}\right)$, so we see that $\langle p_x \rangle = C_r^*\left(\Gamma_{\phi|_A}\right) = C_r^*\left(\Gamma_{\phi|_{\mathrm{Orb}(x)}}\right)$. This implies that

$$C_0\left(\mathrm{Orb}(x)\right) = C(A) \cap C_r^*\left(\Gamma_{\phi|_{\mathrm{Orb}(x)}}\right) = C(A),$$

and hence that $A = \mathrm{Orb}(x)$. Compactness of A implies that $\mathrm{Orb}(x)$ is finite and surjectivity of ϕ that $\mathrm{Orb}(x) = \{\phi^n(x) : \ n \in \mathbb{N}\}$. □

Theorem 4.20. *The maximal ideals in $C_r^*\left(\Gamma_\phi\right)$ consist of the primitive ideals of the form $\ker \pi_F$ for some infinite minimal closed totally ϕ-invariant subset $F \subseteq Y$ and the primitive ideals $P_{A,w}$ for some $w \in \mathbb{T}$, where $A = \mathrm{Orb}(x) = \{\phi^n(x) : \ n \in \mathbb{N}\}$ for a ϕ-periodic point $x \in Y$.*

Proof. This follows from the last two lemmas, after the observation that a primitive ideal $P_{A,w}$ of the form described in the statement is maximal. □

Corollary 4.21. *Let A be a simple quotient of $C_r^*\left(\Gamma_\phi\right)$. Assume A is not finite dimensional. It follows that there is an infinite minimal closed totally ϕ-invariant subset F of Y such that $A \simeq C_r^*\left(\Gamma_{\phi|_F}\right)$.*

To make more detailed conclusions about the simple quotients we need to restrict to the case where Y is of finite covering dimension so that the result of [Th3] applies. For this reason we prove first that finite dimensionality of Y follows from finite dimensionality of X.

5. On the dimension of Y

Let $\mathrm{Dim}\, X$ and $\mathrm{Dim}\, Y$ denote the covering dimensions of X and Y, respectively. The purpose with this section is to establish

Proposition 5.1. $\mathrm{Dim}\, Y \leq \mathrm{Dim}\, X$.

Proof. By definition Y is the Gelfand spectrum of D_{Γ_φ}. Since the conditional expectation $P_{\Gamma_\varphi} : C_r^*\left(\Gamma_\varphi\right) \to D_{\Gamma_\varphi}$ is invariant under the gauge action, in the sense that $P_{\Gamma_\varphi} \circ \beta_\lambda = P_{\Gamma_\varphi}$ for all λ, it follows that

$$D_{\Gamma_\varphi} = P_{\Gamma_\varphi}\left(C_r^*\left(\Gamma_\varphi\right)^{\mathbb{T}}\right).$$

To make use of this description of D_{Γ_φ} we need a refined version of (4.3). Note first that it follows from (4.4) and (4.5) of [Th1] that $V_\varphi C_r^*\left(R\left(\varphi^l\right)\right) V_\varphi^* \subseteq C_r^*\left(R\left(\varphi^{l+1}\right)\right)$ for all $l \in \mathbb{N}$. Consequently

$$V_\varphi^{*k} C_r^*\left(R\left(\varphi^l\right)\right) V_\varphi^k = V_\varphi^{*k+1} V_\varphi C_r^*\left(R\left(\varphi^l\right)\right) V_\varphi^* V_\varphi^{k+1} \subseteq V_\varphi^{*k+1} C_r^*\left(R\left(\varphi^{l+1}\right)\right) V_\varphi^{k+1}$$

for all $k, l \in \mathbb{N}$. It follows therefore from (3.1) and (4.3) that there are sequences $\{k_n\}$ and $\{l_n\}$ in $\mathbb{N}$ such that $l_n \geq k_n$,

$$V_\varphi^{*k_n} C_r^*\left(R\left(\varphi^{l_n}\right)\right) V_\varphi^{k_n} \subseteq V_\varphi^{*k_{n+1}} C_r^*\left(R\left(\varphi^{l_{n+1}}\right)\right) V_\varphi^{k_{n+1}} \tag{5.1}$$

and

$$C_r^*\left(\Gamma_\varphi\right)^{\mathbb{T}} = \overline{\bigcup_n V_\varphi^{*k_n} C_r^*\left(R\left(\varphi^{l_n}\right)\right) V_\varphi^{k_n}}; \tag{5.2}$$

we can for example use $k_n = n$ and $l_n = 2n$.

Let D_n denote the C^*-subalgebra of D_{Γ_φ} generated by

$$P_{\Gamma_\varphi}\left(V_\varphi^{*k_n} C_r^*\left(R\left(\varphi^{l_n}\right)\right) V_\varphi^{k_n}\right)$$

and let Y_n be the character space of D_n. Note that $C(X) \subseteq D_n$ since $V_\varphi^{k_n} g V_\varphi^{*k_n} \in C_r^*\left(R\left(\varphi^{l_n}\right)\right)$ and $g = P_{\Gamma_\varphi}\left(V_\varphi^{*k_n} V_\varphi^{k_n} g V_\varphi^{*k_n} V_\varphi^{k_n}\right)$ when $g \in C(X)$. There is therefore a continuous surjection

$$\pi_n : Y_n \to X$$

defined such that $g\left(\pi_n(y)\right) = y(g),\ g \in C(X)$. We claim that $\#\pi_n^{-1}(x) < \infty$ for all $x \in X$. To show this note that by definition D_n is generated as a C^*-algebra by functions of the form

$$x \mapsto P_{\Gamma_\varphi}\left(V_\varphi^{*k_n} f V_\varphi^{k_n}\right)(x) = \sum_{z,z' \in \varphi^{-k_n}(x)} f(z,z') \prod_{j=0}^{k_n-1} m(\varphi^j(z))^{-\frac{1}{2}} m(\varphi^j(z'))^{-\frac{1}{2}} \tag{5.3}$$

for some $f \in C_r^*\left(R\left(\varphi^{l_n}\right)\right)$. In fact, since $\operatorname{alg}^* R\left(\varphi^{l_n}\right)$ is dense in $C_r^*\left(R\left(\varphi^{l_n}\right)\right)$, already functions of the form (5.3) with

$$f = f_1 \star f_2 \star \cdots \star f_N, \tag{5.4}$$

for some $f_i \in C\left(R\left(\varphi^{l_n}\right)\right), i = 1, 2, \dots, N$, will generate D_n.

Fix $x \in X$ and consider an element $y \in \pi_n^{-1}(x)$. Every $x' \in X$ defines a character $\iota_{x'}$ of D_n by evaluation, viz. $\iota_{x'}(h) = h(x')$, and $\{\iota_{x'} : x' \in X\}$ is dense in Y_n because the implication

$$h \in D_n,\ h(x') = 0\ \forall x' \in X \ \Rightarrow\ h = 0$$

holds. In particular, there is a sequence $\{x_l\}$ in X such that $\lim_{l\to\infty} \iota_{x_l} = y$ in Y_n. Recall now from Lemma 3.6 of [Th1] that there is an open neighbourhood U of x and open sets $V_j, j = 1, 2, \dots, d$, where $d = \#\varphi^{-k_n}(x)$, in X such that

1) $\varphi^{-k_n}\left(\overline{U}\right) \subseteq V_1 \cup V_2 \cup \cdots \cup V_d$,
2) $\overline{V_i} \cap \overline{V_j} = \emptyset,\ i \neq j$, and
3) φ^{k_n} is injective on $\overline{V_j}$ for each j.

Since $\lim_{l\to\infty} x_l = x$ in X we can assume that $x_l \in U$ for all l. For each l, set

$$F_l = \left\{j :\ \varphi^{-k_n}(x_l) \cap V_j \neq \emptyset\right\} \subseteq \{1, 2, \dots, d\}.$$

Note that there is a subset $F \subseteq \{1, 2, \dots, d\}$ such that $F_l = F$ for infinitely many l. Passing to a subsequence we can therefore assume that $F_l = F$ for all l. For each $k \in F$ we define a continuous map $\lambda_k : \varphi^{k_n}\left(\overline{V_k}\right) \to \overline{V_k}$ such that $\varphi^{k_n} \circ \lambda_k(z) = z$. Set $T = \max_{z\in X} \#\varphi^{-1}(z)$. For each $j \in \{1, 2, \dots, T\}$, set

$$A_j = \left\{z \in X : \#\varphi^{-1}\left(\varphi(z)\right) = j\right\} = m^{-1}(j).$$

For each l and each $k \in F$ there is a unique tuple $(j_0(k), j_1(k), \dots, j_{k_n-1}(k)) \in \{1, 2, \dots, T\}^{k_n}$ such that

$$\varphi^{-k_n}(x_l) \cap V_k \cap A_{j_0(k)} \cap \varphi^{-1}\left(A_{j_1(k)}\right) \cap \varphi^{-2}\left(A_{j_2(k)}\right) \cap \cdots \cap \varphi^{-k_n+1}\left(A_{j_{k_n-1}(k)}\right) \neq \emptyset.$$

Since there are only finitely many choices we can arrange that the same tuples, $\left(j_0(k), j_1(k), \dots, j_{k_n-1}(k)\right), k \in F$, work for all l. Then

$$\iota_{x_l}\left(P_{\Gamma_\varphi}\left(V_\varphi^{*k_n} f V_\varphi^{k_n}\right)\right) = \sum_{k,k' \in F} f\left(\lambda_k(x_l), \lambda_{k'}(x_l)\right) \prod_{i=0}^{k_n-1} j_i(k)^{-\frac{1}{2}} j_i(k')^{-\frac{1}{2}} \tag{5.5}$$

for all $f \in C_r^*\left(R\left(\varphi^{l_n}\right)\right)$ and all l.

There is an open neighbourhood U' of $\varphi^{l_n-k_n}(x)$ and open sets $V_j', j = 1, 2, \dots, d'$, where $d' = \#\varphi^{-l_n}\left(\varphi^{l_n-k_n}(x)\right)$, in X such that

1') $\varphi^{-l_n}\left(\overline{U'}\right) \subseteq V'_1 \cup V'_2 \cup \cdots \cup V'_{d'}$,
2') $\overline{V'_i} \cap \overline{V'_j} = \emptyset,\ i \neq j$, and
3') φ^{l_n} is injective on $\overline{V'_j}$ for each j.

Since $\lim_{l\to\infty} \varphi^{l_n-k_n}(x_l) = \varphi^{l_n-k_n}(x)$ we can assume that $\varphi^{l_n-k_n}(x_l) \in U'$ for all l. By an argument identical to the way we found F above we can now find a subset $F' \subseteq \{1, 2, \dots, d'\}$ such that

$$F' = \left\{j : \varphi^{-l_n}\left(\varphi^{l_n-k_n}(x_l)\right) \cap V'_j \neq \emptyset\right\}$$

for all l. For $i \in F'$ we define a continuous map $\mu'_i : \varphi^{l_n}\left(\overline{V'_i}\right) \to \overline{V'_i}$ such that $\mu'_i \circ \varphi^{l_n}(z) = z$ when $z \in \overline{V'_i}$. Set

$$\mu_i = \mu'_i \circ \varphi^{l_n-k_n}$$

on $\varphi^{-(l_n-k_n)}\left(\varphi^{l_n}\left(\overline{V'_i}\right)\right)$. Assuming that f has the form (5.4) we find now that

$$f\left(\lambda_k(x_l), \lambda_{k'}(x_l)\right) = \sum_{i_1,i_2,\dots,i_{N-1}\in F'} f_1\left(\lambda_k(x_l), \mu_{i_1}(x_l)\right) f_2\left(\mu_{i_1}(x_l), \mu_{i_2}(x_l)\right) \cdot \dots \cdot f_N\left(\mu_{i_{N-1}}(x_l), \lambda_{k'}(x_l)\right) \tag{5.6}$$

for all $k, k' \in F$. By combining (5.6) with (5.5) we find by letting l tend to infinity that

$$y\left(P_{\Gamma_\varphi}\left(V_\varphi^{*k_n} f V_\varphi^{k_n}\right)\right) = \sum_{k,k'\in F} H_{k,k'}(x) \prod_{i=0}^{k_n-1} j_i(k)^{-\frac{1}{2}} j_i(k')^{-\frac{1}{2}},$$

where

$$H_{k,k'}(x) = \sum_{i_1,i_2,\dots,i_{N-1}\in F'} f_1\left(\lambda_k(x), \mu_{i_1}(x)\right) f_2\left(\mu_{i_1}(x), \mu_{i_2}(x)\right) \cdot \dots \cdot f_N\left(\mu_{i_{N-1}}(x), \lambda_{k'}(x)\right).$$

Since this expression only depends on F, F' and the tuples

$$\left(j_0(k), j_1(k), \dots, j_{k_n-1}(k)\right), k \in F,$$

it follows that the number of possible values of an element from $\pi_n^{-1}(x)$ on the generators of the form (5.3) does not exceed $2^d 2^{d'} T^{k_n}$, proving that $\#\pi_n^{-1}(x) < \infty$ as claimed.

We can then apply Theorem 4.3.6 on page 281 of [En] to conclude that $\operatorname{Dim} Y_n \leq \operatorname{Dim} X$. Note that $D_n \subseteq D_{n+1}$ and $D_{\Gamma_\varphi} = \overline{\bigcup_n D_n}$ by (5.1) and (5.2). Hence Y is the projective limit of the sequence $Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow \dots$. Since $\operatorname{Dim} Y_n \leq \operatorname{Dim} X$ for all n we conclude now from Theorem 1.13.4 in [En] that $\operatorname{Dim} Y \leq \operatorname{Dim} X$. □

6. The simple quotients

Following [DS] we say that ϕ is *strongly transitive* when for any non-empty open subset $U \subseteq Y$ there is an $n \in \mathbb{N}$ such that $Y = \bigcup_{j=0}^n \phi^j(U)$, cf. [DS]. By Proposition 4.3 of [DS], $C_r^*\left(\Gamma_\phi\right)$ is simple if and only if Y is infinite and ϕ is strongly transitive.

Lemma 6.1. *Assume that ϕ is strongly transitive but not injective. It follows that*

$$\lim_{k\to\infty} \frac{1}{k} \log\left(\inf_{x\in Y} \#\phi^{-k}(x)\right) > 0.$$

Proof. Note that $U = \{x \in Y : \#\phi^{-1}(x) \geq 2\}$ is open and not empty since ϕ is a local homeomorphism and not injective. It follows that there is an $m \in \mathbb{N}$ such that

$$\bigcup_{j=0}^{m-1} \phi^j(U) = Y \tag{6.1}$$

because ϕ is strongly transitive. We claim that

$$\inf_{z \in Y} \#\phi^{-k}(z) \geq 2^{\left[\frac{k}{m}\right]} \tag{6.2}$$

for all $k \in \mathbb{N}$ where $\left[\frac{k}{m}\right]$ denotes the integer part of $\frac{k}{m}$. This follows by induction: Assume that it true for all $k' < k$. Consider any $z \in Y$. If $k < m$ there is nothing to prove so assume that $k \geq m$. By (6.1) we can then write $z = \phi^j(z_1) = \phi^j(z_2)$ for some $j \in \{1, 2, \dots, m\}$ and some $z_1 \neq z_2$. It follows that

$$\#\phi^{-k}(z) \geq \#\phi^{-(k-j)}(z_1) + \#\phi^{-(k-j)}(z_2) \geq 2 \cdot 2^{\left[\frac{k-j}{m}\right]} \geq 2^{\left[\frac{k}{m}\right]}.$$

It follows from (6.2) that $\lim_{k\to\infty} \frac{1}{k} \log\left(\inf_{x \in Y} \#\phi^{-k}(x)\right) \geq \frac{1}{m} \log 2$. □

Let M_l denote the C^*-algebra of complex $l \times l$-matrices. In the following a *homogeneous C^*-algebra* will be a C^*-algebra isomorphic to a C^*-algebra of the form $eC(X, M_l)e$ where X is a compact metric space and e is a projection in $C(X, M_l)$ such that $e(x) \neq 0$ for all $x \in X$.

Definition 6.2. A unital C^*-algebra A is an *AH-algebra* when there is an increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of unital C^*-subalgebras of A such that $A = \overline{\bigcup_n A_n}$ and each A_n is a homogeneous C^*-algebra. We say that A has *no dimension growth* when the sequence $\{A_n\}$ can be chosen such that

$$A_n \simeq e_n C\left(X_n, M_{l_n}\right) e_n$$

with $\sup_n \operatorname{Dim} X_n < \infty$ and $\lim_{n\to\infty} \min_{x \in X_n} \operatorname{Rank} e_n(x) = \infty$.

Note that the no dimension growth condition is stronger than the slow dimension growth condition used in [Th3].

Proposition 6.3. *Assume that* $\operatorname{Dim} Y < \infty$ *and that ϕ is strongly transitive and not injective. It follows $C_r^*\left(R_\phi\right)$ is an AH-algebra with no dimension growth.*

Proof. For each n we have that

$$C_r^*\left(R\left(\phi^n\right)\right) \simeq e_n C\left(Y, M_{m_n}\right) e_n \tag{6.3}$$

for some $m_n \in \mathbb{N}$ and some projection $e_n \in C\left(Y, M_{m_n}\right)$. Although this seems to be well known it is hard to find a proof anywhere so we point out that it can proved by specializing the proof of Theorem 3.2 in [Th1] to the case of a surjective local homeomorphism ϕ. In fact, it suffices to observe that the C^*-algebra A_ϕ which features in Theorem 3.2 of [Th1] is $C(Y)$ in this case. Since $\min_{y \in Y} \operatorname{Rank} e_n(y)$ is the minimal dimension of an irreducible representation of $C_r^*\left(R\left(\phi^n\right)\right)$ it therefore now suffices to show that the minimal dimension of the irreducible representations of $C_r^*\left(R(\phi^n)\right)$ goes to infinity when n does. It follows from Lemma 3.4 of [Th1] that the minimal dimension of the irreducible representations of $C_r^*\left(R(\phi^n)\right)$ is the same as the number $\min_{y \in Y} \#\phi^{-n}(y)$. It follows from Lemma 6.1 that

$$\lim_{n\to\infty} \min_{y \in Y} \#\phi^{-n}(y) = \infty,$$

exponentially fast in fact. □

Lemma 6.4. *Assume that $C_r^*(\Gamma_\phi)$ is simple. Then either ϕ is a homeomorphism or else*

$$\lim_{n\to\infty} \sup_{x\in Y} m(x)^{-1}m(\phi(x))^{-1}m\left(\phi^2(x)\right)^{-1}\dots m\left(\phi^{n-1}(x)\right)^{-1} = 0, \tag{6.4}$$

where $m : Y \to \mathbb{N}$ is the function (4.1).

Proof. Assume (6.4) does not hold. Since ϕ is a local homeomorphism, the function m is continuous so it follows from Dini's theorem that there is at least one x for which

$$\lim_{n\to\infty} m(x)^{-1}m(\phi(x))^{-1}m\left(\phi^2(x)\right)^{-1}\dots m\left(\phi^{n-1}(x)\right)^{-1} \tag{6.5}$$

is not zero. For this x there is a K such that $\#\phi^{-1}\left(\phi^k(x)\right) = 1$ when $k \geq K$, whence the set

$$F = \left\{y \in Y : \ \#\phi^{-1}\left(\phi^k(y)\right) = 1 \ \forall k \geq 0\right\}$$

is not empty. Note that F is closed and that $\phi^{-k}\left(\phi^k(F)\right) = F$ for all k, i.e. F is ϕ-saturated. It follows from Corollary 3.5 that F determines a proper ideal I_F in $C_r^*(R_\phi)$. Since $\phi(F) \subseteq F$, it follows that $\widehat{\phi}(I_F) \subseteq I_F$. Then Theorem 4.10 of [Th1] and the simplicity of $C_r^*(\Gamma_\phi)$ imply that either ϕ is injective or $I_F = \{0\}$. But $I_F = \{0\}$ means that $F = Y$ and thus that ϕ is injective. Hence ϕ is a homeormophism in both cases. ☐

Theorem 6.5. *Let $\varphi : X \to X$ be a locally injective surjection on a compact metric space X of finite covering dimension, and let (Y, ϕ) be its canonical locally homeomorphic extension. Let A be a simple quotient of $C_r^*(\Gamma_\varphi)$. It follows that A is $*$-isomorphic to either*

1) *a full matrix algebra $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, or*
2) *the crossed product $C(F) \times_{\phi|_F} \mathbb{Z}$ corresponding to an infinite minimal closed totally ϕ-invariant subset $F \subseteq Y$ on which ϕ is injective, or*
3) *a purely infinite, simple, nuclear, separable C^*-algebra; more specifically to the crossed product $C_r^*\left(R_{\phi|_F}\right) \times_{\widehat{\phi|_F}} \mathbb{N}$ where F is an infinite minimal closed totally ϕ-invariant subset of Y and $C_r^*\left(R_{\phi|_F}\right)$ is an AH-algebra with no dimension growth.*

Proof. If A is not a matrix algebra it has the form $C_r^*\left(\Gamma_{\phi|_F}\right)$ for some infinite minimal closed totally ϕ-invariant subset $F \subseteq Y$ by (2.2) and Corollary 4.21. If ϕ is injective on F we are in case 2). Assume not. Since $\operatorname{Dim} F \leq \operatorname{Dim} Y \leq \operatorname{Dim} X$ by Proposition 5.1 it follows from Proposition 6.3 that $C_r^*\left(R_{\phi|_F}\right)$ is an AH-algebra with no dimension growth. By [An] (or Theorem 4.6 of [Th1]) we have an isomorphism

$$C_r^*\left(\Gamma_{\phi|_F}\right) \simeq C_r^*\left(R_{\phi|_F}\right) \times_{\widehat{\phi|_F}} \mathbb{N},$$

where $\widehat{\phi|_F}$ is the endomorphism of $C_r^*\left(R_{\phi|_F}\right)$ given by conjugation with $V_{\phi|_F}$. We claim that the pure infiniteness of $C_r^*\left(R_{\phi|_F}\right) \times_{\widehat{\phi|_F}} \mathbb{N}$ follows from Theorem 1.1 of [Th3]. For this it remains only to check that $\widehat{\phi|_F} = \operatorname{Ad} V_{\phi|_F}$ satisfies the two conditions on β in Theorem 1.1 of [Th3], i.e. that $\widehat{\phi|_F}(1) = V_{\phi|_F}V_{\phi|_F}^*$ is a full projection and that there is no $\widehat{\phi|_F}$-invariant trace state on $C_r^*\left(R_{\phi|_F}\right)$. The first thing was observed already in Lemma 4.7 of [Th1] so we focus on the second. Observe that it follows from Lemma 2.24 of [Th1] that $\omega = \omega \circ P_{R_\phi}$ for every trace state ω of $C^*_r(R_\phi)$. By using this, a direct calculation as on page 787 of [Th1] shows that

$$\omega\left(V^n_{\phi|_F} {V^*_{\phi|_F}}^n\right) \leq \sup_{y \in Y} \left[m(y)m(\phi(y))\dots m\left(\phi^{n-1}(y)\right)\right]^{-1}$$

Then Lemma 6.4 implies that $\lim_{n\to\infty} \omega\left(V^n_{\phi|_F} {V^*_{\phi|_F}}^n\right) = 0$. In particlar, ω is not $\widehat{\phi|_F}$-invariant. □

Corollary 6.6. *Assume that $C^*_r(\Gamma_\varphi)$ is simple and that $\operatorname{Dim} X < \infty$. It follows that $C^*_r(\Gamma_\varphi)$ is purely infinite if and only if φ is not injective.*

Proof. Assume first that φ is injective. Then $C^*_r(\Gamma_\varphi)$ is the crossed product $C(X) \times_\varphi \mathbb{Z}$ which is stably finite and thus not purely infinite.

Conversely, assume that φ is not injective. Then a direct calculation, as in the proof of Theorem 4.8 in [Th1], shows that V_φ is a non-unitary isometry in $C^*_r(\Gamma_\varphi)$. Since the C^*-algebras which feature in case 1) and case 2) of Theorem 6.5 are stably finite, the presence of a non-unitary isometry implies that $C^*_r(\Gamma_\varphi)$ is purely infinite. □

Corollary 6.7. *Let S be a one-sided subshift. If the C^*-algebra $\mathcal{O}_S$ associated with S in* [Ca] *is simple, then it is also purely infinite.*

Proof. It follows from Theorem 4.18 in [Th1] that $\mathcal{O}_S$ is isomorphic to $C^*_r(\Gamma_\sigma)$ where σ is the shift map on S. If $\mathcal{O}_S$ is simple, S must be infinite and it then follows from Proposition 2.4.1 in [BS] (cf. Theorem 3.9 in [BL]) that σ is not injective. The conclusion follows then from Corollary 6.6. □

In Corollary 6.7 we assume that the shift map σ on S is surjective. It is not clear if the result holds without this assumption.

For completeness we point out that when X is totally disconnected (i.e. zero dimensional) the algebra $C^*_r\left(R_{\phi|_F}\right)$ which features in case 3) of Theorem 6.5 is approximately divisible, cf. [BKR]. We don't know if this is the case in general, but a weak form of divisibility is always present in $C^*_r(R_\phi)$ when $C^*_r(\Gamma_\varphi)$ is simple and ϕ not injective, cf. [Th3].

Proposition 6.8. *Assume that Y is totally disconnected and ϕ strongly transitive and not injective. It follows that $C^*_r(R_\phi)$ is an approximately divisible AF-algebra.*

Proof. It follows from Proposition 6.8 of [DS] that $C^*_r(R_\phi)$ is an AF-algebra. As pointed out in Proposition 4.1 of [BKR] a unital AF-algebra fails to be approximately divisible only if it has a quotient with a non-zero abelian projection. If $C^*_r(R_\phi)$ has such a quotient there is also a primitive quotient with an abelian projection; i.e. by Proposition 3.6 there is an $x \in Y$ such that $C^*_r\left(R_{\phi|_{\overline{H(x)}}}\right)$ has a non-zero abelian projection p. It follows from (3.1) that every projection of $C^*_r\left(R_{\phi|_{\overline{H(x)}}}\right)$ is unitarily equivalent to a projection in $C^*_r\left(R\left(\phi^n|_{\overline{H(x)}}\right)\right)$ for some n. Since $\overline{H(x)}$ is totally disconnected we can use Proposition 6.1 of [DS] to conclude that every projection in $C^*_r\left(R\left(\phi^n|_{\overline{H(x)}}\right)\right)$ is unitarily equivalent to a projection in $D_{R_{\phi|_{\overline{H(x)}}}} = C\left(\overline{H(x)}\right)$. We may therefore assume that $p \in C\left(\overline{H(x)}\right)$ so that $p = 1_A$ for some clopen

$A \subseteq \overline{H(x)}$. Then $H(x) \cap A \neq \emptyset$ so by exchanging x with some element in $H(x)$ we may assume that $x \in A$. If there is a $y \neq x$ in A such that $\phi^k(x) = \phi^k(y)$ for some $k \in \mathbb{N}$, consider functions $g \in C\left(\overline{H(x)}\right)$ and $f \in C_c(R_\phi)$ such that $g(x) = 1, g(y) = 0$, $\operatorname{supp} g \subseteq A$, $\operatorname{supp} f \subseteq R_\phi \cap (A \times A)$ and $f(x,y) \neq 0$. Then $f, g \in 1_A C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right) 1_A$ and $gf \neq 0$ while $fg = 0$, contradicting that $1_A C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right) 1_A$ is abelian. Thus no such y can exist which implies that $\pi_x(1_A) = 1_{\{x\}}$, where π_x is the representation (2.1), restricted to the subspace of H_x consisting of the functions supported in $\{(x', k, x) \in \Gamma_\phi : \ k = 0\}$. It follows that $\pi_x\left(1_A C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right) 1_A\right) \simeq \mathbb{C}$. Consider a non-zero ideal $J \subseteq \pi_x\left(C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right)\right)$. Then $\pi_x^{-1}(J)$ is a non-zero ideal in $C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right)$ and it follows from Corollary 3.5 that there is an open non-empty subset U of $\overline{H(x)}$ such that $\phi^{-k}\left(\phi^k(U)\right) = U$ for all k and $C_0(U) = \pi_x^{-1}(J) \cap C\left(\overline{H(x)}\right)$. Since $H(x) \cap U \neq \emptyset$, it follows that $x \in U$ so there is a function $g \in \pi_x^{-1}(J) \cap C\left(\overline{H(x)}\right)$ such that $g(x) = 1$. It follows that $\pi_x(g 1_A) = 1_{\{x\}} = \pi_x(1_A) \in J$. This shows that $\pi_x(1_A)$ is a full projection in $\pi_x\left(C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right)\right)$ and Brown's theorem, [Br], shows now that $\pi_x\left(C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right)\right)$ is stably isomorphic to $\pi_x\left(1_A C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right) 1_A\right) \simeq \mathbb{C}$. Since $\pi_x\left(C_r^*\left(R_{\phi|_{\overline{H(x)}}}\right)\right)$ is unital this means that it is a full matrix algebra. In conclusion we deduce that if $C_r^*(R_\phi)$ is not approximately divisible it has a full matrix algebra as a quotient. By Corollary 3.5 this implies that there is a finite set $F' \subseteq Y$ such that $F' = \phi^{-k}\left(\phi^k(F')\right)$ for all $k \in \mathbb{N}$. Since

$$\phi^{-k}\left(\phi^k(x)\right) \subseteq \phi^{-k-1}\left(\phi^{k+1}(x)\right) \subseteq F'$$

for all k when $x \in F'$, there is for each $x \in F$ a natural number K such that $\phi^{-k}\left(\phi^k(x)\right) = \phi^{-K}\left(\phi^K(x)\right)$ when $k \geq K$. Then $\#\phi^{-1}\left(\phi^k(x)\right) = 1$ for $k \geq K+1$, so that $m\left(\phi^k(x)\right) = 1$ for all $k \geq K$, which by Lemma 6.1 contradicts that ϕ is not injective. This contradiction finally shows that $C_r^*(R_\phi)$ is approximately divisible, as desired. □

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