Holes or Empty Pseudo-Triangles in Planar Point Sets

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Abstract. Let $E(k,\ell)$ denote the smallest integer such that any set of at least $E(k,\ell)$ points in the plane, no three on a line, contains either an empty convex polygon with k vertices or an empty pseudo-triangle with ℓ vertices. The existence of $E(k,\ell)$ for positive integers $k,\ell \geq 3$, is the consequence of a result proved by Valtr [*Discrete and Computational Geometry*, Vol. 37, 565–576, 2007]. In this paper, following a series of new results regarding the existence of empty pseudo-triangles in point sets with triangular convex hulls, we determine the exact values of $E(k,5)$ and $E(5,\ell)$, and prove new bounds on $E(k,6)$ and $E(6,\ell)$, for $k,\ell \geq 3$. By dropping the emptiness condition from $E(k,\ell)$, we get another related quantity $F(k,\ell)$, which is the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, no three on a line, contains a convex polygon with k vertices or a pseudo-triangle with ℓ vertices. Extending a result of Bisztriczky and Tóth [*Discrete Geometry, Marcel Dekker*, 49–58, 2003], we obtain the exact values of $F(k,5)$ and $F(k,6)$, and obtain non-trivial bounds on $F(k,7)$.

Keywords. Convex hull, Discrete geometry, Empty convex polygons, Erdős-Szekeres theorem, Pseudo-triangles, Ramsey-type results.

1 Introduction

The famous Erdős-Szekeres theorem [9] states that for every positive integer m, there exists a smallest integer $ES(m)$, such that any set of at least $ES(m)$ points in the plane, no three on a line, contains m points which lie on the vertices of a convex polygon. Evaluating the exact value of $ES(m)$ is a long standing open problem. A construction due to Erdős [10] shows that $ES(m) \geq 2^{m-2}+1$, which is conjectured to be sharp. It is known that $ES(4)=5$ and $ES(5)=9$ [14]. Following a long computer search, Szekeres and Peters [22] recently proved that $ES(6)=17$. The value of $ES(m)$ is unknown for all $m>6$. The best known upper bound for $m \geq 7$ is due to Tóth and Valtr [23]: $ES(m) \leq \binom{2m-5}{m-3}+1$.

In 1978 Erdős [8] asked whether for every positive integer k, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains k points which lie on the vertices of a convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole*. Esther Klein showed $H(4)=5$ and Harborth [12] proved that $H(5)=10$. Horton [13] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that $H(k)$ does not exist for $k \geq 7$. Recently, after a long wait, the existence of $H(6)$ has been proved by Gerken [11] and independently by Nicolás [19]. Later, Valtr [26] gave a simpler version of Gerken's proof.

These problems can be naturally generalized to polygons that are not necessarily convex. In particular, we are interested in pseudo-triangles, which are considered to be the natural counterpart of convex polygons. A pseudo-triangle is a simple polygon with exactly three vertices having interior angles less than 180°. A pseudo-triangle with ℓ vertices is called a ℓ-pseudo-triangle, and a set is said to contain an empty ℓ-pseudo-triangle if there exists a subset of ℓ points forming a pseudo-triangle which contains no point of the set in its interior. A pseudo-triangle with a,b,c as the convex vertices has three concave side chains between the vertices a, b and b, c, and c, a. Based on the length of the three side chains, a pseudo-triangle can be distinguished into three types: a *standard* pseudo-triangle, where each side chain has at least two edges, a *mountain*, where exactly one side chain has only one edge, and a *fan*, where exactly two side chains consists of only one edge (Figure 1). The *apex* of a *fan* pseudo-triangle is the convex vertex having exactly one edge in both its incident side chains.

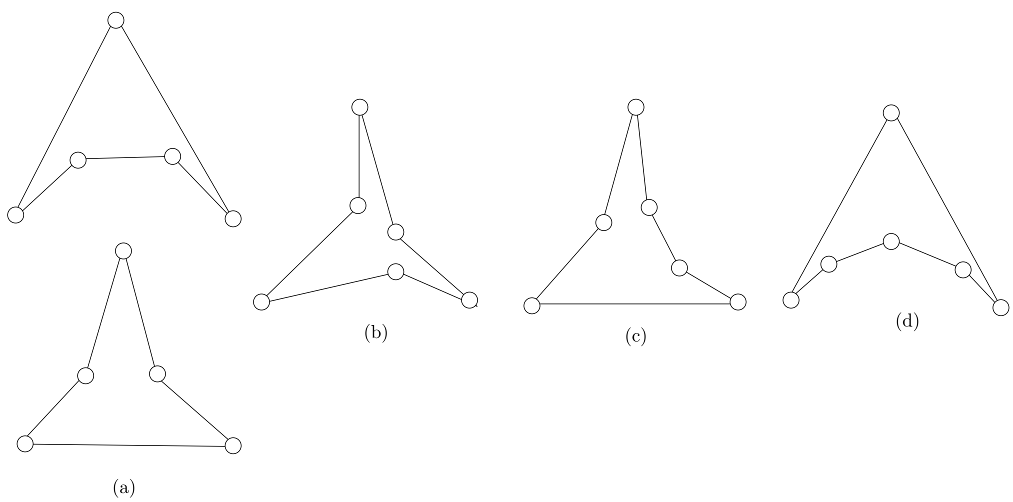

Fig. 1. Pseudo-triangles: (a) Types of 5-pseudo-triangles, (b) Standard 6-pseudo-triangle, (c) 6-mountain, (d) 6-fan.

In spite of the avalanche of research on the various combinatorial and algorithmic aspects of pseudo-triangles in recent times [21], little is known about the existence of empty pseudo-triangles in planar point sets. Kreveld and Speckmann [15] devised techniques to analyze the maximum and minimum number of empty pseudo-triangles defined by any planar point set. Ahn et al. [3] considered the optimization problems of computing an empty pseudo-triangle with minimum perimeter, maximum area, and minimum longest maximal concave chain.

In this paper, analogous to the quantity $H(k)$, we define a Ramsey-type quantity $E(k, \ell)$ as the smallest integer such that any set of at least $E(k, \ell)$ points in the plane, no three on a line, contains a k-hole or an empty ℓ-pseudo-triangle. The existence of $E(k, \ell)$ for all $k, \ell \geq 3$, is a consequence of the following result proved by Valtr [25], and later by Cěrný [7].

Theorem 1. *[7, 25] For any $k,\ \ell \leq 3$, there is a least integer $n(k, \ell)$ such that any point p in any set S of size at least $n(k, \ell)$, in general position, is the apex of an empty k-fan in S or it is one of the vertices of a ℓ-hole in S.*

Clearly, $E(k, \ell) \leq n(k, \ell)$. However, the general upper bound on $n(k, \ell)$ as proved by Valtr [25] is $2^{\binom{k+\ell-2}{k+1}} + 1$, which is double exponential in $k + \ell$. Therefore, following the long and illustrious history of the quantities $ES(k)$ and $H(k)$, evaluating exact values of $E(k, \ell)$ for small values of k, ℓ is an interesting problem, which has not been addressed before. In this paper, following a series of new results regarding the existence of empty pseudo-triangles in point sets with triangular convex hulls, we determine new bounds on $E(k, \ell)$ for small values of k and ℓ. We begin by proving that any set whose convex hull is a triangle and which contains at least two, three, or five interior points always contains an empty 5-pseudo-triangle, an empty 6-pseudo-triangle, or an empty 7-pseudo-triangle, respectively. Using these three results and some existing results in the literature, we determine the exact values of $E(k,5)$ and $E(5,\ell)$, for all $k,\ell \geq 3$. We also obtain some new bounds on $E(k,6)$ and $E(\ell,6)$, for different values of k and ℓ and discuss other implications of our results.

If the condition of emptiness is dropped from $E(k,\ell)$ we get another related quantity $F(k,\ell)$. Let $F(k,\ell)$ be the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, no three on a line, contains a convex k-gon or a ℓ-pseudo-triangle. From the Erdős-Szekeres theorem it follows that $F(k,\ell) \leq ES(k)$ for all $k,\ell \geq 3$. Evaluating non-trivial bounds of $F(k,\ell)$ is also an interesting problem. While addressing a problem related to covering by convex and pseudo-convex polygons, Aichholzer et al. [2] showed that $F(6,6) = 12$. In this paper, using our results on empty-pseudo-triangles and extending a result of Bisztriczky and Fejes Tóth [6], we show that $F(k,5) = 2k-3$, $F(k,6) = 3k-6$, and obtain non-trivial on $F(k,7)$, for $k \geq 3$. As a consequence, we also get the exact value of $F(5,\ell)$ and new bounds on $F(6,\ell)$, for $\ell \geq 3$.

The paper is organized as follows. In Section 2 we introduce notations and definitions. In Section 3 we prove two preliminary observations. The results regarding the existence of empty pseudo-triangles in point sets with triangular convex hulls are presented in Section 4. The new bounds on $E(k,\ell)$ are presented in Sections 5 and 5.4, and bounds on $F(k,\ell)$ are given in Section 6. In Section 7 we summarize our results and give directions for future works.

2 Notations and Definitions

We first introduce the definitions and notations required for the remaining part of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the *convex hull* of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\mathcal{I}(CH(S))$, respectively. A region R in the plane is said to be *empty* in S if R contains no elements of S in its interior. Moreover, for any set T, $|T|$ denotes the cardinality of T.

By $P := p_1p_2\ldots p_k$ we denote a convex k-gon with vertices $\{p_1,p_2,\ldots,p_k\}$ taken in anti-clockwise order. $\mathcal{V}(P)$ denotes the set of vertices of P and $\mathcal{I}(P)$ the interior of P.

For any three points $p,q,r \in S$, $\mathcal{H}(pq,r)$ (respectively $\mathcal{H}_c(pq,r)$) denotes the open (respectively closed) halfplane bounded by the line pq containing the point r. Similarly, $\overline{\mathcal{H}}(pq,r)$ (respectively $\overline{\mathcal{H}}_c(pq,r)$) is the open (respectively closed) halfplane bounded by pq not containing the point r.

The *j-th convex layer* of S, denoted by $L\{j,S\}$, is the set of points that lie on the boundary of $CH(S\backslash\{\bigcup_{i=1}^{j-1} L\{i,S\}\})$, where $L\{1,S\} = \mathcal{V}(CH(S))$. $|L\{j,S\}|$ denotes the number of points of S in j-th convex layer.

Moreover, if $\angle rpq < \pi$, $Cone(rpq)$ denotes the interior of the angular domain $\angle rpq$. A point $s \in Cone(rpq) \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in $Cone(rpq)$ if $Cone(spq)$ is empty in S. Similarly, for any convex region R a point $s \in R \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in R if $Cone(spq) \cap R$ is empty in S. Also, for any convex region R, the point $s \in S$, which has the shortest perpendicular distance to the line segment pq, $p,q \in S$, is called the *nearest neighbor* of pq in R.

3 Empty Pseudo-Triangles: Preliminary Observations

A pseudo-triangle with a, b, c as the convex vertices has three concave side chains between the vertices a, b and b, c, and c, a. We denote the vertices of the pseudo-triangle lying on the concave side chain between a and b by $C(a, b)$. Similarly, we denote by $C(b, c)$ and $C(c, a)$, the vertices on the concave side chains between b, c and c, a, respectively.

In this section, we prove two observations about transformation and reduction of pseudo-triangles.

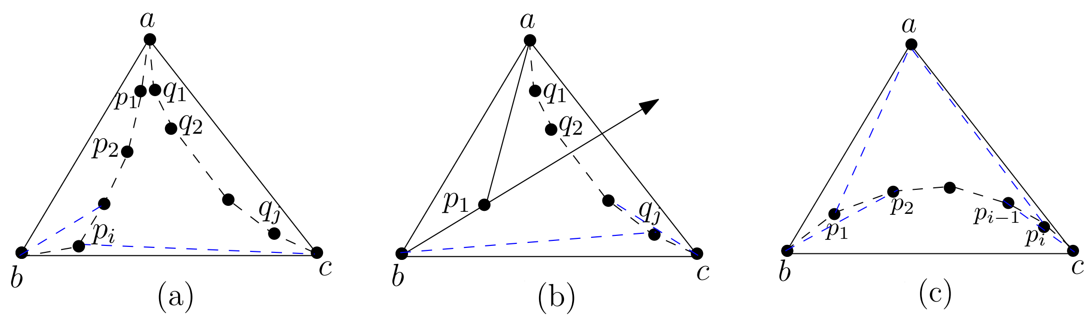

Fig. 2. Illustration for the proofs of Observation 1 and Observation 2.

Observation 1 *Any ℓ-pseudo-triangle can transformed to a standard ℓ-pseudo-triangle, for every $\ell \geq 6$, by appropriate insertion and deletion of edges.*

Proof. Let $\mathcal{P}$ be a ℓ-pseudo-triangle with $\ell \geq 6$, having convex vertices a, b, c, which is not standard. Then, we have the following two cases:

Case **1:** $\mathcal{P}$ is a ℓ-mountain with convex chains $C(a, b) = \{a,\ p_1, p_2, \ldots, p_i, b\}$, $C(b, c) = \{b, c\}$, and $C(a, c) = \{a, q_1, q_2, \ldots, q_j, b\}$, such that $i+j+3 = \ell$, arranged as shown in Figure 2(a). Let s_α be the nearest neighbor of bc in $C(a, b) \cup C(a, c)$. Then, $\{b, s_\alpha, c\}$ are the vertices a concave chain. If $i, j > 1$, then both $|C(a, b) \backslash \{s_\alpha\}| \geq 1$ and $|C(a, c) \backslash \{s_\alpha\}| \geq 1$, and w. l. o. g. we can assume that $s_\alpha \in C(a, b)$. In this case $s_\alpha = p_i$ and $\{a, p_1, p_2, \ldots, p_{i-1}, b\}$, $\{b, p_i, c\}$, and $\{a, q_1, q_2, \ldots, q_j, c\}$ are the vertices of the convex chains which form a standard ℓ-pseudo-triangle as shown in Figure 2(a). Otherwise, w. l. o. g. it suffices to assume that $i = 1$ (Figure 2(b)). If $Cone(p_1bc)$ contains a point of $C(a, c) \backslash \{a, c\}$, then $\{a, p_1, b\}$, $\{b, q_j, c\}$, and $\{a, q_1, q_2, \ldots, q_{j-1}, b\}$ are the vertices of the three concave chains of a standard ℓ-pseudo-triangle. Otherwise, all the points of $C(a, c) \backslash \{a, c\}$ are in $Cone(abp_1)$, and $\{a, q_1, b\}$, $\{b, p_1, c\}$, and $\{a, q_2, q_3, \ldots, q_i, c\}$ are the vertices of the three concave chains of a standard ℓ-pseudo-triangle.

Case **2:** $\mathcal{P}$ is a ℓ-fan with $C(a, b) = \{a, b\}$, $C(b, c) = \{b, p_1, p_2, \ldots, p_i, c\}$ and $C(a, c) = \{a, b\}$, where $i+3 = \ell$, as shown in Figure 2(c). Then, the ℓ-pseudo-triangle with concave chains formed by the set of vertices $\{a, p_1, b\}$, $\{b, p_2, p_3, \ldots, p_{i-1}, c\}$, and $\{a, p_i, b\}$ is standard (Figure 2(c)). □

Observation 2 *An empty ℓ-mountain contains an empty m-mountain whenever $3 \leq m < \ell$.*

Proof. It suffices to show that every empty ℓ-mountain contains an empty $(\ell-1)$-mountain for any $\ell \geq 4$. Let $\mathcal{P}$ be a ℓ-mountain with $\ell \geq 4$, having convex vertices a, b, c. Let

$C(a,b) = \{a, p_1, p_2, \ldots, p_i, b\}$, $C(b,c) = \{b,c\}$, and $C(a,c) = \{a, q_1, q_2, \ldots, q_j, b\}$ be the vertices of the three concave chains of $\mathcal{P}$, such that $i + j + 3 = \ell$, as shown in Figure 2(a). If both $i, j > 1$, an empty $(\ell - 1)$-mountain can be easily obtained by taking the nearest neighbor of bc in $C(a,b) \cup C(a,c)$ and removing either b or c.

Otherwise, w. l. o. g. assume that $i = 1$. If $Cone(p_1bc) \cap (C(a,c) \backslash \{a,c\})$ is non-empty, then $\{a, p_1, b\}$, $\{b, q_j\}$, and $\{a, q_1, q_2, \ldots, q_j\}$ forms an empty $(\ell - 1)$-mountain (Figure 2(b)). Similarly, if $Cone(abp_1) \cap (C(a,c) \backslash \{a,c\})$ is non-empty, then $\{b, p_1, q_1\}$, $\{b, c\}$, and $\{q_1, q_2, \ldots q_j, c\}$ form an empty $(\ell - 1)$-mountain. □

4 Empty Pseudo-Triangles in Point Sets with Triangular Convex Hulls

In this section we prove three results about the existence of empty pseudo-triangles in point sets with triangular convex hulls. These results will be used later to obtain new bounds on $E(k, \ell)$ and $F(k, \ell)$.

4.1 Empty 5-Pseudo-Triangle

Lemma 1. *Any set S of points in the plane, in general position, with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 2$, contains an empty 5-pseudo-triangle.*

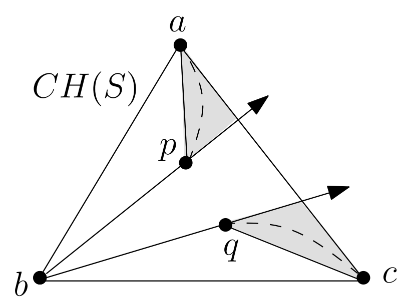

Fig. 3. Illustration for the proof of Lemma 1.

Proof. Let $\mathcal{V}(CH(S)) = \{a, b, c\}$, with the vertices taken in counter-clockwise order. Consider two points $p, q \in \mathcal{I}(CH(S))$, which are consecutive in the radial order around the vertex b of $\mathcal{V}(CH(S))$, that is, $Cone(pbq)$ is empty in S. Let $C_p = \mathcal{V}(CH(\mathcal{H}_c(bp, a) \cap S))$ and $C_q = \mathcal{V}(CH(\mathcal{H}_c(bq, c) \cap S))$ (Figure 3). Observe that $C_p \cup C_q$ form an empty ℓ-mountain with $\ell \geq 5$. The existence of an empty 5-pseudo-triangle now follows from Observation 2. □

4.2 Empty 6-Pseudo-Triangle

Lemma 2. *Any set S of points in the plane, in general position, with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 3$, contains an empty standard 6-pseudo-triangle.*

Proof. Let $\mathcal{V}(CH(S)) = \{a, b, c\}$, with the vertices taken in counter-clockwise order. To begin with, suppose that $|\mathcal{I}(CH(S))| = 3$. Let $p, q, r \in \mathcal{I}(CH(S))$ be such that $\mathcal{I}(qbc)$ is empty in S (Figure 4(a)). When both $\mathcal{I}(qab)$ and $\mathcal{I}(qac)$ are non-empty in S, either $apbqcr$ or $arbqcp$ forms an empty 6-pseudo-triangle. Therefore, w. l. o. g. assume that $\mathcal{I}(qab) \cap S$ is

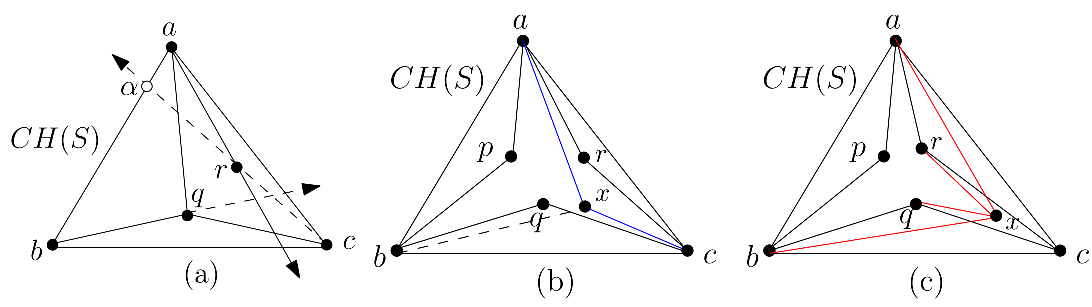

Fig. 4. Illustration for the proof of Lemma 2

empty and $p, r \in \mathcal{I}(qac) \cap S$. Let r be the first angular neighbor of $\overrightarrow{ac}$ in $Cone(qac)$ and α be the point where $\overrightarrow{cr}$ intersects the boundary of $CH(S)$. If $p \in Cone(ar\alpha)$, then $aprcqb$ is an empty 6-pseudo-triangle. Otherwise, $Cone(ar\alpha)$ is empty and either $arcpbq$ or $arcqbp$ is an empty 6-pseudo-triangle. The empty pseudo-triangle thus obtained can be transformed to an empty standard 6-pseudo-triangle by Observation 1.

Next, suppose that there are more than three points in $\mathcal{I}(CH(S))$. It follows from the previous arguments and from Observation 1 that there are three points $p, q, r \in \mathcal{I}(CH(S))$ $\mathcal{A}_1 = apbqcr$ such that is a standard 6-pseudo-triangle. If $\mathcal{A}_1$ is empty, we are done.

If $\mathcal{A}_1$ is not empty, there exists a point $x \in S$ in the interior of $\mathcal{A}_1$. The three line segments xa, xb, and xc may or may not intersect the boundary of $\mathcal{A}_1$. If any two of these line segments, say xa and xc, do not intersect with the edges of $\mathcal{A}_1$, then $\mathcal{A}_2 = apbqcx$ is a standard 6-pseudo-triangle which is contained in $\mathcal{A}_1$ (Figure 4(b)). Otherwise, there are two segments, say xa and xb, which intersect with the edges of $\mathcal{A}_1$. In this case, $\mathcal{A}_2 = apbqxr$ is a standard 6-pseudo-triangle contained in $\mathcal{A}_1$ (Figure 4(c)). If $\mathcal{A}_2$ is not empty, we repeat the above argument and after finitely many such repetitions, we finally obtain an empty standard 6-pseudo-triangle. □

4.3 Empty 7-Pseudo-Triangles

Let S be a set of points in the plane in general position. For $|\mathcal{V}(CH(S))| = 3$, an interior point $p \in S$ is called a $(x, y, z) - splitter$ of $CH(S)$ if the three triangles formed inside $CH(S)$ by the three line segments pa, pb, and pc contain $x \geq y \geq z$ interior points of S, respectively.

We use this definition to establish a sufficient condition for the existence of an empty 7-pseudo-triangle in sets having triangular convex hull.

Theorem 2. *Any set S of points in the plane, in general position, with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 5$, contains an empty 7-pseudo-triangle. Moreover, there exists a set S with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| = 4$, that does not contain a 7-pseudo-triangle.*

Proof of Theorem 2 We begin the proof of Theorem 2 with the following lemma:

Lemma 3. *Any set S of points in the plane, in general position, with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 5$, contains a 7-pseudo-triangle.*

Proof. Let $\mathcal{V}(CH(S)) = \{a, b, c\}$ with the vertices taken in counter-clockwise order. Since we have to find a 7-pseudo-triangle, which is not necessarily empty, it suffices to assume that $|\mathcal{I}(CH(S))| = 5$. First assume that $p \in \mathcal{I}(CH(S))$ is such that $\mathcal{I}(pab)$, $\mathcal{I}(pbc)$, and $\mathcal{I}(pca)$

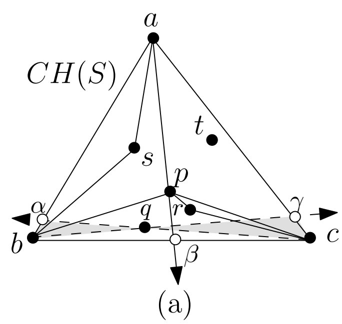

Fig. 5. Illustration for the proof of Lemma 3.

are all non-empty in S. Therefore, p must be a $(2,1,1)$-splitter of $CH(S)$. Without loss of generality, let $q, r \in \mathcal{I}(pbc) \cap S$, $s \in \mathcal{I}(pab) \cap S$, and $t \in \mathcal{I}(pac) \cap S$ be such that q is the nearest angular neighbor of $\overrightarrow{bc}$ in $\mathcal{I}(pbc)$. Let α, β, γ be the points where $\overrightarrow{cq}, \overrightarrow{ap}, \overrightarrow{bq}$ intersect the boundary of $CH(S)$, respectively. Let $R_1 = \mathcal{I}(bq\alpha) \cap \mathcal{I}(bpc)$ and $R_2 = \mathcal{I}(cq\gamma) \cap \mathcal{I}(bpc)$ (see Figure 5(a)). If $r \in R_1 \cup R_2$, then $asbqrcp$ or $asbrqcp$ is a 7-pseudo-triangle. Thus, assume that $(R_1 \cup R_2) \cap S$ is empty. If $r \in \mathcal{I}(\beta pc) \cap S$, then $asbqcrp$ is a 7-pseudo-triangle. Otherwise, $r \in \mathcal{I}(\beta pb) \cap S$, and $aprbqct$ is a 7-pseudo-triangle.

Therefore, suppose that none of the interior points of $CH(S)$ is a $(2,1,1)$-splitter of $CH(S)$. The three vertices of $CH(S)$ along with the interior points p, q, r form a standard 6-pseudo-triangle $\mathcal{P} = apbqcr$ by Lemma 2. Now, there are two cases:

Case **1:** $\mathcal{P}$ is empty in S. The remaining two points s and t in $\mathcal{I}(CH(S))$, must be in either of the three triangles - pab, qbc, and rca. W. l. o. g., assume that $s \in \mathcal{I}(qbc) \cap S$. Since q is not a $(2,1,1)$-splitter, either $\mathcal{I}(qab) \cap S$ or $\mathcal{I}(qac) \cap S$ is empty in S. If $\mathcal{I}(qac) \cap S$ is empty, $apbscqr$ is a 7-pseudo-triangle (Figure 5(b)). Otherwise, $\mathcal{I}(qab)$ is empty in S then $apqbscr$ is a 7-pseudo-triangle. .

Case **2:** $\mathcal{P}$ is non-empty in S. Let $s \in \mathcal{I}(\mathcal{P}) \cap S$. If any one of three line segments sa, sb, or sc intersects the boundary of $\mathcal{P}$ we get a 7-pseudo-triangle. Otherwise, two of these three segments go directly, and we have a smaller 6-pseudo-triangle with a, b, c as its convex vertices (Figure 5(c)). Continuing in this way, we finally get a 7-pseudo-triangle or an empty 6-pseudo-triangle with a, b, c as its convex vertices, which then reduces to *Case* 1. □

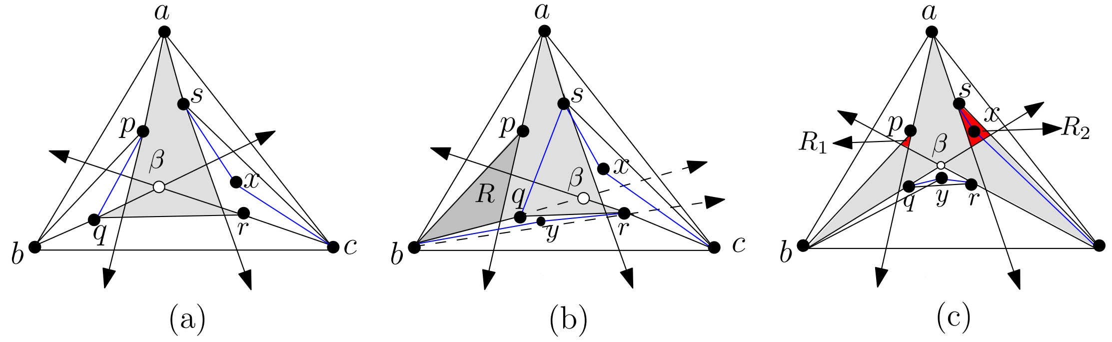

Fig. 6. Existence of an empty 7-pseudo-triangle: (a) $q, r \notin \mathcal{I}(Cone(pas)) \cap S$, (b) $q \in \mathcal{I}(Cone(pas)) \cap S$ and $r \notin \mathcal{I}(Cone(pas)) \cap S$, and (c) $q, r \in \mathcal{I}(Cone(pas)) \cap S$.

The above lemma implies that any triangle with more than 4 interior points contains a standard 7-pseudo-triangle. We now proceed to show that we can, in fact, obtain an empty 7-pseudo-triangle. Let S be a set of points with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 5$. Let $\mathcal{P}_0 = apbqrcs$ be a standard 7-pseudo-triangle contained in S with the least number of interior points, among all the standard 7-pseudo-triangles contained in S. Note that the points a, b, c may not be the vertices of $CH(S)$. Now, we have the following three cases:

Case **1:** $q, r \notin Cone(pas) \cap S$. Let β be the point of intersection of $\overrightarrow{bq}$ and $\overrightarrow{cr}$, and $x \in \mathcal{I}(\mathcal{P}_0) \cap S$. If $x \in \mathcal{I}(qr\beta) \cap S$, then $\mathcal{P}_1 = apqxrcs$ is a smaller 7-pseudo-triangle contained in $\mathcal{P}_0$. Therefore, $\mathcal{I}(qr\beta) \cap S$ can be assumed to be empty. Observe that, if (i) the line segment xa, and either of the line segments xb or xc do not intersect the boundary of $\mathcal{P}_0$, or (ii) both the line segments xb and xc intersect the boundary of $\mathcal{P}_0$, then we can easily construct a 7-pseudo-triangle with lesser interior points than $\mathcal{P}_0$. Therefore, the shaded region inside $\mathcal{P}_0$, shown in Figure 6(a), must be empty. Thus, x lies outside this shaded region and either $apqrcxs$ or $apxbqrs$ is a 7-pseudo-triangle with fewer interior points than $\mathcal{P}_0$ (Figure 6(a)).

Case **2:** $q \in Cone(pas) \cap S$ and $r \notin \mathcal{I}(Cone(pas)) \cap S$. By similar arguments as in *Case* 1, the lightly shaded region inside $\mathcal{P}_0$ shown in Figure 6(b) is empty in S. Moreover, if there exists a point $x \in S$ in the deeply shaded region R shown in Figure 6(b), then $apxbqrs$ is a 7-pseudo-triangle with fewer interior points than $\mathcal{P}$. Therefore, the points of S in $\mathcal{I}(\mathcal{P}_0)$ must lie outside these shaded regions. If $x \in \mathcal{I}(\mathcal{P}_0) \cap S$ is such that it lies below the line $\overrightarrow{br}$, then both xa and xb intersect the boundary of $\mathcal{P}_0$ and $apbqrxs$ is a 7-pseudo-triangle with fewer interior points. If x lies above $\overrightarrow{br}$ but below $\overrightarrow{bq}$, then $apbqxcs$ is a 7-pseudo-triangle with fewer interior points. Therefore, all the interior points of $\mathcal{P}_0$ must be above the line $\overrightarrow{bq}$. If $\mathcal{I}(bqr) \cap S$ is empty, $aqbrcxs$ is a 7-pseudo-triangle with fewer interior points. Otherwise, $\mathcal{I}(bqr) \cap S$ is non-empty. Let $Z = (\mathcal{I}(bqr) \cap S) \cup \{b, r\}$. If $|CH(Z)| \geq 4$, then $\mathcal{V}(CH(Z)) \cup \{q, x_0, c\}$ forms an empty k-mountain, with $k \geq 7$, where x_0 is the nearest angular neighbor of $\overrightarrow{bq}$ in $\mathcal{H}(bq, a) \cap \mathcal{I}(\mathcal{P}_0)$. Thus, $\mathcal{P}_0$ contains an empty 7-pseudo-triangle from Observation 2. Finally, assume that $|CH(Z)| = 3$. Let $\mathcal{V}(CH(S)) = \{b, y, r\}$. In this case, $\mathcal{P}_1 = byrcxsq$ is a 7-pseudo-triangle having fewer interior points than $\mathcal{P}_0$.

Case **3:** $q, r \in Cone(pas) \cap S$. By similar arguments as in *Case* 1 and *Case* 2, the lightly shaded regions inside $\mathcal{P}_0$, shown in Figure 6(c), are empty. At first, assume $\mathcal{I}(qr\beta) \cap S$ is non-empty. If there exists another point $x \in R_1 \cup R_2$ (where R_1 and R_2 are as shown in Figure 6(c)), then either $\mathcal{P}_1 = apxbqzr$ (if $x \in R_1$) or $\mathcal{P}_1 = aqzrcxs$ (if $x \in R_2$) is a 7-pseudo-triangle with $|\mathcal{I}(\mathcal{P}_1) \cap S| < |\mathcal{I}(\mathcal{P}_0) \cap S|$, where z is any point in $\mathcal{I}(qr\beta)$. Therefore, assume that $R_1 \cup R_2$ is empty in S. Let $Z = \mathcal{V}(CH((\mathcal{I}(qr\beta) \cap S) \cup \{q, r\}))$. If $|Z| \geq 4$, then $\{a, p, b\} \cup Z$ is an empty k-mountain, with $k \geq 7$. This can be shortened to obtain an empty 7-mountain by Observation 2. Therefore, assume that $|Z| = 3$ and let $\mathcal{I}(qr\beta) = \{y\}$. If $|\mathcal{I}(qby) \cap S| = 0$ then $aqbyrcs$ is 7-pseudo-triangle contained in $\mathcal{P}_0$ with less interior points. Otherwise, $|\mathcal{I}(qby) \cap S| \geq 1$ and let $Z_1 = \mathcal{V}(CH((\mathcal{I}(b\beta r) \cap S) \cup \{b, r\}))$. Now, as $|\mathcal{I}(qby) \cap S| \geq 1$, we have $|Z_1| \geq 4$. If $|Z_1| \geq 5$, $Z_1 \cup \{a, q\}$ forms an empty k-mountain, with $k \geq 7$. Thus, $\mathcal{P}_0$ contains an empty 7-pseudo-triangle from Observation 2. Therefore, $|Z_1| = 4$, which implies that $|\mathcal{I}(qby) \cap S| = 1$. Similarly, we can assume that $|\mathcal{I}(rcy) \cap S| = 1$. Let $\mathcal{I}(qby) \cap S = \{z_1\}$ and $\mathcal{I}(rcy) \cap S = \{z_2\}$. Then, depending upon the location of z_1, either aqz_1yz_2cr or az_1byz_2cr is a 7-pseudo-triangle with fewer interior points than $\mathcal{P}_0$. Finally, if $\mathcal{I}(qr\beta) \cap S$ is empty, we have a 7-pseudo-triangle with fewer interior points from arguments similar to those in *Case* 2.

Lemma 3 together with the discussions in the above three cases prove that any set S, of points in the plane, in general position, with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 5$, contains an empty 7-pseudo-triangle.

To show that this is tight, observe that one of the side chains of a 7-pseudo-triangle must have at least three edges. Therefore, any set S with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| = 4$ containing a 7-pseudo-triangle must contain a 4-hole with exactly two consecutive vertices belonging to the vertices of $\mathcal{V}(CH(S))$. It is easy to see that this condition is violated in the point set shown in Figure 7(a), and the result follows.

5 $E(k, \ell)$

As mentioned earlier, $E(k, \ell)$ is the smallest integer such that any set of at least $E(k, \ell)$ points in the plane, no three on a line, contains a k-hole or an empty ℓ-pseudo-triangle. The existence of $E(k, \ell)$ for all $k, \ell \geq 3$, is a consequence of a result of Valtr [25] and Čěrný [7] (Theorem 1). However, the general upper bound on $E(k, \ell)$ obtained from Valtr's [25] result is double exponential in $k + \ell$. In this section we obtain new bounds on $E(k, \ell)$ for small values of k and ℓ.

It is clear that $E(k, 3) = E(3, \ell) = 3$, for all $k, \ell \geq 3$. Also, $E(k, 4) = k$ for $k \geq 4$ and $E(4, \ell) = 5$, $\ell \geq 5$, since $H(4) = 5$. Using the results proved in the previous section we now proceed to obtain new bounds on $E(k, \ell)$ for several small values of $k, \ell \geq 5$.

We begin by introducing the notion of λ-convexity, where λ is a non-negative integer. A set S of points in the plane, in general position, is said to be λ-convex if every triangle determined by S contains at most λ points of S. Valtr [24, 25] and Kun and Lippner [18] proved that for any $\lambda \geq 1$ and $\nu \leq 3$, there is a least integer $N(\lambda, \nu)$ such that any λ-convex point set of size at least $N(\lambda, \nu)$ contains a ν-hole. The best known upper bound on $N(\lambda, \nu)$ for general λ and ν, due to Valtr [25], is $N(\lambda, \nu) \leq 2^{\binom{\lambda+\nu}{\lambda+2}-1} + 1$, which is double-exponential in $\lambda + \nu$. No lower bound on $N(\lambda, \nu)$ better than exponential in $\lambda + \nu$ is known.

5.1 $E(k, 5)$

In this section we determine the exact value of $E(k, 5)$ by using Lemma 1 and a result of Károlyi et al. [16].

Although, in general, the there is a gap of an exponential factor of $\lambda + \nu$ between the best known upper and lower bounds of $N(\lambda, \nu)$, in the special when $\lambda = 1$ much more can be said. Kun and Lippner [18] proved the general upper bound $N(1, \nu) \leq 2^{\lceil (2\nu+5)/3 \rceil}$. Károlyi et al. [17] proved that $N(1, \nu) \geq M_\nu$ for odd values of ν, where

$$M_\nu := \begin{cases} 2^{(\nu+1)/2} - 1, & \text{for } \nu \geq 3 \text{ odd;} \\ \frac{3}{2} 2^{\nu/2} - 1, & \text{for } \nu \geq 4 \text{ even.} \end{cases}$$

Finally, Károlyi et al. [16] proved that for any $\nu \geq 3$, $N(1, \nu) = M_\nu$.

Using this result, now we prove the following theorem:

Theorem 3. *For every positive integer $k \geq 3$, $E(k, 5) = M_k$.*

Proof. Let S be a set of M_k points in the plane, in general position. If there are three points in S such that the triangle determined by them contains more than 1 point of S in its interior, then by Lemma 1 S contains an empty 5-pseudo-triangle. Therefore, S contains a empty 5-pseudo-triangle unless S is 1-convex. However, the maximum size of a 1-convex set not containing a 5-hole is $N(1,k) - 1 = M_k - 1$. Therefore, if S is 1-convex, it always contains a 5-hole. This implies that $E(k,5) \leq M_k$.

Moreover, if a set is 1-convex, it does not contain any empty 5-pseudo-triangle. This implies that $E(k,5) \geq N(1,k) - 1 = M_k - 1$, which together with the upper bound mentioned above proves that for every $k \geq 3$, $E(k,5) = M_k$. □

5.2 $E(5, \ell)$

It is obvious that $E(5,3) = 3$ and $E(5,4) = 5$. It follows from Theorem 3 that $E(5,5) = 7$. In this section using the following result, proved by the authors in [5], we determine the values of $E(5,\ell)$, for $\ell \geq 6$

Theorem 4. *[5]Any set Z of 9 points in the plane in general position, with $|CH(Z)| \geq 4$, contains a 5-hole.* □

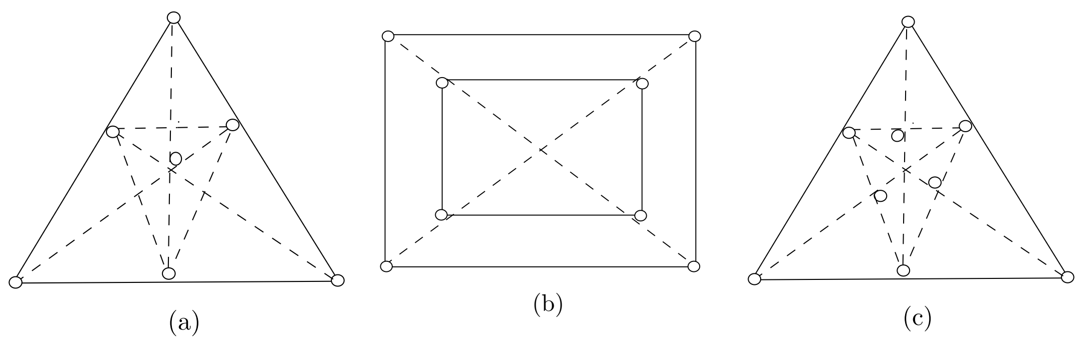

Fig. 7. (a) Triangle with 4 interior points and no 7-pseudo-triangle, (b) 8 points with no 5-hole and no 6-pseudo triangle or 7-pseudo-triangle, and (c) 9 points with no 5-hole and no 8-pseudo-triangle.

Using Lemma 1 and the above theorem, we now determine the exact values of $E(5,\ell)$ for $\ell \geq 6$.

Theorem 5. $E(5,6) = E(5,7) = 9$, *and* $E(5,\ell) = 10$, *for* $\ell \geq 8$.

Proof. The set of 8 points shown in Figure 7(b) contains no 5-hole and no empty 6-pseudo-triangle and no empty 7-pseudo-triangle. This implies that $E(5,6) > 8$ and $E(5,7) > 8$.

Now, consider a set S of 9 points in general position. It follows from Theorem 4 that S contains a 5-hole whenever $|CH(S)| \geq 4$. Now, if $|CH(S)| = 3$, then $|\mathcal{I}(CH(S))| = 5$, and the existence of an empty 6-pseudo-triangle and an empty 7-pseudo-triangle in S follows from Lemma 2 and Lemma 3, respectively. Therefore, $E(5,6) \leq 9$ and $E(5,7) \leq 9$, which together with the lower bound mentioned above implies that $E(5,6) = E(5,7) = 9$.

We know that for $\ell \geq 3$, $E(5,\ell) \leq H(5) = 10$, since every set of 10 points in general position, contains a 5-hole. The set of 9 points shown in Figure 7(c) contains no 5-hole and no empty ℓ-pseudo-triangle for $l \geq 8$. This implies that for $\ell \geq 8$, $E(5,\ell) = 10$. □

5.3 $E(k, 6)$

In Lemma 2 it was proved that any set S of points in the plane, in general position, with $|CH(S)| = 3$ and $|\mathcal{I}(CH(S))| \geq 3$, contains an empty standard 6-pseudo-triangle. This together with the fact that any 2-convex point set cannot contain a 6-pseudo-triangle, implies that $E(k,6) = N(2,k) \le 2^{\binom{k+2}{4}-1} + 1$.

However, in the special case when $k = 6$, we can obtain better bounds. For this reason, we need the following technical lemma:

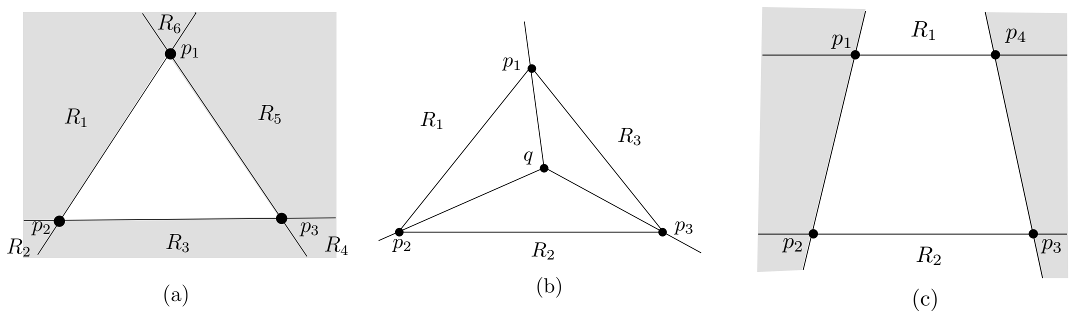

Fig. 8. Illustration for the proof of Lemma 4: (a) $|\mathcal{I}(CH(Z))| = 3$, (b) $|\mathcal{I}(CH(Z))| = 4$ and $|L\{2,Z\}| = 3$, and (c) $|\mathcal{I}(CH(Z))| = 4$ and $|L\{2,Z\}| = 4$.

Lemma 4. *If Z is a set of points in the plane in general position, with $|CH(Z)| \ge 8$ and $|\mathcal{I}(CH(Z))| \ge 4$, then Z contains a 6-hole.*

Proof. As it is always possible to reduce a convex 9-gon to a convex 8-gon with at most as many interior points, it suffices to prove the theorem for $|CH(Z)| = 8$.

If $|\mathcal{I}(CH(Z))| = 1$, then a 6-hole can be obtained easily. Now, if $|\mathcal{I}(CH(Z))| = 2$, then the line joining these two points divides the plane into two halfplanes one of which must contain at least four points of $\mathcal{V}(CH(Z))$. These 4 points along with the two points in $\mathcal{I}(CH(Z))$ form a 6-hole.

The remaining two cases are dealt with separately as follows:

Case **1:** $|\mathcal{I}(CH(Z))| = 3$. Consider the partition of the exterior of the triangle formed in the second layer into disjoint regions R_i as shown in Figure 8 (a). Clearly, Z contains 6-hole, unless the following inequalities hold:

$$|R_1| \le 2, \qquad |R_3| \le 2, \qquad |R_5| \le 2, \tag{1}$$

$$\begin{aligned} |R_6| + |R_1| + |R_2| &\le 3, \\ |R_2| + |R_3| + |R_4| &\le 3, \\ |R_4| + |R_5| + |R_6| &\le 3. \end{aligned} \tag{2}$$

Adding the inequalities of (2) and using the fact $|\mathcal{V}(CH(Z))| = 8$ we get $|R_2| + |R_4| + |R_6| \le 1$. On adding this inequality together with those of (1) we finally get $\sum_{i=1}^{6} |R_i| \le 7 < 8 = |\mathcal{V}(CH(Z))|$, which is a contradiction.

Case **2:** $|\mathcal{I}(CH(Z))| = 4$. We have the following two subcases based on the size of the second layer.

Case **2.1:** $|L\{2,Z\}| = 3$. Then $|L\{3,Z\}| = 1$, and consider the partition of the exterior of $CH(L\{2,Z\})$ into three disjoint regions R_i as shown in Figure 8(b). Clearly, S contains a 6-hole whenever $|R_i| \ge 3$, for $i \in \{1,2,3\}$. Otherwise, $|R_1| + |R_2| + |R_3| \le 6 < 8 = |\mathcal{V}(CH(Z))|$, which is a contradiction.

Case **2.2:** $|L\{2, Z\}| = 4$. Let $L\{2, Z\} = \{p_1, p_2, p_3, p_4\}$ be the vertices of the second layer taken in counter-clockwise order. Let R_1 and R_2 be the shaded regions as shown in Figure 8(c). It is easy to see that S contains a 6-hole unless $|R_1| + |R_2| \leq 1$, $|\overline{\mathcal{H}}(p_1p_2, p_3) \cap S| \leq 3$, and $|\overline{\mathcal{H}}(p_1p_2, p_3) \cap S| \leq 3$. However, by adding these three inequalities together we get $|\mathcal{V}(CH(Z))| \leq 7 < 8$, which is a contradiction. □

Using this lemma we now prove the following theorem:

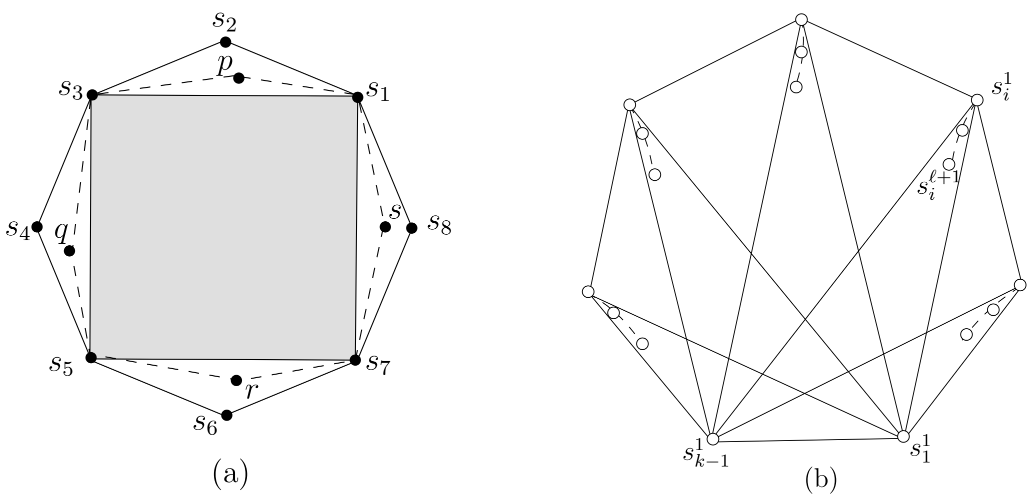

Fig. 9. (a) Illustration for the proof of Theorem 6 and (b) Illustration for the proof of Lemma 5.

Theorem 6. $12 \leq E(6, 6) \leq 18$.

Proof. Using the order-type database, Aichholzer et al. [2] obtained a set of 11 points that contains neither a convex hexagon nor a 6-pseudo-triangle [2]. This implies that $E(6, 6) \geq 12$.

Now, consider a set S of 18 points in general position. Suppose $|CH(S)| = k \leq 7$ and partition $CH(S)$ into $k - 2$ triangles whose vertex set is $\mathcal{V}(CH(S))$. Since, there are $18 - k$ points inside $CH(S)$, there exists a triangle which has at least $\lceil \frac{18-k}{k-2} \rceil$ points of S inside it. Observe that $\lceil \frac{18-k}{k-2} \rceil \geq 3$, since $k \leq 7$. Therefore, whenever $|CH(S)| \leq 7$, a triangle with at least three interior points exists and Lemma 2 ensures the existence of an empty 6-pseudo-triangle.

Next, suppose that $|CH(S)| = 8$. Let $\mathcal{V}(CH(S)) = \{s_1, s_2, \ldots, s_8\}$, where the vertices are taken in counter-clockwise order. If $|\mathcal{I}(s_1s_3s_5s_7) \cap S| \geq 5$, a triangle with at least three interior points exists and the existence of an empty 6-pseudo-triangle follows from Lemma 2. Therefore, suppose that $|\mathcal{I}(s_1s_3s_5s_7) \cap S| \leq 4$. Let p be the nearest neighbor of the line segment s_1s_3 in $\mathcal{H}(s_1s_3, s_2) \cap S$. Note that p can be the same as s_2, whenever $\mathcal{I}(s_1s_2s_3) \cap S$ is empty. Similarly, let $q, r, s \in S$ be the nearest neighbors of the line segments s_3s_5, s_5s_7, and s_7s_1, respectively as shown in Figure 9. Observe that the convex octagon $s_1ps_3qs_5rs_7s$ can have at most four points of S inside it. Lemma 4 now implies that this convex octagon always contains a 6-hole.

Finally, if $|CH(S)| \geq 9$, then $CH(S)$ can be reduced to a convex octagon with at most as many interior points, and the same argument as before works. Therefore, we have $E(6, 6) \leq 18$. □

Remark 1: Using the order type data-base Aichholzer et al. [2] observed that there exist precisely 9 out of over 2.33 billion realizable order types of 11 points which do not contain a convex hexagon nor a pseudo-triangle with 6 vertices. Experimenting with Overmars' *empty 6-gon program* [20] we were unable to find a set of 12 points which contains no 6-hole and empty 6-pseudo-triangle. In fact, it follows from Lemma 2 and the proof of Theorem 6 that a set S of 12 points contains an empty 6-pseudo-triangle or a 6-hole whenever $|CH(S)| \leq 5$ or $|CH(S)| \geq 8$. Therefore, a set of 12 points not containing a 6-hole or an empty 6-pseudo-triangle must have $|CH(S)| = 6$ or $|CH(S)| = 7$. Although we were unable to geometrically show the existence of a 6-hole or an empty 6-pseudo-triangle in these two cases, experimental evidence motivates us to conjecture that $E(6,6) = 12$. We believe that a very detailed analysis for the different cases that arise when $|CH(S)|$ is either 6 or 7, or some computer-aided enumeration method might be useful in settling the conjecture.

5.4 Other Improvements and Remarks

We now turn our attention to $E(6,\ell)$. Clearly, $E(6,\ell) \leq H(6)$ and $E(6,\ell) \geq N(\ell-4,6)$, since an $(\ell-4)$-convex set cannot contain an ℓ-pseudo-triangle. However, when $\ell = 7$, Theorem 2 and a result of Gerken [11] can be used to obtain a better upper bound. Consider a set S of 33 points in general position. Gerken [11] proved that any set which contains a 9-gon contains a 6-hole. Therefore, it suffices to assume that $|CH(S)| = k \leq 8$. $CH(S)$ can be partitioned into $k-2$ triangles whose vertex set is exactly $\mathcal{V}(CH(S))$. Since $|\mathcal{I}(CH(S)| = 33-k$, one of these $k-2$ triangles contains at least $\lceil \frac{33-k}{k-2} \rceil$ interior points. As $k \leq 8$, we have $\lceil \frac{33-k}{k-2} \rceil \geq 5$, and the existence of an empty 7-pseudo-triangle in S follows from Theorem 2.

Remark 2: Note that Theorem 2 gives a proof of the existence of $E(7,7)$, which does not use Theorem 1. Valtr's result [24, 25] implies that any 4-convex set without a 7-hole has at most $N(4,7)-1$ points. This together with Theorem 2 proves that, $E(7,7) \leq N(4,7)$. Moreover, a three convex set cannot contain a 7-pseudo-triangle, which implies that $E(7,7) \geq N(3,7)$.

Observe that if it is possible to show that for every integer $k \geq 3$, there exists a smallest integer $\Delta(k)$ such that any triangle with more than $\Delta(k)$ interior points contains an empty k-pseudo-triangle, then from Valtr's $\Delta(k)$-convexity result it will follow that $E(k) \leq N(\Delta(k),k)$.

The bounds obtained on the values $E(k,5), E(5,\ell), E(k,6)$, and $E(6,\ell)$ for different values of k and ℓ are summarized in Table 1.

6 $F(k,\ell)$

In the previous sections we have discussed about the existence of *empty* convex polygons or pseudo-triangles in point sets. If the empty condition is dropped, we get another related quantity $F(k,\ell)$, which we define as the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, in general position, contains a convex k-gon or a ℓ-pseudo-triangle. From the Erdős-Szekeres theorem it follows that $F(k,\ell) \leq ES(k)$ for all $k,\ell \geq 3$. Evaluating non-trivial bounds on $F(k,\ell)$ is also an interesting problem. While addressing problems related to partitions and decompositions of planar point sets, Aichholzer et al. [2] showed that $F(6,6) = 12$. Moreover, Aichholzer et al. [2] claim that $21 \leq F(7,7) \leq 23$, though the result is still unpublished. In this section, using our results on empty pseudo-triangles and the

Table 1. Bounds on $E(k, \ell)$

$E(k,5) = M_k := \begin{cases} 2^{(k+1)/2} - 1, & \text{for } k \geq 3 \text{ odd;} \\ \frac{3}{2}2^{k/2} - 1, & \text{for } k \geq 4 \text{ even.} \end{cases}$
$E(5,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ 9 & \text{for } \ell = 6, \\ 9 & \text{for } \ell = 7, \\ 10 & \text{for } \ell \geq 8. \end{cases}$
$E(k,6) = N(2,k) = \begin{cases} 3 & \text{for } k = 3, \\ 5 & \text{for } k = 4, \\ 9 & \text{for } k = 5, \\ [12,18] & \text{for } k = 6, \end{cases}$
$E(6,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ [12,18] & \text{for } \ell = 6, \\ [N(3,6),33] & \text{for } \ell = 7, \\ [N(\ell-4,6),H(6)] & \text{for } \ell \geq 8. \end{cases}$

extending a result of Bisztriczky and Fejes Tóth [6], we obtain the exact values of $F(k,5)$ and $F(k,6)$, and obtain non-trivial bounds on $F(k,7)$.

We shall see that any ℓ-convex point set with at least $(k-3)(\ell+1)+3$ points contains a convex k-gon. Relaxing the general position assumption Bisztriczky and Fejes Tóth [6] proved that, this bound is, in fact, tight. This means that there exists a set of $(k-3)(\ell+1)+2$ points, not necessarily in general position, which is ℓ-convex but has no convex k-gon.

In the following lemma, we generalize the construction of Bisztriczky and Fejes Tóth [6] to obtain a set of $(k-3)(\ell+1)+2$ points, *in general position*, which is ℓ-convex but has no convex k-gon, if $k < \ell/2$.

Lemma 5. *Let k, ℓ denote natural numbers such that $k \geq 3$ and $\ell < k/2$. Any set of at least $(k-3)(\ell+1)+3$ points in general position in the plane, which is ℓ-convex, contains k points in convex position, and in this respect the bound tight.*

Proof. Consider a set S of $(k-3)(\ell+1)+3$ points in the plane, in general position, which is ℓ-convex. If $|CH(S)| = k$, we are done. Otherwise, let $|CH(S)| = m \leq k-1$, and consider a triangulation of $CH(S)$ into $m-2$ triangles. Since S is ℓ-convex, this implies that $|S| \leq m + (m-2)\ell \leq (k-3)(\ell+1)+2$, which is a contradiction.

We now construct an ℓ-convex set Z of $(k-3)(\ell+1)+2$ points in general position, which contains no k-gon. Refer to Figure 9(b). Let $s_1^1, s_2^1, \ldots, s_{k-1}^1$ be a set of $k-1$ lying on the vertices of a convex $k-1$-gon in counter-clockwise direction. Consider, $Z = \{s_i^j \mid i = 2,3,\ldots,k-2; j = 1,2,\ldots,\ell+1\}$, where s_i^j is inside the triangle $s_{i-1}^1 s_i^1 s_{i+1}^1$, for $j = 2,3,\ldots,\ell+1$. Moreover, depending on whether $k-1$ is even or odd the points in Z satisfy the following property.

Case **A:** $k-1 = 2m$ is even. The set of points $\{s_i^j \mid j = 2,3,\ldots,\ell+1\}$ lies on a concave chain $C(s_i^1, s_1^1)$ from s_i^1 to s_1^1, for $i = 2,3,\ldots,m$. Similarly, the set of points $\{s_i^j \mid j = 2,3,\ldots,\ell+1\}$ lies on a concave chain $C(s_i^1, s_{k-1}^1)$ from s_i^1 to s_{k-1}^1, for $i = m+1, m+2, \ldots, 2m-1$ $(= k-2)$.

Case **B:** $k-1 = 2m+1$ is odd. The set of points $\{s_i^j \mid j = 2,3,\ldots,\ell+1\}$ lies on a concave chain $C(s_i^1, s_1^1)$ from s_i^1 to s_1^1, for $i = 2,3,\ldots,m$. Similarly, the set of points $\{s_i^j \mid j =$

$2, 3, \ldots, \ell+1\}$ lies on a concave chain $C(s_i^1, s_{k-1}^1)$ from s_i^1 to s_{k-1}^1, for $i = m+1, m+2, \ldots, 2m$ $(= k-2)$.

Clearly, $|Z| = (k-3)(\ell+1)+2$. We shall now show that the set Z constructed above is ℓ-convex. Consider three distinct points s_i^p, s_j^q, and s_k^r in S. To begin with let $p < q < r$, and consider the following three different cases:

Case **1:** $i = j = k$. Then $\mathcal{I}(s_i^p s_j^q s_k^r)$ is empty in Z.
Case **2:** $i = j \neq k$. Then the points of Z contained in $\mathcal{I}(s_i^p s_j^q s_k^r)$ are $s_i^{p+1}, s_i^{p+2}, \ldots, s_i^{q-1}$. Therefore, $|\mathcal{I}(s_i^p s_j^q s_k^r) \cap S| = q - p - 1 \leq \ell - 1$.
Case **3:** $i \neq j \neq k$. This implies, the points of S contained in $\mathcal{I}(s_i^p s_j^q s_k^r)$ are $s_j^{q+1}, s_j^{q+2}, \ldots, s_j^{\ell+1}$. Hence, $|\mathcal{I}(s_i^p s_j^q s_k^r) \cap S| = \ell - q + 1 \leq \ell$.

From the above three cases, we conclude that the set Z is ℓ-convex. It remains to show that it contains no convex k-gon. Let $\mathcal{P} \subset Z$ be a set of points which lie on the vertices of a convex polygon. Let $\mathcal{P}_i \subset \mathcal{P}$ be the set of points in $\mathcal{P}$ which has subscript i, for $i \in \{2, 3, \ldots, k-2\}$.

If for all $i \in \{2, 3, \ldots, k-2\}$, $|\mathcal{P}_i| \leq 1$, then clearly $|\mathcal{P}| \leq k-1 < k$. Otherwise assume that $|\mathcal{P}_i| \geq 2$, for at least some $i \in \{2, \ldots, k-2\}$. Note that due to the orientations of the arrangements of the points in $\mathcal{P}_i$ along concave chains as described above, there can be at most one subscripts i for which $|\mathcal{P}_i| \geq 3$. Next, observe that there can be at most two subscripts i for which $|\mathcal{P}_i| \geq 2$, since the set $\mathcal{P}_i$ is contained in triangle $s_{i-1}^1 s_i^1 s_{i+1}^1$. If there are two subscripts i and j such that both $|\mathcal{P}_i|, |\mathcal{P}_j| \geq 2$, then none of the points s_1^1 and s_{k-1}^1 can be in $\mathcal{P}$. If there is one such subscript i, then only one of the points s_1^1 or s_{k-1}^1 can be in $\mathcal{P}$.

With these observations, we have the following two cases:

Case **1:** $|\mathcal{P}_{i_0}| \geq 3$, for some i_0. We now have the following two cases:
Case **1.1:** For all $i \neq i_0$, $|\mathcal{P}_i| \leq 1$. In this case the largest size of a convex polygon in Z can be obtained by taking all the points in $\mathcal{P}_{i_0}$, where $i_0 = (k-1)/2$ or $i_0 = k/2$, depending on whether $k-1$ is even or odd, and one point from each $\mathcal{P}_i$ on one side of P_{i_0}, depending upon the curvature of the concave chain at $\mathcal{P}_{i_0}$. Therefore, the largest size of a convex polygon possible is $|\mathcal{P}| \leq (k-1)/2 + \ell$ for $k-1$ even, and $|\mathcal{P}| \leq k/2 + \ell$ for $k-1$ odd. Now, since $\ell < k/2$, by assumption, it follows that $|\mathcal{P}| < k$.
Case **1.2:** There exists some $j_0 \neq i_0$ such that $|\mathcal{P}_{j_0}| = 2$. In this case the largest size of the convex polygon can be obtained by taking i_0 as in *Case* 1.1, $j_0 = 2$ or $j_0 = k-2$, and one point each from every $\mathcal{P}_i$ between $\mathcal{P}_{i_0}$ and $\mathcal{P}_{j_0}$. Now, as none of the points s_1^1 or s_{k-1}^1 can be in $\mathcal{P}$, it follows that $|\mathcal{P}| \leq (k-1)/2 + \ell$ for $k-1$ even, and $|\mathcal{P}| \leq k/2 + \ell$ for $k-1$ odd.
Case **2:** $|\mathcal{P}_{i_0}| = 2$, for some i_0, and $|P_{j_0}| \leq 2$. If there exits some other $j_0 \neq i_0$ such that $|\mathcal{P}_{j_0}| = 2$, then size of a convex polygon that can be found in Z is obtained by taking $i_0 = 2$ and $j_0 = k-2$ (or vice versa) and one point from each $\mathcal{P}_i$ between $\mathcal{P}_{i_0}$ and $\mathcal{P}_{j_0}$. Clearly, the size of the largest convex polygon that can be obtained in this way is $|\mathcal{P}| \leq k-1$. Otherwise, for all $i \neq i_0$, $|P_{i_0}| = 1$, and it is easy to see that $|\mathcal{P}| \leq k-1$. □

Using this lemma, we now obtain the exact values of $F(k,5)$ and $F(k,6)$ in the following theorem:

Theorem 7. *For any positive integer $k \geq 3$, we have*

(i) $F(k,5) = 2k-3$ *for* $k \geq 3$.
(ii) $F(k,6) = 3k-6$ *for* $k \geq 3$.

Proof. Lemma 1 implies that any set which has a triangle with 2 interior points has a 5-pseudo-triangle. Moreover, a 1-convex set cannot contain a 5-pseudo-triangle. Therefore, part (i) follows from Lemma 5 by putting $\ell = 1$.

Similarly, Lemma 2 implies that any set which has a triangle with 3 interior points has a 6-pseudo-triangle. Moreover, a 2-convex set cannot contain a 5-pseudo-triangle. Therefore, part (ii) follows from Lemma 5 by putting $\ell = 2$. □

In the following theorem, using Lemma 5 and the results on 7-pseudo-triangles, we obtain new bounds on $F(k,7)$.

Theorem 8.

$$F(k,7) = \begin{cases} 3 & \text{for } k = 3, \\ 5 & \text{for } k = 4, \\ 9 & \text{for } k = 5, \\ [16,17] & \text{for } k = 6, \\ [21,23] & \text{for } k = 7, \\ [4k-9, 5k-12] & \text{for } k \geq 8. \end{cases}$$

Proof. Using the fact that $ES(4) = 5$ and $ES(5) = 9$, it is easy to obtain $F(4,7) = 5$ and $F(5,7) = 9$, respectively. For $k = 6$ we slightly modify the construction in Lemma 5 to obtain a set of 15 points, shown in Figure 10(a) which contains no 6-gon or 7-pseudo-triangle. This example and the fact that $ES(6) = 17$ [22], implies $16 \leq F(6,7) \leq 17$.

Theorem 2 implies that any triangle with 5 or more points in its interior contains a 7-pseudo-triangle. Lemma 5 with $\ell = 4$ implies that any 4-convex set of $5k-12$ points contains a k-hole, thus proving that $F(k,7) \leq 5k-12$. Moreover, any 3-convex point set cannot contain a 7-pseudo-triangle. The lower bound on $F(k,7)$ now follows from the tightness part of Lemma 5, with $\ell = 3$ and $k \geq 7$. Therefore, for $k \geq 7$ we have $4k-9 \leq F(k,7) \leq 5k-12$.

For $k = 7$, the above inequalities give $19 \leq F(7,7) \leq 23$. As mentioned earlier, the improved lower bound of 21 on $F(7,7)$ follows from a claim of Aichholzer et al. [1]. □

Remark 3: The set of 16 points shown in Figure 10(b) is clearly 4-convex. This implies that it cannot any ℓ-pseudo-triangle, for $\ell \geq 8$. Moreover, from arguments similar to those in Lemma 5 it is easy to see that it contains no convex 6-gon. Moreover, since $ES(6) = 17$, we have $F(6,\ell) = 17$, for $\ell \geq 8$.

Remark 4: Since an ℓ-convex point set cannot not contain any $(\ell+4)$ pseudo-triangle, it follows from Lemma 5 that $F(k,\ell+4) \geq (k-3)(\ell+1)+3$, whenever $\ell < k/2$.

The bounds obtained on the values $F(k,5), F(5,\ell), F(k,6), F(6,\ell)$, and $F(k,7)$ for different values of k and ℓ are summarized in Table 2.

7 Conclusions

In this paper we have introduced the quantity $E(k,\ell)$, which denotes the smallest integer such that any set of at least $E(k,\ell)$ points in the plane, no three on a line, contains either an empty convex polygon with k vertices or an empty pseudo-triangle with ℓ vertices. Though

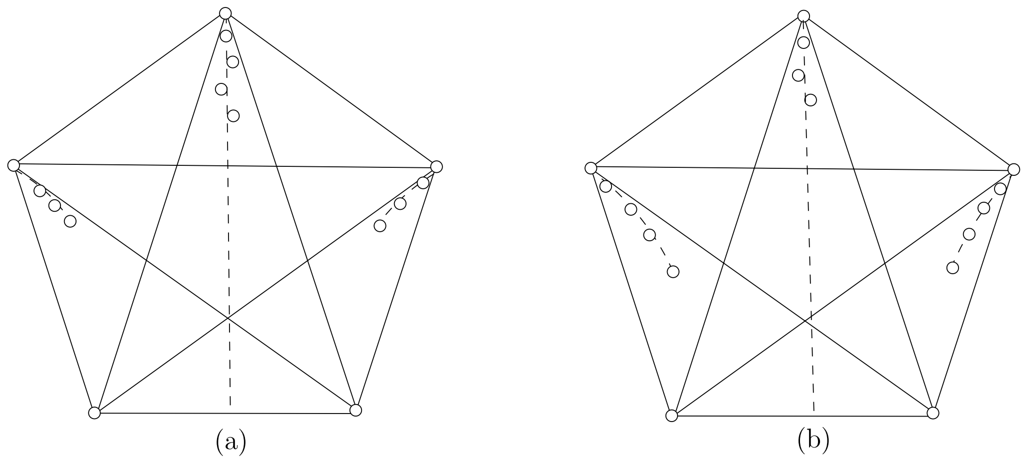

Fig. 10. (a) A set of 15 points not containing a 6-gon or a 7-pseudo-triangle, (b) A set of 16 points not containing a 6-gon or an ℓ-pseudo-triangle for $\ell \geq 8$.

Table 2. Summary of the results

$F(k,5) = 2k - 3$
$F(5,\ell) = \begin{cases} 3 \text{ for } \ell = 3, \\ 4 \text{ for } \ell = 4, \\ 7 \text{ for } \ell = 5, \\ 9 \text{ for } \ell \geq 6. \end{cases}$
$F(k,6) = 3k - 6$
$F(6,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ 12 & \text{for } \ell = 6, \\ [16,17] & \text{for } \ell = 7, \\ 17 & \text{for } \ell \geq 8. \end{cases}$
$F(k,7) = \begin{cases} 3 & \text{for } k = 3, \\ 5 & \text{for } k = 4, \\ 9 & \text{for } k = 5, \\ [16,17] & \text{for } k = 6, \\ [21,23] & \text{for } k = 7, \\ [4k-9, 5k-12] & \text{for } k \geq 8. \end{cases}$

the existence of $E(k,\ell)$ for positive integers $k, \ell \geq 3$, is the consequence of a result proved by Valtr [25], the general upper bound on $E(k,\ell)$ is double-exponential in $k + \ell$. In this paper following a series of new results regarding the existence of empty pseudo-triangles in point sets with triangular convex hulls, we determine the exact values of $E(k,5)$ and $E(5,\ell)$, and prove new bounds on $E(k,6)$ and $E(6,\ell)$, for $k,\ell \geq 3$. In particular, we show that $12 \leq E(6,6) \leq 18$ and conjecture the lower bound is, in fact, an equality. Proving this conjecture and tightening the bounds on $E(6,\ell)$, for $\ell \geq 7$ are interesting problems.

We have also introduced another related quantity $F(k,\ell)$, which is the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, no three on a line, contains a convex polygon with k vertices or a pseudo-triangle with ℓ vertices. Extending a result of Bisztriczky and Tóth [6] we have proved that $F(k,5) = 2k-3$, $F(k,6) = 3k-6$, and obtained new bounds on $F(k,7)$. Obtaining the exact values of $F(k,7)$ for $k \geq 6$ is another interesting problem.

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