Non-contextuality and free will in modal quantum theory

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Abstract

Modal quantum theory (MQT) is a simplified cousin of ordinary Hilbert space quantum theory. We show that two important theorems of actual quantum theory, the Kochen-Specker theorem excluding non-contextual hidden variables and the Conway-Kochen "free will theorem" about entangled systems, have direct analogues in MQT. The proofs of these analogue theorems are similar to, but much simpler than, the originals. We also show that the structure of possible measurement results for an entangled system in MQT cannot be represented by probability assignments satisfying the no-signaling principle, such as those given by ordinary quantum theory.

1   Modal quantum theory

Modal quantum theory, or MQT, is a simplified mathematical model having many affinities with the Hilbert space structure actual quantum theory (AQT). In [1] it was shown that, though the state space in MQT may be discrete and lacks an inner product, it is nevertheless possible to give a usable interpretation of the model based on the "modal" distinction between *possible* and *impossible* measurement outcomes. MQT systems have superposition and interference effects, and entangled systems have properties that are inconsistent with local hidden variable theories.

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In this paper, we show that MQT also supports analogues of two important results from actual quantum theory, namely the Kochen-Specker argument about contextuality and hidden variable theories [2], and the Conway-Kochen "free will theorem" about the properties of entangled systems [3]. In each case, the proof of the MQT version of the theorem is substantially simpler. We are not giving new proofs of these important results in AQT, but rather deriving analogue results in MQT, a very different theory. To emphasize this, we show that the predictions of MQT cannot always be encompassed by a probabilistic theory such as AQT.

The basic rules of MQT are easy to summarize.

States. The state of a system is a non-zero vector $|\psi)$ in a vector space $\mathcal{V}$ over a field $\mathcal{F}$ of scalars. $\mathcal{F}$ may be any field; many interesting examples arise when $\mathcal{F}$ is chosen to be finite.

Effets and measurements. An effect is an element $(e|$ in the dual space $\mathcal{V}^*$. The effect $(e|$ is possible for the state $|\psi)$ if $(e|\psi) \neq 0$ and impossible if $(e|\psi) = 0$. A measurement is a basis for $\mathcal{V}^*$ whose elements correspond to the potential results of the measurement process.

Composite systems. If two systems have state spaces $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$, then the composite system has a state space $\mathcal{V}^{(12)} = \mathcal{V}^{(1)} \otimes \mathcal{V}^{(2)}$.

Time evolution. The time evolution of the state of an isolated system in MQT is described by an invertible linear operator T:

$$|\psi) \longrightarrow |\psi') = T |\psi). \qquad (1)$$

For finite $\mathcal{F}$, this evolution cannot be continuous in time.

(The time evolution rule is included here for completeness; we will require only the first three for our discussion.)

Let us illustrate these rules by considering a system with $\dim \mathcal{V} = \dim \mathcal{V}^* = 2$, a "modal bit" or *mobit*. The dual space contains at least three distinct non-zero vectors, which we can designate $(a|$, $(b|$, and $(c| = (a| + (b|$. (For the simplest case, when $\mathcal{F} = \mathbb{Z}_2$, these are in fact the only three non-zero vectors in $\mathcal{V}^*$.)

Any pair of these forms a basis for $\mathcal{V}^*$, and thus represents a possible measurement on the system. We shall call the three measurement bases X,
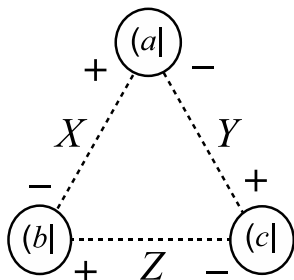
Figure 1: Three measurement bases for a mobit system..

Y and Z, and designate the two potential results for each measurement by $+$ or $-$. Thus,

$$X: \begin{array}{l} (+_x| = (a| \\ (-_x| = (b| \end{array} \qquad Y: \begin{array}{l} (+_y| = (c| \\ (-_y| = (a| \end{array} \qquad Z: \begin{array}{l} (+_z| = (b| \\ (-_z| = (c| \end{array}. \qquad (2)$$

These are shown in Figure 1.

2 Non-contextuality

Kochen and Specker [2] considered a single spin-1 system in AQT together with measurements of the squares of spin components along various axes. These measurements have three important properties:

- The possible results of any measurement are 0 and 1 (in units of $\hbar^2$).

- The squared spin components along any three orthogonal directions are compatible observables and thus may be measured simultaneously.

- If the squared spin components along three orthogonal directions are measured, then exactly two of these directions will yield the result 1 and one direction will yield 0.

We can view the collection of squared spin components along an orthogonal triad of directions as a single observable, whose result is the particular direction along which the measurement yields 0. Each possible result corresponds

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to a projection effect operator on the spin 1 Hilbert space. The orthogonal triad of directions (in real space) gives rise to a set of projections onto orthogonal one-dimensional subspaces (in Hilbert space).

Kochen and Specker asked whether the behavior of the spin-1 system could be explained by a hidden-variable model. In such a model, the result of any measurement would be predetermined by the values of one or more hidden variables, whose underlying statistical distribution would give rise to the the observed probabilistic properties of the quantum system. Kochen and Specker further required that the hidden-variable predictions be *non-contextual*. That is, for a particular effect—i.e., whether or not the squared spin component is 0 for a given direction—the hidden-variable "yes/no" prediction cannot depend on which other two directions are included in the orthogonal triad that is being measured. Could such a model of non-contextual hidden variables account for the properties of spin-1 systems listed above?

The question can be translated into a question of graph-coloring. Given some set of directions in real three-dimensional space, can we color those directions red (for 1) and green (for 0) such that any orthogonal triad of directions contains exactly two red and one green? Kochen and Specker showed that this was not in general possible. They found a set of 117 directions in space that could not be colored in this way, since each direction was part of more than one orthogonal triad. (Later proofs of the theorem, such as the one given in [5], reduce the number of directions required, but all such constructions remain fairly complicated.)

We can construct an analogue of the Kochen-Specker argument in MQT, showing that the predictions of this theory are also incompatible with non-contextual hidden variables. Consider the three mobit measurements listed in Equation 2 and diagrammed in Figure 1. In a hidden-variable model for this system, the results of all measurements are predetermined by the hidden variables, but because the values of these variables are unknown, more than one result may be deemed possible in a given situation. The requirement of non-contextuality is that the hidden variables determine whether or not a given effect will occur, independent of what other effects are included in the measurement. That is, the hidden-variable "yes/no" prediction is a function only of the effect and not of the measurement basis to which that effect belongs.

Therefore, our question is once again translated into a graph-coloring problem. Consider the triangle in Figure 1. Each vertex is an effect, and each side corresponds to a measurement basis. We wish to color the vertices
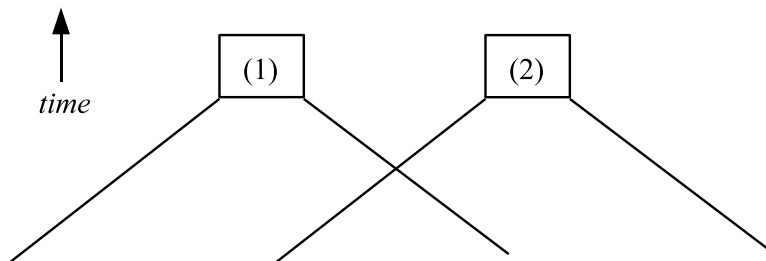
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Figure 2: Spacetime diagram of two near-simultaneous measurement processes.

red and green so that each side contains exactly one green vertex. This is obviously impossible. Therefore, the properties of a mobit system in MQT cannot be explained by an model of non-contextual hidden variables.

The essential idea is the same, both in the Kochen-Specker proof in AQT and in its MQT analogue. However, the construction in MQT is almost trivial, in sharp contrast to that of Kochen and Specker.

3 The free will theorem

Conway and Kochen have used the Kochen-Specker construction to prove the remarkable "free will theorem" about entangled quantum systems [3], [4]. This theorem links the freedom of observers to choose their measurements to the "free will" (indeterminacy) displayed by quantum systems in producing the outcomes of those measurements. Conway and Kochen introduce three basic axioms for the physical world, which they designate SPIN, TWIN and FIN. The SPIN axiom states that spin-1 quantum systems behave as we have described in the last section. The TWIN axiom describes the correlations between two spin-1 systems in a "singlet" (total spin zero) state. In this entangled quantum state, measurements of the squared spin components along two parallel directions must always yield identical results.

The FIN axiom states that the speed of information transfer has a finite limit (which might be, but is not necessarily, the speed of light). Consider two measurement processes on spatially separated subsystems, as shown in a spacetime diagram in Figure 2. The two measurements are nearly simultaneous, and so by the FIN axiom it is not possible for information to travel

from one measurement to the other. This means that the result of the measurement on system (1) can only depend on physical conditions within the causal past of the measurement, shown as the shaded region in Figure 2. The "free will" of observers means that we can arrange the choice of system (2) measurement to lie entirely outside the causal past of the system (1) measurement, so the system (1) result cannot depend on this choice. The reverse is also true.

These facts are the only consequences of the FIN axiom that are used in the proof of the free will theorem. Because of this, in a later version of the theorem Conway and Kochen replace the FIN axiom with a weaker axiom (called MIN) that asserts the independence of measurement results from distant measurement choices [4].

The MQT version of the free will theorem is based on the properties of an entangled state of two mobits. If $|0\rangle$ and $|1\rangle$ are a pair of basis states, this state can be written[1]

$$|S) = |0, 1) - |1, 0) . \tag{3}$$

This state has the property that, if measurements are made on the two mobit systems, the same effect can never occur for both systems. This is because, for any effect $(e|$,

$$(e, e \,|S) = (e \,|0) \,(e \,|1) - (e \,|1) \,(e \,|0) = 0. \tag{4}$$

Let $\langle A^{(1)} = a : B^{(2)} = b \rangle$ denote the situation in which a measurement of A on system (1) yields result a and a measurement of B on system (2) yields result b. Then for the state $|S)$, any joint result of the form $\langle A^{(1)} = a : A^{(2)} = a \rangle$ is impossible. The impossible results include:

$$\begin{array}{ll} \langle X^{(1)} = + : X^{(2)} = + \rangle & \langle X^{(1)} = - : X^{(2)} = - \rangle \\ \langle Y^{(1)} = + : Y^{(2)} = + \rangle & \langle Y^{(1)} = - : Y^{(2)} = - \rangle \\ \langle Z^{(1)} = + : Z^{(2)} = + \rangle & \langle Z^{(1)} = - : Z^{(2)} = - \rangle \end{array} \tag{5}$$

Recalling the three measurement bases in Equation 2, we see that the following joint results are also impossible for $|S)$:

$$\begin{array}{ll} \langle X^{(1)} = + : Y^{(2)} = - \rangle & \langle Y^{(1)} = - : X^{(2)} = + \rangle \\ \langle Y^{(1)} = + : Z^{(2)} = - \rangle & \langle Z^{(1)} = - : Y^{(2)} = + \rangle \\ \langle Z^{(1)} = + : X^{(2)} = - \rangle & \langle X^{(1)} = - : Z^{(2)} = + \rangle \end{array} \tag{6}$$

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[1]The negative sign in the definition of $|S)$ means that the second term is multiplied by the additive inverse of the scalar 1. If $\mathcal{F} = \mathbb{Z}_2$, then $-1 = 1$, and we have $|S) = |0, 1) + |1, 0)$.

We are now ready to state the MQT version of the three Conway-Kochen axioms. The FIN axiom (or its minimal replacement MIN) is retained without change. The MQT version of SPIN (which we may denote SPIN*) asserts the existence of the three overlapping measurement bases X, Y and Z shown in Figure 1. The MQT TWIN* axiom states that a possible state of the system has the properties of the MQT state $|S\rangle$—that is, that the joint results listed in Equations 5 and 6 are impossible.

Now we are ready to prove our theorem. Assume that FIN (or MIN), SPIN* and TWIN* hold for a composite system. Suppose also that observers are freely able to choose X, Y and Z measurements on the separated subsystems. Now imagine that the actual results of all measurements are fully determined by physical factors in the causal pasts of those measurements.

If the measurements $Z^{(1)}$ and $Z^{(2)}$ are made, then the results cannot agree. The only two possible joint results are $\langle Z^{(1)} = + : Z^{(2)} = -\rangle$ or $\langle Z^{(1)} = - : Z^{(2)} = +\rangle$. Without loss of generality (since the two situations are symmetric), suppose the actual result is the first of these.

If the measurement on system (2) is $X^{(2)}$ instead, then the result of the $Z^{(1)}$ measurement is unchanged, and we have $\langle Z^{(1)} = + : X^{(2)} = x\rangle$ for some x. According to Equation 6, the only possible joint result is $\langle Z^{(1)} = + : X^{(2)} = +\rangle$. We can therefore conclude that $X^{(2)} = +$ for any choice of measurement on system (1).

Given that $\langle Z^{(1)} = + : Z^{(2)} = -\rangle$, what if the measurement on system (1) is actually $Y^{(1)}$? The $Z^{(2)}$ result is unchanged, and Equation 6 tells us that only one $Y^{(1)}$ result is possible. We must have $\langle Y^{(1)} = - : Z^{(2)} = -\rangle$. From here, we can inquire how things change if we alter the system (2) measurement to $X^{(2)}$. Again, the $Y^{(1)}$ result cannot change, and Equation 6 restricts us to the single possible joint result $\langle Y^{(1)} = - : X^{(2)} = -\rangle$. We can therefore conclude that $X^{(2)} = -$ for any choice of measurement on system (1). This contradicts our previous conclusion about $X^{(2)}$.

Our hypothesis that the actual results of the measurements are predetermined is therefore faulty. Given the axioms FIN (or MIN), SPIN* and TWIN*, the freedom of the observers to perform X, Y or Z measurements on the mobits implies that the results of those measurements are not predetermined by the causal pasts of the systems. If observers in MQT have "free will", then so do the systems they observe.

The proof of the MQT version of the free will theorem has the same structure as the corresponding proof by Conway and Kochen in AQT. As in the Kochen-Specker theorem, the mathematical construction involved for

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		$X^{(2)}$		$Y^{(2)}$		$Z^{(2)}$	
		+	−	+	−	+	−
$X^{(1)}$	+	0	#	#	0	#	#
	−	#	0	#	#	0	#
$Y^{(1)}$	+	#	#	0	#	#	0
	−	0	#	#	0	#	#
$Z^{(1)}$	+	#	0	#	#	0	#
	−	#	#	0	#	#	0

Figure 3: Table of joint measurement results for a pair of mobits in the $|S)$ state. In this table, impossible results are designated by 0 and possible ones by #.

the MQT version is very much simpler.

4   Modal and actual quantum theories

We wish to emphasize that the proofs presented here are *not* alternate proofs for the Kochen-Specker and Conway-Kochen theorems in actual quantum theory. These are proofs of analogous results in modal quantum theory. Though MQT has many affinities to AQT, the two are not the same.

It is conceivable, however, that systems in one theory might be able to simulate the other. For example, it might be that the possibility relations among various measurements in an MQT system could be realized as probabilistic relations among measurements in an appropriately chosen AQT system. Then the MQT proof would apply to the corresponding AQT situation. However, as we will now show, this cannot always be done. In fact, the possibility relations for an MQT system need not be consistent with *any* reasonable assignment of probabilities for the system, much less one derived from AQT.

Consider a pair of mobits in the entangled MQT state $|S) = |0,1) - |1,0)$. We summarize the predictions of this state for various joint measurements in the table shown in Figure 3. This table contains three rows and three columns corresponding to the X, Y and Z measurements on each mobit. For each joint measurement, we obtain a 2×2 sub-table showing whether each joint result is possible (designated by #) or impossible (designated by

0). The 0's in Figure 3 are exactly those impossible results enumerated in Equations 5 and 6.

We now wish to create a table of probabilities with the same essential structure as Figure 3. The entries of the table will be of the form $P(a, b|A^{(1)}, B^{(2)})$, the probability that a joint measurement of the observable $(A^{(1)}, B^{(2)})$ will yield the joint result (a, b). Such a table should meet the following requirements

I. Each probability must be between 0 and 1. Furthermore, for any $A^{(1)}$ and $B^{(2)}$,

$$\sum_{a,b} P(a, b|A^{(1)}, B^{(2)}) = 1. \tag{7}$$

II. The marginal probability distribution for the results of a subsystem measurement is independent of the choice of measurement on the other subsystem. For instance, given measurements $A^{(1)}$, $A^{(1)\prime}$ and $B^{(2)}$,

$$\sum_{a} P(a, b|A^{(1)}, B^{(2)}) = \sum_{a'} P(a', b|A^{(1)\prime}, B^{(2)}) = P(b|B^{(2)}). \tag{8}$$

This is called the *no-signaling principle* [6]. If it were not true, then a choice of measurement on system (1) could cause an immediate change in the statistical properties of system (2), allowing information to be transmitted from one to the other instantaneously. The no-signaling principle holds for measurements on composite systems in actual quantum theory.

III. Each impossible joint outcome in Figure 3 is assigned a probability $P(a, b|A^{(1)}, B^{(2)}) = 0$.

IV. Each possible joint outcome in Figure 3 is assigned a probability $P(a, b|A^{(1)}, B^{(2)}) > 0$.

Requirements I–III almost completely specify the probability assignment for Figure 3. The general result is shown in Figure 4. Only three real parameters, denoted by q, r and s in Figure 4, determine all of the probabilities in the table.

From the table, however, we note that

$$
\begin{aligned}
q + r &= P(+,-|Y^{(1)}, X^{(2)}) = -P(-,+|X^{(1)}, Y^{(2)}) \\
q + s &= P(+,-|X^{(1)}, Z^{(2)}) = -P(-,+|Z^{(1)}, X^{(2)}) \\
r + s &= P(+,-|Z^{(1)}, Y^{(2)}) = -P(-,+|Y^{(1)}, Y^{(2)}).
\end{aligned} \tag{9}
$$

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		$X^{(2)}$		$Y^{(2)}$		$Z^{(2)}$	
		$+$	$-$	$+$	$-$	$+$	$-$
$X^{(1)}$	$+$	0	$\frac{1}{2}+q$	$\frac{1}{2}+q$	0	$\frac{1}{2}-s$	$q+s$
	$-$	$\frac{1}{2}-q$	0	$-q-r$	$\frac{1}{2}+r$	0	$\frac{1}{2}-q$
$Y^{(1)}$	$+$	$\frac{1}{2}-q$	$q+r$	0	$\frac{1}{2}+r$	$\frac{1}{2}+r$	0
	$-$	0	$\frac{1}{2}-r$	$\frac{1}{2}-r$	0	$-r-s$	$\frac{1}{2}+s$
$Z^{(1)}$	$+$	$\frac{1}{2}+s$	0	$\frac{1}{2}-r$	$r+s$	0	$\frac{1}{2}+s$
	$-$	$-q-s$	$\frac{1}{2}+q$	0	$\frac{1}{2}-s$	$\frac{1}{2}-s$	0

Figure 4: Probability table consistent with the possibility data given in Figure 3, consistent with the no-signaling principle. All of the entries are determined by just three real parameters q, r and s.

This can only be satisfied if these six probabilities, which correspond to possbile results in Figure 3, are assigned probability zero, in violation of requirement IV. Therefore, we cannot find a probability assignment with the same structure as Figure 3 that satisfies all four requirements. The predictions of MQT in this case cannot be simulated by AQT, or indeed by any probabilistic theory satisfying the no-signaling principle.

What if we relax the troublesome requirement IV and permit an assignment of probability zero to a result designated "possible" in MQT? Then the probability table can be completed by letting $q = r = s = 0$, yielding the table shown in Figure 5. It is interesting to note that, with the selection of two measurements for each system ($X^{(1)}$, $Y^{(1)}$ for system (1) and $X^{(2)}$, $Z^{(2)}$ for system (2), for example), the probabilities from Figure 5 describe a *PR box*, a type of nonlocal correlation known to be inconsistent with AQT [7], [6]. This is also shown in Figure 5.

This illustrates in the most definitive way that modal quantum theory is not reducible to actual quantum theory. On the other hand, there is a sense in which AQT can be regarded a "special case" of MQT with $\mathcal{F} = \mathbb{C}$. The inner product structure of AQT Hilbert space motivates us to consider only normalized state vectors, orthonormal measurement bases, and unitary time evolution operators. All of these are restrictions of the larger class of states, measurements and evolution operators permitted in MQT. If we

		$X^{(2)}$		$Y^{(2)}$		$Z^{(2)}$	
		$+$	$-$	$+$	$-$	$+$	$-$
$X^{(1)}$	$+$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	$-$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$Y^{(1)}$	$+$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	$-$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
$Z^{(1)}$	$+$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
	$-$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0

$\Longrightarrow$

		$X^{(2)}$		$Z^{(2)}$	
		$+$	$-$	$+$	$-$
$X^{(1)}$	$+$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	$-$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
$Y^{(1)}$	$+$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	$-$	0	$\frac{1}{2}$	0	$\frac{1}{2}$

Figure 5: Unique probability table consistent with Figure 3, satisfying only Requirements I–III. Choosing only two rows and two columns, we arrive at the probability table of a PR box.

consider only the distinction between possible ($p \neq 0$) and impossible ($p = 0$) measurement results, then any situation in AQT can be viewed from a modal point of view.

The highly simplified structure of MQT is thus a generalization of some aspects of AQT. As we have seen, it is possible to use this generalized framework to prove analogues to important AQT results about hidden variable models. The MQT proofs are much easier to understand and can be used to shed light on the essential structure of these theorems.

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