RIEMANN HYPOTHESIS AND SOME NEW ASYMPTOTICALLY MULTIPLICATIVE INTEGRALS WHICH CONTAIN THE REMAINDER OF THE PRIME-COUNTING FUNCTION $\pi(x)$

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ABSTRACT. A new parametric integral is obtained as a consequence of the Riemann hypothesis. An asymptotic multiplicability is the main property of this integral.

1. The result

1.1. Let us remind that

$$\pi(x)=\int_0^x \frac{\mathrm{d}t}{\ln t}+P(x)=\mathrm{li}(x)+P(x). \tag{1.1}$$

The following theorem holds true.

Theorem. On the Riemann hypothesis

$$\int_2^\infty \frac{\ln(xe^{-2})}{x^{3/2+\delta}}P(x)\mathrm{d}x=-\frac{1}{\delta}+\mathcal{O}(1),\ \delta\in(0,\Delta) \tag{1.2}$$

where Δ is a sufficiently small fixed value and the $\mathcal{O}(1)$-function is bounded on $[0,\Delta]$.

1.2. Let

$$\Omega(\delta)=\int_2^\infty \frac{\ln(xe^{-2})}{x^{3/2+\delta}}\{-P(x)\}\mathrm{d}x. \tag{1.3}$$

Then we obtain from (1.2)

Corollary 1.

$$\Omega\left(\prod_{k=1}^n \delta_k\right)\sim\prod_{k=1}^n \Omega(\delta_k),\ \delta_k\to 0,\ k=1,\dots,n, \tag{1.4}$$

especially,

$$\Omega(\delta)\sim\sqrt[n]{\Omega(\delta^n)},$$

i.e., the function $\Omega(\delta)$ possesses the property of the asymptotic multiplicability.

Let

$$n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k},\ p_1,\dots,p_k\to\infty$$

Key words and phrases. Riemann zeta-function.

be the factorization of a natural number n. Then we have (see (1.4)

$$\frac{1}{n}=\prod_{l=1}^{k}\left(\frac{1}{p_l}\right)^{\alpha_l} \ \Rightarrow \ \Omega\left(\frac{1}{n}\right)\sim\prod_{l=1}^{k}\left\{\Omega\left(\frac{1}{p_l}\right)\right\}^{\alpha_l},$$

for example.

2. The main formula

2.1. Let us remind that

$$\begin{aligned} \pi(x)&=\int_2^x\frac{\mathrm{d}t}{\ln t}+\bar{P}(x)=\mathrm{Li}(x)+\bar{P}(x), \\ \bar{P}(x)&=P(x)+\mathrm{V.p.}\int_0^2\frac{\mathrm{d}t}{\ln t}=P(x)+K;\ K\approx 1.04, \end{aligned} \tag{2.1}$$

(see (1.1), (2.1) and [1], p. 3). The following lemma holds true.

Lemma 1.

$$\int_2^\infty \frac{\partial}{\partial x}\{\ln(1-x^{-s})\}\bar{P}(x)\mathrm{d}x=\ln\zeta(s)+\int_2^\infty\frac{\ln(1-x^s)}{\ln x}\mathrm{d}x, \tag{2.2}$$

for $\sigma>1,\ s=\sigma+it$.

Proof. We apply the formula ([2], p. 2)

$$\ln\zeta(s)=s\int_2^x\frac{\pi(x)}{x(x^s-1)}\mathrm{d}x,\ \sigma>1. \tag{2.3}$$

Since

$$\frac{s}{x(x^s-1)}=\frac{\partial}{\partial x}\{\ln(1-x^{-s})\}, \tag{2.4}$$

then we obtain from (2.3)

$$\ln\zeta(s)=\int_2^\infty\frac{\partial}{\partial x}\{\ln(1-x^{-s})\}\pi(x)\mathrm{d}x,\ \sigma>1. \tag{2.5}$$

Next

$$\frac{\mathrm{d}\mathrm{Li}(x)}{\mathrm{d}x}=\frac{1}{\ln x},\ \mathrm{Li}(2)=0,$$

(see (2.1)) and

$$\begin{aligned} &\int_2^\infty\frac{\partial}{\partial x}\{\ln(1-x^{-s})\}\mathrm{Li}(x)\mathrm{d}x=\mathrm{Li}(x)\ln(1-x^{-s})\Big|_{x=2}^{x=\infty}- \\ &-\int_2^\infty\frac{\ln(1-x^{-s})}{\ln x}\mathrm{d}x=-\int_2^\infty\frac{\ln(1-x^{-s})}{\ln x}\mathrm{d}x, \end{aligned} \tag{2.6}$$

then from (2.5) by (2.1), (2.6) the formula (2.2) follows. □

3. The differentiation of the formula (2.2)

Let $\prod_1$ denote the open rectangle generated by the points $1+\delta\pm i,\ \frac{3}{2}\pm i$ $(0<\delta$ is the sufficiently small fixed value). The following lemma holds true.

Lemma 2.

$$-\int_2^\infty F(x;s)\bar{P}(x)\mathrm{d}x=\frac{\zeta'(s)}{\zeta(s)}+G(s),\ s\in\Pi_1, \tag{3.1}$$

where

$$\begin{split} & F(x;s)=\sum_{n=0}^\infty\{s(n+1)\ln x-1\}x^{-(n+1)s-1}, \\ & G(s)=\sum_{n=0}^\infty\frac{1}{(n+1)s-1}\frac{1}{2^{(n+1)s-1}}. \end{split} \tag{3.2}$$

Proof. We use the formula (see (2.2), (2.4))

$$\int_2^\infty\frac{s}{x^s-1}\frac{\bar{P}(x)}{x}\mathrm{d}x=\sum_{n=1}^\infty\int_{p_n}^{p_{n+1}}\frac{s}{x^s-1}\frac{\bar{P}(x)}{x}\mathrm{d}x=\sum_{n=1}^\infty w_n(s). \tag{3.3}$$

Since $\bar{P}(x)=\mathcal{O}(x)$, and $\bar{P}(x),\ x\in(p_n,p_{n+1})$ is continuous (p_n is a prime number, $p_1=2$) then $w_n(s),\ s\in\Pi_1$ is an analytic function. Next, the uniform convergence of the series in the set Π_1 follows from the uniform convergence of the integral (see (3.1) in the set Π_1. Thus, by the theorem of Weierstrass, the integral in (3.3) is an analytic function in Π_1. Since

$$\begin{split} & \frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{x^s-1}=\frac{1}{x^s-1}-\frac{sx^s\ln x}{(x^s-1)^2}=\frac{x^{-s}}{1-x^{-s}}-\frac{sx^{-s}\ln x}{(1-x^{-s})^2}= \\ & -\sum_{n=0}^\infty\{s(n+1)\ln x-1\}x^{-(n+1)s}, \end{split} \tag{3.4}$$

then we have (see (3.3), (3.4))

$$\begin{split} & \frac{\mathrm{d}}{\mathrm{d}s}\int_2^\infty\frac{s}{x^s-1}\frac{\bar{P}(x)}{x}\mathrm{d}x=-\int_2^\infty F(x;s)\bar{P}(x)\mathrm{d}x,\ x\in\Pi_1, \\ & F(x;s)=\sum_{n=0}^\infty\{s(n+1)\ln x-1\}x^{-(n+1)s-1}. \end{split} \tag{3.5}$$

Similarly, we obtain

$$\begin{split} & \frac{\mathrm{d}}{\mathrm{d}s}\int_2^\infty\frac{\ln(1-x^{-s})}{\ln x}\mathrm{d}x=\int_2^\infty\frac{x^{-s}}{1-x^{-s}}\mathrm{d}x=\int_2^\infty\left\{\sum_{n=0}^\infty x^{-(n+1)s}\right\}\mathrm{d}x= \\ & =\sum_{n=0}^\infty\frac{1}{(n+1)s-1}\frac{1}{2^{(n+1)s-1}},\ s\in\Pi_1. \end{split} \tag{3.6}$$

Finally, we obtain the formula (3.1) by the differentiation of (2.2), (see (3.5), (3.6)). $\square$

4. Cancelation of the singular parts corresponding to the pole at $s=1$. The pole at $s=\frac{1}{2}$

Let Π_2 denote the open rectangle generated by the points $\frac{1}{2}\pm i, \frac{3}{2}\pm i$. The following lemma holds true.

Lemma 3.

$$\int_2^\infty F(x;s)\bar{P}(x)\mathrm{d}x = -\frac{1}{2s-1}+g(s),\ s\in\Pi_1 \tag{4.1}$$

where $g(s),\ s\in\Pi_2$ is the analytic and bounded function.

Proof. Let (see (3.1))

$$H(s)=\frac{\zeta'(s)}{\zeta(s)}+G(s),\ s\in\Pi_1. \tag{4.2}$$

First of all (see (3.2))

$$\begin{aligned} G(s)&=\frac{1}{s-1}\frac{1}{2^{s-1}}+\frac{1}{2s-1}\frac{1}{2^{2s-1}}+g_1(s), \\ g_1(s)&=\sum_{n=2}^\infty \frac{1}{(n+1)s-1}\frac{1}{2^{(n+1)s-1}},\ s\in\Pi_1. \end{aligned} \tag{4.3}$$

Since

$$\frac{1}{s-1}\frac{1}{2^{s-1}}=\frac{1}{s-1}e^{-(s-1)\ln 2}=\frac{1}{s-1}\left\{1-(s-1)\ln 2+\mathcal{O}(|s-1|^2)\right\}= \\ =\frac{1}{s-1}-\ln 2+\mathcal{O}(|s-1|)=\frac{1}{s-1}+g_2(s),$$

and similarly

$$\frac{1}{2s-1}\frac{1}{2^{s-1}}=\frac{1}{2s-1}-\ln 2+\mathcal{O}(|2s-1|)=\frac{1}{2s-1}+g_3(s)$$

then (see (4.2), (4.3))

$$\begin{aligned} H(s)&=\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}+\frac{1}{2s-1}+g_4(s),\ s\in\Pi_1 \\ g_4(s)&=g_1(s)+g_2(s)+g_3(s), \end{aligned} \tag{4.4}$$

where $g_4(s), s\in\Pi_2$ is the analytic and bounded function. Next, by the known formula

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}=b-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)+\sum_{n=1}^\infty\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}\right)=g_5(s) \tag{4.5}$$

where $\zeta(\rho_n)=0$, and $g_5(s),\ s\in\Pi_2$ is the analytic and bounded function. Finally, from (3.1) by (4.2)-(4.5) the asertion of the Lemma 3 follows. □

5. The analytic continuation of the formula (3.1)

5.1. The following lemma holds true.

Lemma 4. *On the Riemann hypothesis*

$$\int_2^\infty F(x;\sigma)\bar{P}(x)\mathrm{d}x=-\frac{1}{2\sigma-1}+g(\sigma),\ \sigma\in\left(\frac{1}{2},\frac{3}{2}\right) \tag{5.1}$$

where $g(\sigma),\ \sigma\in[\frac{1}{2},\frac{3}{2}]$ is the bounded function.

Proof. Let $\Pi_2(\delta)$ denote the open rectangle generated by the points $\frac{1}{2}+\delta\pm i; \frac{3}{2}\pm i$. We obtain from (3.2)

$$|F(x;s)| = \mathcal{O}\left(\frac{\ln x}{x^{\sigma+1}}\right). \tag{5.2}$$

Next, on the Riemann hypothesis, the following estimate of von Koch

$$P(x), \bar{P}(x) = \mathcal{O}(\sqrt{x}\ln x) \tag{5.3}$$

holds true (comp. (2.1)). Then we have (see (5.2), (5.3))

$$\int_2^\infty |F(x;s)\bar{P}(x)|\mathrm{d}x = \mathcal{O}\left(\int_2^\infty x^{-1-\frac{\delta}{2}}\mathrm{d}x\right) = \mathcal{O}\left(\frac{1}{\delta}\right),$$

i.e. the integral

$$\int_2^\infty F(x;s)\bar{P}(x)\mathrm{d}x,\ s\in\Pi_2(\delta)$$

is the analytic function. Finally, from the formula (4.1), $s\in\Pi_1$, ($g(s),\ s\in\Pi_2$ is the analytic function) by the method of analytic continuation we obtain the formula

$$\int_2^\infty F(x;s)\bar{P}(x)\mathrm{d}x = -\frac{1}{2s-1}+g(s),\ s\in\Pi_2(\delta),$$

from which the formula (5.1) follows. □

5.2. Since (see (5.2))

$$\int_2^\infty F(x;\sigma)\mathrm{d}x = \mathcal{O}\left(\int_2^\infty x^{-\frac{3}{2}+\delta}\mathrm{d}x\right) = \mathcal{O}(1),\ \sigma\in\left[\frac{1}{2},\frac{3}{2}\right],$$

we obtain, putting $\bar{P}(x) = P(x)+K$ (see (2.1)) in (5.1), the following lemma

Lemma 5. *On the Riemann hypothesis*

$$\int_2^\infty F(x;\sigma)P(x)\mathrm{d}x = -\frac{1}{2\sigma-1}+\tilde{g}(\sigma),\ \sigma\in\left(\frac{1}{2},\frac{3}{2}\right) \tag{5.4}$$

where $\tilde{g}(\sigma),\ \sigma\in[\frac{1}{2},\frac{3}{2}]$ *is the bounded function.*

6. Proof of the Theorem

Since (see (3.2), comp. (5.2), (5.3))

$$F(x;\sigma) = \frac{\sigma\ln x-1}{x^{\sigma+1}}+\mathcal{O}\left(\frac{\ln x}{x^{2\sigma+1}}\right),$$

$$\int_2^\infty \frac{\ln x}{x^{2\sigma+1}}|P(x)|\mathrm{d}x = \mathcal{O}\left(\int_2^\infty x^{-\frac{3}{2}+\delta}\mathrm{d}x\right) = \mathcal{O}(1),\ \sigma\in\left[\frac{1}{2},\frac{3}{2}\right],$$

$$\int_2^\infty F(x;\sigma)P(x)\mathrm{d}x = \int_2^\infty \frac{\sigma\ln x-1}{x^{\sigma+1}}P(x)\mathrm{d}x+\mathcal{O}(1),\ \sigma\in\left(\frac{1}{2},\frac{3}{2}\right)$$

then (see (5.4))

$$\int_2^\infty \frac{\sigma\ln x-1}{x^{\sigma+1}}P(x)\mathrm{d}x = -\frac{1}{2\sigma-1}+\mathcal{O}(1),\ \sigma\in\left(\frac{1}{2},\frac{3}{2}\right).$$

Putting here $\sigma=\frac{1}{2}+\delta,\ \delta\in(0,\Delta),\ \Delta<1$, we obtain the formula

$$\int_2^\infty \frac{\left(\frac{1}{2}+\delta\right)\ln x-1}{x^{\frac{3}{2}+\delta}}P(x)\mathrm{d}x = -\frac{1}{\delta}+\mathcal{O}(1). \tag{6.1}$$

However, (see (5.3))

$$\begin{aligned}
&\int_2^\infty \frac{\left(\frac{1}{2}+\delta\right)\ln x-1}{x^{\frac{3}{2}+\delta}}P(x)\mathrm{d}x-\int_2^\infty \frac{\frac{1}{2}\ln x-1}{x^{\frac{3}{2}+\delta}}P(x)\mathrm{d}x= \\
&=\delta\int_2^\infty \frac{\ln x}{x^{\frac{3}{2}+\delta}}P(x)\mathrm{d}x=\mathcal{O}\left(\delta\int_2^\infty\frac{\ln^2 x}{x^{1+\delta}}\mathrm{d}x\right)= \\
&\mathcal{O}\left(\delta\int_2^\infty x^{-1-\frac{\delta}{2}}\mathrm{d}x\right)=\mathcal{O}\left(\delta\frac{1}{\delta}\right)=\mathcal{O}(1),\ \delta\in(0,\Delta).
\end{aligned} \tag{6.2}$$

The formula (1.2) follows from (6.1) by (6.2).

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

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