# Operational Entanglement Families of Symmetric Mixed $N$-Qubit States

T. Bastin,[1] P. Mathonet,[2] and E. Solano[3,4]

[1]*Institut de Physique Nucléaire, Atomique et de Spectroscopie, Université de Liège, B-4000 Liège, Belgium*
[2]*Mathematics research unit, FSTC, University of Luxembourg, L-1359 Luxembourg, Luxembourg*
[3]*IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Spain*
[4]*Departamento de Química Física, Universidad del País Vasco - Euskal Herriko Unibertsitatea, E-48080 Bilbao, Spain*
(Dated: November 5, 2010)

We introduce an operational entanglement classification of symmetric mixed states for an arbitrary number of qubits under stochastic local operations and classical communication (SLOCC). We define families of entanglement classes successively embedded into each other, prove that they are of non-zero measure, and construct witness operators to distinguish them. Moreover, we discuss how arbitrary symmetric mixed states can be realized in the lab via a one-to-one correspondence between well-defined sets of controllable parameters and the corresponding entanglement families.

Entanglement is at the heart of quantum information theory [1] and its classification is expected to characterize potential important applications [2, 3]. Bipartite entanglement is well understood and the case of pure three-qubit states was successfully studied via the concept of SLOCC invariance [4]. For more than three qubits there is an infinite number of SLOCC classes and the introduction of families of entanglement classes [5] has been instrumental to shed light into the increasing level of complexity [6]. Recently, an effort has been done to approach entanglement classification to operational schemes, where one is able to associate univocally physical knobs in given setups with the classes and families determined by specific mathematical criteria [7]. The case of mixed states is more elaborate and, for three qubits, the notions of compactness and convexity proved to be useful [8]. Here, we introduce a classification of mixed symmetric $N$-qubit states into different families of entanglement classes, generalizing the pure state case [7]. Our proposal is based on embedded compact convex sets, allowing for the construction of witness operators for each entanglement family and introducing a natural hierarchy between them. In this sense, it offers a full generalization of the three-qubit mixed state classification of Acín *et al.* [8] to the $N$-qubit symmetric case.

We first review recent results on symmetric $N$-qubit pure states, where they can be written in the so-called Majorana representation form [7, 9]

$$|\psi_S\rangle = \mathcal{N} \sum_{1 \leqslant i_1 \neq \ldots \neq i_N \leqslant N} |\epsilon_{i_1}, \ldots, \epsilon_{i_N}\rangle. \tag{1}$$

Here, the sum is over all $N!$ possible tuples $i_1, \ldots, i_N$, $\mathcal{N}$ is a normalization prefactor and the $|\epsilon_i\rangle$'s are single qubit states $\alpha_i|0\rangle + \beta_i|1\rangle$ with $|\alpha_i|^2 + |\beta_i|^2 = 1$. The expression (1) offers the advantage of allowing for an easy operational classification of all symmetric pure states based on families of SLOCC entanglement classes. The latter are identified by the *degeneracy configuration* and the *diversity degree* of the set of states $\{|\epsilon_1\rangle, \ldots, |\epsilon_N\rangle\}$ [7]. The degeneracy configuration $\mathcal{D}$ is the decreasing order list of the number of $|\epsilon_i\rangle$ states identical to each other, this

number being 1 for each state occurring once. The diversity degree $d$ is the dimension of this list, that is the number of distinct $|\epsilon_i\rangle$ states in Eq. (1). The degeneracy configuration is nothing else than a partition of $N$ and the number of different degeneracy configurations for an $N$-qubit system is given by the partition function $p(N)$. Two states with different degeneracy configurations belong necessarily to different SLOCC classes, the converse being only true when the diversity degree is equal or smaller than 3 [7]. The Greenberger-Horne-Zeilinger (GHZ) SLOCC class for symmetric states is always contained in the $\mathcal{D}_{1,\ldots,1}$ family ($d = N$), the Dicke state $|D_N^{(k)}\rangle$ SLOCC classes ($k = 1, \ldots, [N/2]$) [10] are identified with the $\mathcal{D}_{N-k,k}$ families ($d = 2$), while the symmetric separable states define the $\mathcal{D}_N$ family ($d = 1$) [7]. Consequently, we denote also these families by GHZ, $\mathrm{W}_k$ and S, respectively. In a sense, the $\mathcal{D}_{1,\ldots,1}$ family gathers the most complex states characterized with all distinct $\epsilon_i$'s, while $\mathcal{D}_N$ the simplest states with all $\epsilon_i$'s identical.

Mixed state entanglement classification can be obtained generalizing the case of pure states. This was achieved in the three-qubit case [8] by defining successive compact and convex classes of states embedded into each other, allowing for the construction of witness operators that are able to distinguish them. In this sense, a prior identification of a hierarchy between all pure three-qubit classes was required. This yielded a scheme of successive closed sets of mixed states embedded into each other, namely $S \subset B \subset W \subset GHZ$, B denoting the set of biseparable states. We fully generalize here this approach to the $N$-qubit case and focus on the symmetric subspace. To this aim, we identify the unique hierarchy between all aforementioned $\mathcal{D}$ families of symmetric pure states that can lead to a comprehensive classification of mixed states. This family hierarchy arises naturally from their topology of sets of states lying successively at the boundary of each other, where the $\mathcal{D}_{1,\ldots,1}$ family determines the primary set at the boundary of which all others reside and $\mathcal{D}_N$ is located in the other extreme at the boundary of all other families. This hierarchy yields a

direct ordering relation between all entanglement families. Topologically, $\mathcal{D}_{1,\ldots,1}$ is the only family to form an open set and $\mathcal{D}_N$ a closed one. Let us exemplify this for $N = 4$. In this case, the symmetric states (1) are fully determined by the knowledge of four $\epsilon_i$'s, yielding the five entanglement families $\mathcal{D}_{1,1,1,1}$, $\mathcal{D}_{2,1,1}$, $\mathcal{D}_{2,2}$, $\mathcal{D}_{3,1}$, and $\mathcal{D}_4$ [7]. The first family gathers states characterized with four distinct $\epsilon_i$'s and is spanned by considering all quadruplets $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ fulfilling the condition of distinctness that makes the set open. Any state having two or more identical $\epsilon_i$'s lies at the topological boundary of the $\mathcal{D}_{1,1,1,1}$ open set while belonging to any of the four remaining families. Along the same lines, one observes how the other successive families fit into each other: $\mathcal{D}_{2,2}$, $\mathcal{D}_{3,1}$, and $\mathcal{D}_4$ lie at the boundary of the $\mathcal{D}_{2,1,1}$ family, and $\mathcal{D}_4$ is at the boundary of $\mathcal{D}_{2,2}$ and $\mathcal{D}_{3,1}$. Note that neither $\mathcal{D}_{2,2}$ is at the boundary of $\mathcal{D}_{3,1}$ nor the converse. This highlights the natural hierarchy and the topology of the entanglement families when they are ordered according to their boundary imbrication, a family $\mathcal{D}'$ being said to *descend* from $\mathcal{D}$ ($\mathcal{D} \to \mathcal{D}'$) if $\mathcal{D}'$ lies at the boundary of $\mathcal{D}$ [11]. This topology implies in particular that any *continuous* entanglement measure that would vanish for all states of a given family would also vanish for all states in all descending families.

The family hierarchy can be advantageously illustrated using *entanglement family graphs* (Hasse diagrams), as shown in Fig. 1 for $N = 4$. In these graphs, any downward arrow or path of downward arrows materializes a descending relation between two families. In the general $N$ case, $\mathcal{D}_{1,\ldots,1}$ is always the highest level family with respect to the defined hierarchy and $\mathcal{D}_N$ the lowest level one with no descendants. Lower level families have necessarily lower diversity degrees. Any family with a given diversity degree $d$ never descends from another one with same $d$, while it always descends from at least a family with diversity degree $d+1$ (for $d < N$). Consequently, the diversity degree is a good hierarchy marker and it makes sense to display the entanglement family graphs by layers of families of up-to-bottom diversity degree. The number of layers is exactly $N$ and the layers $d = N$, $d = N - 1$, and $d = 1$ never contain more than one entanglement family, namely $\mathcal{D}_{1,\ldots,1}$, $\mathcal{D}_{2,1,\ldots,1}$, and $\mathcal{D}_N$, respectively.

Symmetric $N$-qubit mixed states are states that can be written as convex sums of projectors onto symmetric pure states. It follows that they can be classified generalizing the $\mathcal{D}$ family classification of the latter and taking into account the hierarchy of these families. To this aim, we define the $\mathcal{D}$ families for mixed states as being the sets of symmetric mixed states that can be written as convex sums of projectors onto symmetric pure states of the $\mathcal{D}$ families in question *and any of their descendants*. This ensures the creation of closed sets of mixed states since all descendants of a family, as from top to bottom in Fig. 1, form its complete boundary. In this manner, the formed sets are compact and convex, allowing for the construc-
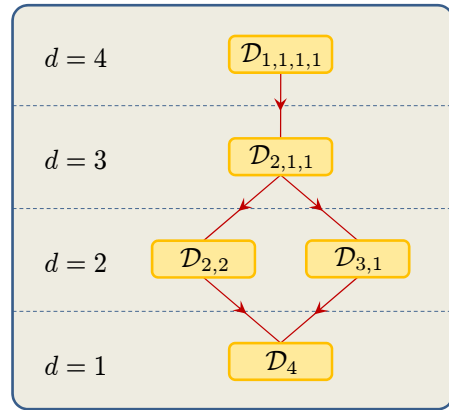


FIG. 1. Entanglement family graph for $N = 4$.

tion of witness operators [3] detecting states outside the sets according to the Hahn-Banach separation theorem. We have furthermore

$$\mathcal{D}' \subset \mathcal{D} \qquad (2)$$

for all $\mathcal{D}'$ families descending from $\mathcal{D}$. This results in an onion-like layer structure of the different families with a complexity growing with that of the entanglement family graph. The physical interpretation of the classification is as follows: a mixed state $\rho$ that belongs to a $\mathcal{D}$ family and to none of lower levels is a mixed state whose simplest decompositions into projectors onto symmetric pure states require at least a pure symmetric state of the $\mathcal{D}$ family. For any $N$, $\mathcal{D}_N$ is the only family that has no descendants and it gathers consequently mixed states that can be written as convex sums of only $\mathcal{D}_N$ pure states, that is of only separable states. This family is nothing else than the usual set of separable mixed states restricted to the symmetric subspace. It is also not excluded that a state belongs to several families not descending from each other, as is the case for nonsymmetric three-qubit states where mixed states can be found biseparable with respect to any bipartition without being fully separable [3, 8].

Let us make explicit our mixed state classification for $N = 2, 3$ and 4, while the case of higher $N$'s extrapolates straightforwardly. For up to three qubits, it fits with well-known results of mixed state entanglement classification, namely with the non-substructure of two-qubit entanglement and with the Acín *et al.* [8] classification for mixed three-qubit states. Beyond these cases, it yields a general entanglement classification of higher number of qubits when focusing on mixed symmetric states. For $N = 2$, there are only two pure state families, $\mathcal{D}_{1,1}$ ($d = 2$) and $\mathcal{D}_2$ ($d = 1$) [7], gathering symmetric entangled and separable states, respectively, with hierarchy $\mathcal{D}_{1,1} \to \mathcal{D}_2$. Accordingly, for mixed states, the $\mathcal{D}_2$ family contains all separable symmetric mixed states, while $\mathcal{D}_{1,1}$ contains all mixed states that can be expressed as a convex sum of

projectors onto $\mathcal{D}_2$ and $\mathcal{D}_{1,1}$ states, that is onto any symmetric state. $\mathcal{D}_{1,1} \setminus \mathcal{D}_2$ corresponds merely to the set of entangled symmetric mixed states. We have $\mathcal{D}_2 \subset \mathcal{D}_{1,1}$ and this set structure reflects merely the usual simple structure of 2-qubit mixed state entanglement according to which the separable state set is surrounded by the entangled state set, the latter one showing no substructure.

For $N = 3$, we have the three pure-state entanglement families, $\mathcal{D}_{1,1,1}$ $(d = 3)$, $\mathcal{D}_{2,1}$ $(d = 2)$, and $\mathcal{D}_3$ $(d = 1)$, identifying the GHZ, W, and separable SLOCC classes for symmetric states, respectively [7]. The hierarchy of the 3 families forms the chain $\mathcal{D}_{1,1,1} \to \mathcal{D}_{2,1} \to \mathcal{D}_3$. Accordingly, the $\mathcal{D}_3$ mixed state family contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_3$ states only; the $\mathcal{D}_{2,1}$ mixed state family contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_3$ and $\mathcal{D}_{2,1}$ states; and the $\mathcal{D}_{1,1,1}$ mixed state family contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_3$, $\mathcal{D}_{2,1}$ and $\mathcal{D}_{1,1,1}$ states, that is onto any symmetric state. The $\mathcal{D}_{1,1,1}$, $\mathcal{D}_{2,1}$, and $\mathcal{D}_3$ families for mixed states are nothing else than the GHZ, W, and separable S classes of mixed states identified in Ref. [8], respectively, disregarding the biseparable sets nonexistent in the symmetric subspace. The resulting structure $\mathcal{D}_3 \subset \mathcal{D}_{2,1} \subset \mathcal{D}_{1,1,1}$ is just the translation of the relation S $\subset$ W $\subset$ GHZ [8] for symmetric states.

For $N = 4$, we have the five entanglement pure state $\mathcal{D}$ families as shown in Fig. 1, along with their hierarchy which starts to be more involved than the simple direct structure of the cases $N = 2$ and 3. Consequently, for mixed states, the $\mathcal{D}_4$ family contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_4$ states; the $\mathcal{D}_{3,1}$ family contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_4$ and $\mathcal{D}_{3,1}$ states; $\mathcal{D}_{2,2}$ contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_4$ and $\mathcal{D}_{2,2}$ states; $\mathcal{D}_{2,1,1}$ contains all states that can be expressed as a convex sum of projectors onto $\mathcal{D}_4$, $\mathcal{D}_{3,1}$, $\mathcal{D}_{2,2}$, and $\mathcal{D}_{2,1,1}$ states; and, finally, the $\mathcal{D}_{1,1,1,1}$ mixed state family contains all states that can be expressed as a convex sum of projectors onto any symmetric state. It follows the family set structure as shown in Fig. 2. The convex hull of $\mathcal{D}_{3,1}$ and $\mathcal{D}_{2,2}$ collects mixed states that can be written as convex sums of $\mathcal{D}_4$, $\mathcal{D}_{3,1}$, and $\mathcal{D}_{2,2}$ states, that is on S and W-kind states. This is why it is denoted by W in Fig. 2.

For pure states, all $\mathcal{D}$ entanglement families form zero-measure sets with the exception of $\mathcal{D}_{1,\ldots,1}$ at the boundary of which all families reside. For mixed states, the situation is totally different and all $\mathcal{D}$ families are of non-zero measure. This can be proven as follows. The symmetric $N$-qubit mixed state density operators belong to the $(N + 1)^2$-dimensional real Hilbert space of symmetric hermitic operators. A natural basis in this space is given by the set of operators

$$\{\hat{\sigma}_\lambda \equiv \hat{\sigma}_{N,k}, \hat{\sigma}_{N,k,j}^{(r)}, \hat{\sigma}_{N,k,j}^{(i)}, \ j, k = 0, \ldots, N, j > k\}, \quad (3)$$
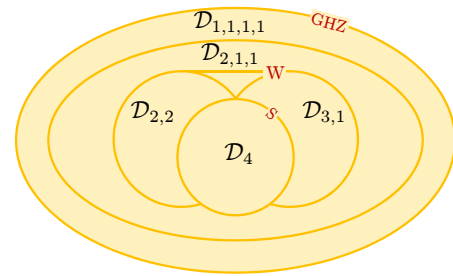


FIG. 2. (Color online) Symmetric mixed state family structure for $N = 4$ (see text).

with $\hat{\sigma}_{N,k} = |D_N^{(k)}\rangle\langle D_N^{(k)}|$, $\hat{\sigma}_{N,k,j}^{(r)} = |D_N^{(k)}\rangle\langle D_N^{(j)}| + |D_N^{(j)}\rangle\langle D_N^{(k)}|$, and $\hat{\sigma}_{N,k,j}^{(i)} = i(|D_N^{(k)}\rangle\langle D_N^{(j)}| - |D_N^{(j)}\rangle\langle D_N^{(k)}|)$. In this basis, any pure symmetric separable state $|\epsilon\rangle^{\otimes N}\langle\epsilon|$ with $|\epsilon\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ reads $|\epsilon\rangle^{\otimes N}\langle\epsilon| = \sum_\lambda f_\lambda(\theta, \phi)\hat{\sigma}_\lambda$ with $f_{N,k,j}^{(r)}(\theta, \phi)$ and $f_{N,k,j}^{(i)}(\theta, \phi)$ the real and imaginary part of $(C_N^k C_N^j)^{1/2} \cos(\theta/2)^{2N-(k+j)} \sin(\theta/2)^{k+j} e^{i(k-j)\phi}$, respectively, and $f_{N,k}(\theta, \phi) = f_{N,k,k}^{(r)}(\theta, \phi)$. Expressing, similarly, $(N + 1)^2$ such separable states $|\epsilon_i\rangle^{\otimes N}\langle\epsilon_i|$ yields the system

$$(|\epsilon_i\rangle^{\otimes N}\langle\epsilon_i|) = F(\hat{\sigma}_\lambda), \quad (4)$$

where $F$ is the matrix with elements $F_{i,\lambda} = f_\lambda(\theta_i, \phi_i)$. Since the $f_\lambda$ functions are linearly independent in $\theta$ and $\phi$, it is always possible to find $(N + 1)^2$ distinct $(\theta_i, \phi_i)$ for which the $F$ matrix is nonsingular and the system (4) is invertible. In this case the corresponding $(N+1)^2$ separable state density operators form a (non-orthonormal) basis in the symmetric hermitian operator space and their affine hull (set of all their linear combinations with coefficients adding up to 1) yields the subset of all trace one operators containing in particular all symmetric mixed states. Since the convex hull of a set is of non-zero measure inside its affine hull, the convex hull of the $(N+1)^2$ separable basis states is a non-zero measure set of symmetric separable states inside the whole symmetric mixed state space. The conclusion follows that the $\mathcal{D}_N$ family is of non-zero measure. All other families being convex and compact, embedded into each others and containing $\mathcal{D}_N$, each successive family adds a non-zero measure set of states with respect to all their descendants, as a direct consequence of the Hahn-Banach separation theorem.

Having established the structure of the set of symmetric mixed $N$-qubit states, we now show how witness operators can be used to distinguish these different families of multipartite entanglement. Witness operators are useful tools that allow one to detect states lying outside compact convex sets [3]. Thanks to this property of the defined $\mathcal{D}$ families, witnesses can always be built to detect mixed states lying out of them and thus belonging to

their complements, that is to any higher or similar level family. For a given $\mathcal{D}$ family, such witnesses are observables whose expectation values are positive for any state belonging to $\mathcal{D}$ and strictly negative for at least a state outside $\mathcal{D}$, that is inside the complement set $\overline{\mathcal{D}}$. We call these observables $\overline{\mathcal{D}}$-witnesses and denote them by $\mathcal{W}_{\overline{\mathcal{D}}}$. Any symmetric state $\rho$ fulfilling $\text{Tr}(\mathcal{W}_{\overline{\mathcal{D}}}\rho) < 0$ is said to be detected by the witness and is guaranteed to belong to $\overline{\mathcal{D}}$ and therefore not to be a $\mathcal{D}$ state. As witnesses cannot detect states lying inside compact convex sets, $\mathcal{D}$-witness operators do not exist. Since $\overline{\mathcal{D}}_N$-witnesses detect non-separable states they are just entanglement witnesses in the symmetric subspace [12].

Projector-based operators provide a convenient way for building $\overline{\mathcal{D}}$-witnesses. Any observable of the form

$$\mathcal{W} = \alpha_{\mathcal{D}}\mathbb{1} - |\psi\rangle\langle\psi|, \tag{5}$$

with $|\psi\rangle$ a $\overline{\mathcal{D}}$ state and

$$\alpha_{\mathcal{D}} = \max_{|\phi\rangle\in\mathcal{D}}|\langle\phi|\psi\rangle|^2 \tag{6}$$

is a $\overline{\mathcal{D}}$-witness. Let us exemplify different family witnesses for $N = 4$. First we consider witness operators built using projectors onto the GHZ state. In this state, we do have $\alpha_{\mathcal{D}_4} = 1/2$ [13], $\alpha_{\mathcal{D}_{3,1}} = 1/2$, $\alpha_{\mathcal{D}_{2,2}} = 3/4$ and $\alpha_{\mathcal{D}_{2,1,1}} \simeq 0.875$. Since $\alpha_{\mathcal{D}_{3,1}} = \alpha_{\mathcal{D}_4}$, the GHZ state does not provide distinct $\overline{\mathcal{D}}_{3,1}$- and $\overline{\mathcal{D}}_4$-witnesses. A better option with that concern is to consider witnesses based on projectors onto the "tetrahedron state" $|T_4\rangle = 1/\sqrt{3}|D_4^{(0)}\rangle + \sqrt{2/3}|D_4^{(3)}\rangle$, so called because the Bloch sphere representation of its four $|\epsilon_i\rangle$ states [see Eq. (1)] do form a regular tetrahedron [14]. For $|T_4\rangle$, the maximal overlaps with the different family states read $\alpha_{\mathcal{D}_4} = 1/3$ [14], $\alpha_{\mathcal{D}_{3,1}} = 2/3$, $\alpha_{\mathcal{D}_{2,2}} = 1/2$ and $\alpha_{\mathcal{D}_{2,1,1}} \simeq 0.749$, allowing ones to witness distinctly all four-qubit family complement sets.

The proposed classification of mixed entangled states in the symmetric subspace can be implemented in the lab with a one-to-one correspondance between given experimental parameters and the entanglement families. In the experimental setups proposed in Refs. [15], $N$-qubit symmetric pure states are produced in atomic and photonic systems by initial preparation or projective measurements of $N$ individual photon polarization states using adequate elliptical polarizers. There, a final atomic or photonic state of the form (1) is produced, where each qubit state $|\epsilon_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$ is directly determined by the polarization orientation $\boldsymbol{\epsilon}_i = \alpha_i\boldsymbol{\sigma}_- + \beta_i\boldsymbol{\sigma}_+$ of the $i$th polarization filter. A one-to-one correspondance between the degeneracy configuration of the polarizer orientations (list of numbers of identical polarizers) and the $\mathcal{D}$ family of the pure state produced in the atomic or photonic $N$-qubit system is obtained. This correspondance is generalized to the symmetric mixed states by considering the polarizer orientations to be unknown within the perimeter imposed by a given degeneracy configuration. For instance, in a four-qubit set-up, three identical polarizer orientations distinct from a last one while ignoring their exact orientations produces a $\mathcal{D}_{3,1}$ symmetric mixed state, whose decomposition onto projectors of symmetric pure states is known from the probability distribution of the allowed polarizer orientations, each orientation set determining a well-defined pure state [15]. It is noteworthy to mention that with this approach all symmetric mixed multiqubit states can be produced.

[1] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Reb. Mod. Phys. **81**, 865 (2009).

[3] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).

[4] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[5] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschede, Phys. Rev. A **65**, 052112 (2002).

[6] L. Lamata, J. León, D. Salgado, and E. Solano, Phys. Rev. A **75**, 022318 (2007).

[7] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. **103**, 070503 (2009); P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, Phys. Rev. A **81**, 052315 (2010).

[8] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).

[9] E. Majorana, Nuovo Cimento **9**, 43 (1932).

[10] Here $|D_N^{(k)}\rangle$ denotes the Dicke state with $k$ $|1\rangle$ excitations and $[.]$ denotes the integer part.

[11] Mathematically speaking, this is just the definition of a *partial order* in the $\mathcal{D}$ family set.

[12] See also G. Tóth and O. Gühne, Appl. Phys. B **98**, 617 (2010).

[13] T.-C. Wei and P. M. Goldbart, Phys. Rev. A **68**, 042307 (2003).

[14] J. Martin, O. Giraud, P. A. Braun, D. Braun, and T. Bastin, Phys. Rev. A **81** 062347 (2010).

[15] T. Bastin, C. Thiel, J. von Zanthier, L. Lamata, E. Solano, and G.S. Agarwal, Phys. Rev. Lett. **102**, 053601 (2009); N. Kiesel, W. Wieczorek, S. Krins, T. Bastin, H. Weinfurter, and E. Solano, Phys. Rev. A **81**, 032316 (2010); A. Maser, R. Wiegner, U. Schilling, C. Thiel, and J. von Zanthier, Phys. Rev. A **81**, 053842 (2010).