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Comparing globalness of bipartite unitary operations acting on quantum information: delocalization power, entanglement cost, and entangling power

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We compare three different characterizations of the globalness of bipartite unitary operations based on different tasks, namely, delocalization power, entanglement cost for LOCC implementation, and entangling power. We present extended analysis on the globalness in terms of delocalization in two ways. First, we show that the delocalization power differs whether the global operation is applied on one piece of quantum information or two pieces. Second, by introducing the concept of dislocation, we prove that the local unitary equivalents of controlled-unitary operations assisted by LOCC cannot dislocate one piece of quantum information when applied on two pieces of quantum information. This confirms that the local unitary equivalents of controlled-unitary operations, which are LOCC one-piece relocalizeable, belong to a class of global operations with relatively weak globalness in terms of dislocation of quantum information.

1. Introduction

Understanding the source of quantum advantage in quantum computation is a long-standing issue in quantum information science. Previous researches have shown that certain quantum computation is 'classical', for the reason that it is efficiently simulateable by classical computers. One example is any computation performed just by local operations and classical communication (LOCC) without using any entangled resources. All models of quantum computation outperforming classical counterparts use entanglement resources (such as measurement-based quantum computation (Raussendorf and Briegel 2001)) or some kind of non-LOCC operation.

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Non-LOCC operations are called 'global' operations. The source of quantum speedup must be due to the properties of the global operations. In this paper we refer to the properties exclusive to global opeartions as *globalness* of quantum operations.

It is also known that not all global operations result in quantum speedup for quantum computation. There must be a specific globalness that differentiates the quantum operations leading to quantum speedup from those do not. The difference may be due to more than one kind of globalness, but even this is not clear at this point. For this reason, having a good understanding of the globalness of quantum operations is important. The main purpose of this paper is to deepen our understanding on the globalness of quantum operations, specifically by comparing different ways to characterize the globalness.

In the rest of the paper, we will focus on three characterizations of the globalness of quantum operations – entangling power, entanglement cost for LOCC implementation, and delocalization power of quantum information. The rest of the paper is organized as following. In Section 2, we present an overview on the three characterizations. We summarize the comparison of different aspects of the globalness of bipartite unitary operations presented in the previous works in Section 3. We extend the analysis of globalness in terms of the delocalization power in Sections 4 and 5. In Section 4, we discuss how the globalness of certain bipartite unitary operations reduces when a part of the input state is known. In Section 5, we introduce the concept of dislocation of quantum information. Finally, in Section 6, we conclude by comparing the globalness of bipartite unitary operations identified by the three different characterizations.

2. Three characterizations of the globalness of quantum operations

One way to quantify the globalness of a quantum operation is to analyze how much it 'outperforms' LOCC in a given task. One typical example is entanglement generation. Entanglement is defined as a property of a quantum state which cannot increase on average under LOCC (Horodecki *et al.* 2009; Plenio and Virmani 2007). Thus, under this definition, the maximum amount of entanglement that LOCC can generate is zero. On the other hand, global operations can generate nonzero entanglement when they are performed on appropriately chosen input states. Clearly, global operations outperform LOCC in entanglement generation. The maximum amount of entanglement that a single-shot use of a given global operation can generate is unique. Hence, we may use this amount to characterize the globalness of quantum operations, which is referred to as the entangling power of quantum operations (Kraus and Cirac 2001; Wolf *et al.* 2003; Linden *et al.* 2009).

If entanglement is supplied as an extra resource to LOCC, then the set of operations that can be implemented becomes larger. The minimal amount of entanglement that must be supplied to deterministically implement a given global operation is unique, based on the fact that the entanglement cannot be generated by LOCC. Therefore, the (minimal) entanglement cost of deterministic LOCC implementation can be used to characterize the globalness of the given global operation.

Different tasks reveal different aspects of the globalness. The entangling power mentioned above is based on the entanglement generation task which is based on protocol involving application of quantum operations on *known* input states. One of the key features of quantum mechanics is that the states cannot be distinguished perfectly if they are unknown. In classical information processing, information is represented by physical states which can be perfectly distinguished. In contrast, quantum information processing allows the information to be represented by non-perfectly distinguishable states. In the most extreme case, the input state of the task is completely unknown, as in quantum teleportation (Bennett *et al.* 1993) and quantum error correction (Knill and Laflamme 1997). Quantum computation involves a subtlety in the sense that the many famous quantum computation algorithms start from a fixed input state, *e.g.* Shor's and Grover's algorithm. In this case, the quantum state remains in a known state throughout the whole process. Nevertheless, the number of classical bits required for the exact description of the quantum state during the computation is strongly believed to scale exponentially. Thus, even for such quantum computation algorithms, it makes sense to assume that the operations during the computation is applied on (at least computationally) unknown states.

If we want to understand how the globalness of quantum operations contributes to advantages in quantum information processing, it is important to analyze the globalness that is relevant to the operational power on quantum information. In this aspect, the entangling power has one shortcoming, since entaglement generation is based on fixed known input states. A characterization based on protocols involving unknown quantum states will be more appropriate in this aspect.

Based on these considerations, the authors have proposed a classification of global operations based on the 'delocalization' power of global operations on quantum information, where this quantum information is defined as a completely unknown quantum state (Soeda and Murao 2010). The task they considered is as follows. Imagine that there are two d-level quantum systems (or, in the quantum information science terminology, *qudits*), each possessed by two different parties, namely Alice and Bob. Alice and Bob do not know their initial state but are promissed that it is a product state $|\psi_{\mathrm{A}}\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle$ and $|\psi_B\rangle$ are d-dimensional state vectors corresponding to Alice's and Bob's quantum system, respectively. The authors called this situation that each system contains one *piece* of quantum information. Then, a global operation known to both Alice and Bob is applied on these two quantum systems.

Before applying the global operation, we may interpret the situation as the case where two pieces of quantum information are localized in their respective minimal Hilbert space. After the global operation, the (reduced) state of each quantum system no longer equals that of the initial one. In a sense, the quantum information has been *delocalized* out of their original Hilbert space. To *relocalize* both of the delocalized two pieces of quantum information requires the reverse operation of the applied global operation, where the relocalization of quantum information is defined as the task of restoring the state of each quantum system back to its *original* product state. Note that the quantum information under this definition is an unknown quantum state, therefore relocalization cannot be achieved by re-preparing the state of the system back to its original one.

Relocalization of the both pieces of quantum information at the same time cannot be performed by LOCC because the reverse operation of a global operaiton is also global.

However, we can consider a slightly more relaxed task of relocalizing just one of the two pieces of quantum information by sacrificing the other piece of quantum information. This LOCC one-piece relocalization has been used to classify the bipartite unitary operations.

3. Comparison of globalness by different characterizations

In the previous section, we introduced three different characterizations for the globalness of quantum operations: entangling power, entanglement cost for LOCC implementation, and delocalization power on quantum information. We investigate whether the globalness characterized by each method is same to or different from that of by the others. For simplicity, we restrict ourselves to the case where global operations are chosen to be bipartite unitary operations.

The formulation of entangling power depends on the set of allowed input states and the measure of entanglement. Entangling power is usually difficult to calculate because it involves two optimizations. One is the maximization over all possible input states (usually taken to be separable or product states). The other is the calculation of the amount of generated entanglement according to the chosen entanglment measure. Even when the quantum operation is restricted to bipartite unitary operations, the exact value is obtained for only limited cases (Kraus and Cirac 2001; Chefles 2005).

Nevertheless, we can make a relatively generic statement about entangling power if the entanglement measure is continuous. The statement is as follows. The identity operation clearly generates no entanglement at all, hence its entangling power should be zero. Invoking a continuity argument, there should be a set of operations in the neighborhood of the identity operation such that their entangling power is arbitrarily small.

The second characterization of globalness, entanglement cost for LOCC implementation, can be calculated as follows. First, choose the global operation that we want to deterministically implement by entanglement assisted LOCC. Then, we choose an entangled state for the resource. For this particular combination of the entangled state and the global operation, we check if there exists a LOCC protocol that accomplishes the implementation. The existence of such a protocol is determined by the entanglement resource. We then search for the smallest amount of entanglement among those entangled states that enabled the implementation. The amount of entanglement of this state will give us the entanglement cost for the LOCC implementation of the particular global operation.

Again, we see that this requires a very difficult optimization, which is one of the main reasons why entanglement cost has been obtained for limited number of cases. The authors and coworker obatined the entanglement cost for all two-qubit controlled-unitary operations (Soeda *et al.* 2010) given by the form $|0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes u$, where $|0\rangle, |1\rangle$ are an orthonormal basis of a qubit Hilbert space and u is a two-by-two uniatry matrix. According to their result, any such unitary operation requires 1 ebit of entanglement, no matter how small their entangling power is. When u is almost equal to the identity matrix, the entire controlled-unitary operation becomes very close to the identity operation, therefore, the entangling power is also very close to zero; but the entanglement cost for LOCC implementation is 1 ebit.

Finally, the last characterization of globalness based on the delocalization power reveals that there are two classes of globalness for bipartite unitary operations. We refer the reader to Soeda and Murao 2010 for the proof and simply state the result in this paper. Bipartite unitary operations can be classified into two classes according to LOCC one-piece relocalizability. It is proven that one piece of quantum information can be relocalized by LOCC if and only if the delocalizing unitary operation is local unitarily equivalent to a controlled-unitary operation. The result holds for any unitary operation on a bipartite d-level system. A general form of a controlled-unitary opeartion U_u on this system can be expressed as $\sum_k |k\rangle\langle k| \otimes u_k$, where $|k\rangle$ forms an orthonormal basis for one of the subsystems and u_k are unitary operations on the other subsystem. The continuity argument proves that the classification is irrelevant of the entangling power of each unitary operation. For details, see Soeda and Murao 2010.

This analysis shows that there are several aspects of globalness in quantum operations. Interestingly, as for two-qubit unitary operations, all LOCC one-piece relocalizable unitary operations require 1 ebit of entanglement for LOCC implementation.

4. Delocalization power for one piece of quantum information

In the classification by delocalization, it is crucial that the both quantum systems are set in unknown state. If more information is available about the input state, then Alice and Bob can exploit it to perform a LOCC one-piece relocalization for a class of global operations wider than local unitary equivalents of controlled-unitary operations.

To see this property, we present an example of a two-qubit unitary operation, where LOCC one-piece relocalization is impossible if both qubits are in unknown states, but it becomes possible if one of the qubits is promised to be in a particular pure state. Let us denote the Hilbert space of each qubit by $\mathcal{H}_A$ and $\mathcal{H}_B$, where $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ will be an orthonormal basis for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The orthonormal basis of the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ will be chosen as $|i\rangle_A \otimes |j\rangle_B$, called the computational basis. The matrix elements of the example unitary operation U_{ex} are given by

$$U_{\text{ex}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1}$$

According to Anders *et al.* 2010, U_{ex} is local unitarily equivalent to

$$e^{i\pi/4(\sigma_{x,A}\otimes\sigma_{x,B}+\sigma_{y,A}\otimes\sigma_{y,B})}, \tag{2}$$

where $\sigma_{x,A}$ and $\sigma_{y,A}$ are the Pauli X and Pauli Y matrix on $\mathcal{H}_A$, respectively, and $\sigma_{x,B}$ and $\sigma_{y,B}$ are that of $\mathcal{H}_B$.

Any unitary matrix has an *operator Schmidt decomposition* (Nielsen *et al.* 2003). The operator Schmidt decomposition of a two-qubit unitary operation is given by

$$\sum_{k=0}^{3} \lambda_k A_k \otimes B_k, \tag{3}$$

where $\{\lambda_k\}$ is a set of four positive numbers satisfying $\sum_{k=0}^{3} \lambda_k^2 = 1$, and A_k and B_k are operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, satisfying the orthonormal relation

$$\frac{1}{2}\mathrm{Tr}[A_k^\dagger A_j] = \delta_{k,j} \tag{4}$$

and

$$\frac{1}{2}\mathrm{Tr}[B_k^\dagger B_j] = \delta_{k,j}. \tag{5}$$

The operator Schmidt number is the number of the non-zero coefficents λ_k. Two-qubit unitary operations can have the operator Schmidt number of 1,2, or 4, but not 3. Any two-qubit unitary operator has the following Cartan decomposition,

$$u_A \otimes u_B \cdot e^{i(\alpha_x \sigma_{x,A} \otimes \sigma_{x,B} + \alpha_y \sigma_{y,A} \otimes \sigma_{y,B} + \alpha_z \sigma_{z,A} \otimes \sigma_{z,B})} \cdot v_A \otimes v_B, \tag{6}$$

where, u_A, v_A are unitary operators on $\mathcal{H}_A$; u_B, v_B are on $\mathcal{H}_B$; and $\sigma_{z,A}$ and $\sigma_{z,B}$ are the Pauli Z matrices on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively (Kraus and Cirac 2001). In this paper, we call α_x, α_y, and α_z the Cartan coefficients. A two-qubit unitary operator has the operator Schmidt number of 2 if and only if it has one non-zero Cartan coefficient (Nielsen *et al.* 2003).

Any two-qubit controlled unitary operation given by

$$U_u = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes u \tag{7}$$

has the operator Schmidt number of 2, which can be seen as follows. The two-by-two unitary matrix u can be decomposed as follows

$$u = e^{i\theta}|\varphi\rangle\langle\varphi| + e^{i\theta'}\left|\varphi^\perp\right\rangle\left\langle\varphi^\perp\right|, \tag{8}$$

where $|\varphi\rangle$ and $\left|\varphi^\perp\right\rangle$ are normalized, orthogonal 2-dimensional vectors. Defining a unitary matrix v by

$$v = |0\rangle\langle\varphi| + |1\rangle\left\langle\varphi^\perp\right|, \tag{9}$$

the spectral decomposition of U_u can be expressed as follows,

$$U_u = (\mathbb{I} \otimes v) \cdot \mathrm{diag}(1, 1, e^{i\theta}, e^{i\theta'}) \cdot (\mathbb{I} \otimes v^\dagger). \tag{10}$$

We set four numbers a, b, c, and d by

$$a = \frac{1}{4}(2 + \theta + \theta') \tag{11}$$

$$b = \frac{1}{4}(2 - \theta - \theta') \tag{12}$$

$$c = \frac{1}{4}(\theta - \theta') \tag{13}$$

$$d = \frac{1}{4}(-\theta + \theta'). \tag{14}$$

Direct substitution will reveal that

$$U_u = e^{ia} \cdot (\mathbb{I} \otimes v) \cdot (e^{ib\sigma_{z,A}} \otimes e^{ic\sigma_{z,B}}) \cdot e^{id\sigma_{z,A} \otimes \sigma_{z,B}} \cdot (\mathbb{I} \otimes v^\dagger). \tag{15}$$

Noting that

$$e^{id\sigma_{z,A} \otimes \sigma_{z,B}} = \cos d\, \mathbb{I} \otimes \mathbb{I} + i \sin d\, \sigma_{z,A} \otimes \sigma_{z,B}, \tag{16}$$

the operator Schmidt decomposition of U_u is given by

$$U_u = \cos d(e^{ia} e^{ib\sigma_{z,A}}) \otimes (v e^{ic\sigma_{z,B}} v^\dagger) + \sin d(ie^{ia} e^{ib\sigma_{z,A}} \sigma_{z,A}) \otimes (v e^{ic\sigma_{z,B}} \sigma_{z,B} v^\dagger), \quad (17)$$

which shows that the operator Schmidt number of any two-qubit controlled-unitary operation is 2. Note that we assumed $\sin d$ and $\cos d$ are both positive; if not, we can have the sign be absorbed into the local operators.

Because only controlled-unitary operations and their local unitary equivalents are LOCC one-piece relocalizable, two-qubit unitary operations are LOCC one-piece relocalizable if and only if their operator Schmidt number is 2. U_{ex} has two non-zero Cartan coefficients (Anders *et al.* 2010), therefore its operator Schmidt number is four. Thus, U_{ex} is not LOCC one-piece relocalizable.

When the input state of one of the parties (say Alice's) is fixed to

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad (18)$$

the action of U_{ex} becomes equal to that of a certain local unitary equivalent of a controlled-unitary operation. Let H denote the Hadamard gate, which is defined by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (19)$$

It can be checked by direct calculation that for arbitrary $|\psi\rangle \in \mathcal{H}_B$,

$$U_{\mathrm{ex}} |+\rangle \otimes |\psi\rangle = (H \otimes \mathbb{I})(|0\rangle \langle 0| \otimes \mathbb{I} + |1\rangle \langle 1| \otimes \sigma_{x,B}) |+\rangle \otimes |\psi\rangle \quad (20)$$

holds. The same calculation can be done using the stabilizer formalism (Gottesman 1997) by exploiting the fact that U_{ex} is a Clifford operation. In any case, the action of U_{ex} is same as that of a local unitary equivalent of a controlled-not when one of the parties' input is fixed to a particular state, implying that certain unitary operations that are not LOCC one-piece relocalizable become LOCC one-piece relocalizable when there is only one piece of quantum information.

Note that some unitary operations remain LOCC one-piece *un*relocalizable even when one of the input state is fixed to any particular state. Such an example is the swap operation U_{SWAP}, whose action is given by

$$U_{\mathrm{SWAP}} |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_2\rangle \otimes |\psi_1\rangle . \quad (21)$$

In this case, even if one of the parties' input is fixed to a particular state, the other party's piece of quantum information completely moves to the other side. This phenomena is an example of what we call a *perfect dislocation* of quantum information. The relocalization of the perfectly dislocated piece of quantum information is essentially quantum teleportation. In order to relocalize a perfectly dislocated piece of quantum information, there must be at least as much operational resources to perform quantum teleporation. Because quantum teleporation is impossible by LOCC alone, LOCC one-piece relocalization is impossible for the swap operation even if there is just one piece of quantum information.

5. Dislocation of quantum information by controlled-unitary operations

5.1. *One-piece and two-piece dislocation*

In the previous section, we briefly introduced the concept of *perfect dislocation* of quantum information. By applying a global operation on pieces of localized quantum information, we can displace the quantum information out of its original Hilbert space. Dislocation is in a sense the maximum delocalization of quantum information. The swap operation is the extreme case where the reduced state of each local subsystem becomes completely irrelevant to the original quantum information stored in that subsystem.

Without any other resource, the swap operation is the only unitary operation that can dislocate two-pieces of quantum information simultaneously. This two-piece dislocateability is an exclusive property of the swap operation. On the other hand, a wider class of unitary operations, namely the local unitary equivalents of the swap operation, also becomes two-piece dislocateable if local operations are allowed as an extra operational resource. It seems reasonable to allow LOCC as a free extra resource for dislocation, because we are only interested in the global effect of global operations on quantum information.

With this modification, it will be more appropriate to call the new task, *LOCC assisted dislocation*. We may also be interested in the case where one of the two pieces of quantum information can be sacrificed in order to dislocate the other piece of quantum informaton. In view of this, the terms 'LOCC assisted *two-piece* dislocation' and 'LOCC assisted *one-piece* dislocation' will be used in this paper to distigunish between the simultaneous dislocation of two pieces of quantum information from one-piece of dislocation with possible sacrifice of the other. From here on, we will use the simplified version of these two terminologies, namely, 'two-piece dislocation' and 'one-piece dislocation'.

We may use this LOCC assisted dislocation as a new task to create a new characterization of the globalness of quantum operations. Because two-piece dislocateable unitary operations are necessarily one-piece dislocateable, the former task is more difficult than the latter, indicating that those unitary operations that achieve two-piece dislocation has more globalness than those do not.

As we have seen in the previous sections, controlled-unitary operations did not exhibit the highest class of globalness in terms of delocalization power. While there are unitary operations like the swap operation and their local unitary equivalents that allow two-piece dislocation, we will show that all bipartite controlled-unitary operations cannot achieve even one-piece dislocation.

Let us focus on two-qubit controlled-unitary operations for simplicity. The following argument can be extended to arbitrary two-*qudit* controlled-unitary operations. Mathematically, performing LOCC one-piece dislocation for a given global unitray operation U is finding a CPTP map implementable only by LOCC Γ^U_{LOCC} such that for any $|\varphi\rangle \in \mathcal{H}_A$ and any $|\varphi'\rangle \in \mathcal{H}_B$,

$$\mathrm{Tr}_A \Gamma^U_{\mathrm{LOCC}}(U \cdot |\varphi\rangle\langle\varphi| \otimes |\varphi'\rangle\langle\varphi'| \cdot U^\dagger) = |\varphi\rangle\langle\varphi| . \tag{22}$$

Here, we assumed that we want to dislocate Alice's piece of quantum information to Bob's Hilbert space, which is why there is the partial trace over Alice's Hilbert space

Tr_A. The dislocation of Bob's quantum information to Alice's side can be defined similarly by replacing the partial trace over Alice's Hilbert space with that of over Bob's Hilbert space. Although we are only interested in controlled-unitary operations, this formulation applies for any bipartite unitary operation.

5.2. *Formulation of LOCC using accumulated operators*

We adopt the standard formulation of LOCC (Donald *et al.* 2002). In the two-party scenario (as in our case), Alice and Bob perform one local measurement operation in turns while exchanging the outcome of each measurement operation by classical communication. The measurement operation at a particular turn is chosen according to all the outcomes by the both parties up to that turn, where the decision is made following a protocol agreed beforehand by the parties. Strictly speaking, we may consider LOCC protocols which cannot be expressed in this form. These protocols, however, can always be substituted by the protocols in this standard form.

Each local quantum operation can be described as a generalized measurement, which is represented by a set of operators $\{M^{(r)}\}$ satisfying the completeness relation

$$\sum_r M^{(r)\dagger} M^{(r)} = \mathbb{I}. \tag{23}$$

There exists one operator for each outcome in the measurement, which is denoted by the superscript r.

We add a subscript to the outcome index, for example r_k, to specify to which measurement operation the index belongs. In this notation, r_k belongs to the k-th measurement operation in the sequence. We use $\vec{R}_k = (r_1, r_2, \ldots, r_k)$ to denote the set of measurement outcomes of the first k measurement operations in the sequence. The $(k+1)$-th measurement operation is then a function of $\vec{R}_k$. The set of operators describing this measurement operation is denoted by,

$$\{M^{(r_{k+1}|\vec{R}_k)}\}_{r_{k+1}}. \tag{24}$$

Let us denote Alice's measurement operations by $M_A^{(r_n|\vec{R}_{n-1})}$ and Bob's by $M_B^{(r_n|\vec{R}_{n-1})}$. We set

$$M_A^{(r_1|\vec{R}_0)} = M_A^{(r_1)} \tag{25}$$

and

$$M_B^{(r_1|\vec{R}_0)} = M_B^{(r_1)}. \tag{26}$$

Note that $M_A^{(r_n|\vec{R}_{n-1})}$ is an operator on $\mathcal{H}_A$ and $M_B^{(r_n|\vec{R}_{n-1})}$ is on $\mathcal{H}_B$. When n-th turn is Alice's turn then $(n+1)$-th turn is Bob's turn, which implies that Alice does not perform any operation during this $(n+1)$-th turn. In this case, we set Alice's measurement operation to the identity operation, *i.e.*

$$\{M_A^{(r_{n+1}|\vec{R}_n)}\} = \{\mathbb{I}\}. \tag{27}$$

If this $(n+1)$-th turn happened to be Bob's, then his measurement operation is set to the identity operation.

The effect of the measurement operations accumulates as a LOCC protocol proceeds. The accumulated effect up to a particular turn is expressed by the product product of all the measurement operators corresponding to all the measurement outcomes obtained up to that point. Given a particular sequence of measurement outcomes $\vec{R}_n$, we represent the accumulated effect corresponding to this sequence by an *accumulated operator* $A^{\vec{R}_n}$ defined by

$$A^{\vec{R}_n} = \prod_{k=1}^{n} M_A^{(r_k|\vec{R}_{k-1})}. \tag{28}$$

Bob's accumulated operator will be denoted by $B^{\vec{R}_n}$.

5.3. *Impossibility of dislocation*

We prove by contradiction that LOCC one-piece dislocation is impossible for any controlled-unitary operation. Now, consider the following scenario where Alice has an extra aniclla qubit, whose Hilbert space is denoted by $\mathcal{H}_a$. Let $|\Phi\rangle_{Aa}$ denote a maximally entangled state between Alice's input qubit and the ancilla qubit; more specifically, it is defined by

$$|\Phi\rangle_{Aa} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_a + |1\rangle_A \otimes |1\rangle_a) \in \mathcal{H}_A \otimes \mathcal{H}_a. \tag{29}$$

Suppose there is a LOCC one-piece dislocation protocol for the given controlled-unitary operation U_u described by Eq. (7). Let Alice set her input qubit and the ancilla in the state of $|\Phi\rangle_{Aa}$ while Bob's input remains arbitrary. Alice and Bob perform U_u and the LOCC protocol to complete the dislocation. Because the one-piece dislocation is defined for arbitrary product input states, Alice's ancilla qubit and Bob's input qubit should be now in the state of

$$|\Phi\rangle_{aB} = \frac{1}{\sqrt{2}}(|0\rangle_a \otimes |0\rangle_B + |1\rangle_a \otimes |1\rangle_B) \in \mathcal{H}_a \otimes \mathcal{H}_B, \tag{30}$$

which implies that

$$\mathrm{Tr}_A[\Gamma_{\mathrm{LOCC}}^{U_u}(U_u |\Phi\rangle_{Aa} \langle\Phi| \otimes |\varphi'\rangle \langle\varphi'| U_u^\dagger)] = |\Phi\rangle_{aB} \langle\Phi| \tag{31}$$

holds for arbitrary $|\varphi'\rangle \in \mathcal{H}_B$. Using the accumulated operator representation of Γ_{LOCC}, we have

$$\mathrm{Tr}_A[\sum_{\vec{R}_n}(A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u |\Phi\rangle_{Aa} \langle\Phi| \otimes |\varphi'\rangle \langle\varphi'| U_u^\dagger)(A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger] = |\Phi\rangle_{aB} \langle\Phi| . \tag{32}$$

We modify the LOCC protocol $\Gamma_{\mathrm{LOCC}}^{U_u}$ by adding an extra measurement operation by Alice, namely

$$\{|0\rangle_A \langle 0| , |0\rangle_A \langle 1|\}. \tag{33}$$

We denote this modified protocol by $\Gamma'^{U_u}_{\mathrm{LOCC}}$. Direct substitution reveals that

$$\begin{aligned}&\mathrm{Tr}_A[\Gamma'^{U_u}_{\mathrm{LOCC}}(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|\varphi'\right\rangle \left\langle\varphi'\right| U_u^\dagger)]\\ &= \mathrm{Tr}_A[\sum_{\vec{R}_n}(\left|0\right\rangle_A \left\langle 0\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|\varphi'\right\rangle \left\langle\varphi'\right| U_u^\dagger)(\left|0\right\rangle_A \left\langle 0\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger]\\ &+ \mathrm{Tr}_A[\sum_{\vec{R}_n}(\left|0\right\rangle_A \left\langle 1\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|\varphi'\right\rangle \left\langle\varphi'\right| U_u^\dagger)(\left|0\right\rangle_A \left\langle 1\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger].\end{aligned} \tag{34}$$

By definition, the partial trace Tr_A on any density matrix ρ can be re-expressed as

$$\mathrm{Tr}_A[\rho] = {}_A\left\langle 0\right| \rho \left|0\right\rangle_A + {}_A\left\langle 1\right| \rho \left|1\right\rangle_A, \tag{35}$$

which implies that the right hand side of Eq. (34) equals to

$$\mathrm{Tr}_A[\left|0\right\rangle_A \left\langle 0\right| \otimes \mathrm{Tr}_A[\Gamma^{U_u}_{\mathrm{LOCC}}(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|\varphi'\right\rangle \left\langle\varphi'\right| U_u^\dagger)]]. \tag{36}$$

When combined with Eq. (32), we see that

$$\mathrm{Tr}_A[\Gamma'^{U_u}_{\mathrm{LOCC}}(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|\varphi'\right\rangle \left\langle\varphi'\right| U_u^\dagger)] = \left|\Phi\right\rangle_{aB} \left\langle\Phi\right|. \tag{37}$$

Because the right hand side is a pure state, it must be true that

$$\begin{aligned}&\mathrm{Tr}_A[(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|\varphi'\right\rangle \left\langle\varphi'\right| U_u^\dagger)(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger]\\ &\qquad = p^{\vec{R}_n,k,\varphi'} \left|\Phi\right\rangle_{aB} \left\langle\Phi\right|\end{aligned} \tag{38}$$

for all $\vec{R}_n$ and $k = 0, 1$, where $p^{\vec{R}_n,k,\varphi'}$ is a positive coefficient normalized by

$$\sum_{\vec{R}_n,k} p^{\vec{R}_n,k,\varphi'} = 1. \tag{39}$$

Since Eq. (38) holds for any $\left|\varphi'\right\rangle \in \mathcal{H}_B$, we have

$$\begin{aligned}&\mathrm{Tr}_A[(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \cdot \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \frac{1}{2}\mathbb{I} \cdot U_u^\dagger)(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger]\\ &= \frac{1}{2}\mathrm{Tr}_A[(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|0\right\rangle \left\langle 0\right| U_u^\dagger)(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger]\\ &\quad + \frac{1}{2}\mathrm{Tr}_A[(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \left|\Phi\right\rangle_{Aa} \left\langle\Phi\right| \otimes \left|1\right\rangle \left\langle 1\right| U_u^\dagger)(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger]\\ &\qquad = (p^{\vec{R}_n,k,0} + p^{\vec{R}_n,k,1}) \left|\Phi\right\rangle_{aB} \left\langle\Phi\right|.\end{aligned} \tag{40}$$

Note that $\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n}$ acts only on Alice's input qubit. Taking the partial trace over Alice's *aniclla* qubit Tr_a, Eq. (40) gives

$$\mathrm{Tr}_A[(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u \cdot \frac{1}{2}\mathbb{I} \otimes \frac{1}{2}\mathbb{I} \cdot U_u^\dagger)(\left|0\right\rangle_A \left\langle k\right| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger] = \frac{(p^{\vec{R}_n,k,0} + p^{\vec{R}_n,k,1})}{2}\mathbb{I}, \tag{41}$$

where we have used the relation

$$\mathrm{Tr}_a \left|\Phi\right\rangle_{aA} \left\langle\Phi\right| = \mathrm{Tr}_a \left|\Phi\right\rangle_{aB} \left\langle\Phi\right| = \frac{1}{2}\mathbb{I}. \tag{42}$$

Noting that the identity operator commutes with any unitary operators, after performing the partial trace Tr_A, we have

$$ {}_A\langle k| A^{\vec{R}_n} A^{\vec{R}_n \dagger} |k\rangle_A B^{\vec{R}_n} B^{\vec{R}_n \dagger} = \frac{(p^{\vec{R}_n,k,0} + p^{\vec{R}_n,k,1})}{2} \mathbb{I}. \tag{43} $$

This equation guarantees that Bob's accumulated operator $B^{\vec{R}_n}$ for each sequence of measurement outcomes $\vec{R}_n$ is proportional to a unitary operator, *i.e.*

$$ B^{\vec{R}_n} = c^{\vec{R}_n} u^{\vec{R}_n}, \tag{44} $$

where the coefficient $c^{\vec{R}_n}$ is set to satisfy

$$ (c^{\vec{R}_n})^2 = \frac{(p^{\vec{R}_n,k,0} + p^{\vec{R}_n,k,1})}{2\, {}_A\langle k| A^{\vec{R}_n} A^{\vec{R}_n \dagger} |k\rangle_A}. \tag{45} $$

For any operator T on $\mathcal{H}_A$, it holds that

$$ T \otimes I |\Phi\rangle_{Aa} = I \otimes {}^tT |\Phi\rangle_{Aa}, \tag{46} $$

where ${}^tT$ denotes the transpose of T in the computational basis. Let $\{S_x^{(i)}\}$ denote the set of operators forming a basis for the operators on $\mathcal{H}_x$ (where $x = a, A,$ or B), that is for any operator T on $\mathcal{H}_x$ there exists a set of complex numbers $c^{(i)}$ such that

$$ T = \sum_i c^{(i)} S_x^{(i)}. \tag{47} $$

An example of such a basis is the set of Pauli operators and the identity operator, if the Hilbert space in question has the dimension of 2. With this basis, U_u on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed as a linear combination of $S_A^{(i)} \otimes S_B^{(j)}$, namely,

$$ U_u = \sum_{i,j} u_{ij} S_A^{(i)} \otimes S_B^{(j)}, \tag{48} $$

where u_{ij} denotes the coefficient of $S_A^{(i)} \otimes S_B^{(j)}$. Let us choose $S_a^{(i)}$ to satisfy

$$ S_a^{(i)} = {}^tS_A^{(i)} \tag{49} $$

in the computational basis and define $\tilde{U}_u$ on $\mathcal{H}_a \otimes \mathcal{H}_B$ by

$$ \tilde{U}_u = \sum_{i,j} u_{ij} S_a^{(i)} \otimes S_B^{(j)}. \tag{50} $$

Under these conventions, we have

$$ \begin{aligned} &(|0\rangle_A \langle k| A^{\vec{R}_n} \otimes B^{\vec{R}_n})(U_u |\Phi\rangle_{Aa} \langle\Phi| \otimes |\varphi'\rangle\langle\varphi'| U_u^\dagger)(|0\rangle_A \langle k| A^{\vec{R}_n} \otimes B^{\vec{R}_n})^\dagger \\ &\qquad = |0\rangle_A \langle 0| \otimes \tilde{U}_u {}^tA^{\vec{R}_n} |k\rangle_a \langle k|_a A^{\vec{R}_n *} \otimes B^{\vec{R}_n} |\varphi'\rangle\langle\varphi'| B^{\vec{R}_n \dagger} \tilde{U}_u^\dagger. \end{aligned} \tag{51} $$

Comparing this equation to Eq. (38), it must be that

$$ \tilde{U}_u {}^tA^{\vec{R}_n} |k\rangle_a \langle k|_a A^{\vec{R}_n *} \otimes B^{\vec{R}_n} |\varphi'\rangle\langle\varphi'| B^{\vec{R}_n \dagger} \tilde{U}_u^\dagger = p^{\vec{R}_n,k,\varphi'} |\Phi\rangle_{aB} \langle\Phi|, \tag{52} $$

which is equivalent to

$$ \tilde{U}_u {}^tA^{\vec{R}_n} |k\rangle_a \otimes B^{\vec{R}_n} |\varphi'\rangle = \sqrt{p^{\vec{R}_n,k,\varphi'}} |\Phi\rangle_{aB}. \tag{53} $$

Let an ancilla state (not necessarily normalized) $\left|v^{\vec{R}_n,k}\right\rangle$ be defined by

$$\left|v^{\vec{R}_n,k}\right\rangle = {}^{t}A^{\vec{R}_n} \left|k\right\rangle_a . \quad (54)$$

By substituting Eq. (44) into Eq. (52), we conclude that

$$\tilde{U}_u \cdot (\mathbb{I} \otimes u^{\vec{R}_n}) \left|v^{\vec{R}_n,k}\right\rangle \otimes \left|\varphi'\right\rangle = \sqrt{p^{\vec{R}_n,k,\varphi'}}/c^{\vec{R}_n} \left|\Phi\right\rangle_{aB} \quad (55)$$

holds for all $|\varphi'\rangle$. The right hand side is collinear to $|\Phi\rangle_{aB}$ for all $|\varphi'\rangle$. On the other hand, because $\tilde{U}_u \cdot (\mathbb{I} \otimes u^{\vec{R}_n})$ is invertible, the left hand side returns linearly independent vectors when $|\varphi'\rangle$ are chosen linearly independently. This, however, is a contradiction proving that the assumption that the given controlled-unitary operation U_u is dislocateable must not hold.

This proof strongly depends on the fact that Bob's input state is kept arbitrary. Indeed, if we are allowed to choose Bob's input state, one-piece dislocation is possible for certain controlled-unitary operations. An example is the controlled-not operation on two qubits.

6. Conclusion and discussion

In this paper, we first reviewed three different characterizations of the globalness of bipartite unitary operations, which were entangling power, entanglement cost for LOCC implementation, and delocalization power. It was shown that the entangling power of a certain bipartite unitary operation has little relation to the globalness revealed by the other two characterizations.

Next, we extended our analysis on characterization in terms of delocalizing power. We investigated global unitary operations which are not LOCC one-piece relocalizeable for two pieces of delocalized quantum information, and shown that these unitary operaitons belonging to the higher globalness class than the local unitary equivalents of controlled-unitary operations can be further divided into two subclasses, by considering LOCC one-piece relocalization of just *one* piece of delocalized quantum information.

We also introduced the concept of dislocation of quantum information, which was the maximum delocalization on quantum information. In particular, we proved that LOCC one-piece dislocation is impossible for any controlled-unitary operations. This confirms that the local unitary equivalents of controlled-unitary operations, which are LOCC one-piece relocalizeable, belong to a class of global operations with relatively weak globalness also in terms of dislocation of quantum information.

The bipartie unitary operations that are LOCC one-piece *un*relocalizable can be further divided into different classes by analyzing whether LOCC one-piece/two-piece dislocation is possible. It is intresting to see which bipartite unitary operations are both LOCC one-piece unrelocalizable and LOCC one-piece/two-piece dislocateable at the same time. These properties shall be investigated further in the future.

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