Information propagation for interacting particle systems

Sarah K. Harrison,[1] Norbert Schuch,[2] Tobias J. Osborne,[3,4] and Jens Eisert[4,5]

[1] *Department of Mathematics, Royal Holloway University of London, Egham, Surrey, TW20 0EX, UK*
[2] *Institute for Quantum Information, California Institute of Technology, MC 305-16, Pasadena CA 91125, USA*
[3] *Institut für Theoretische Physik, Leibniz-Universität Hannover, Appelstr. 2, 30167 Hannover, Germany*
[4] *Institute for Advanced Study Berlin, 14193 Berlin, Germany*
[5] *Institute of Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany*

We show that excitations of interacting quantum particles in lattice models always propagate with a finite speed of sound. Our argument is simple yet general and shows that by focusing on the physically relevant observables one can generally expect a bounded speed of information propagation. The argument applies equally to quantum spins, bosons such as in the Bose-Hubbard model, fermions, anyons, and general mixtures thereof, on arbitrary lattices of any dimension. It also pertains to dissipative dynamics on the lattice, and generalizes to the continuum for quantum fields. Our result can be seen as a meaningful analogue of the Lieb-Robinson bound for strongly correlated models.

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How fast can information propagate through a system of interacting particles? The obvious answer seems: No faster than the speed of light. While certainly correct, this is not the answer one is usually looking for. For instance, in a classical solid, liquid, or gas, perturbations rather propagate at the speed of sound, which is determined by the way the particles in the system locally interact with each other, without any reference to relativistic effects. We would like to understand whether a similar "speed of sound" exists for interacting quantum systems, limiting the propagation speed of localized excitations, i.e., (quasi-)particles. For interacting quantum spin systems, such a maximal velocity, known as the Lieb-Robinson bound [1–4], has indeed been shown. While it seems appealing that there should always be such a bound, systems of interacting bosons can show counter-intuitive effects, in particular since the interpretation of excitations in terms of particles is no longer fully justified; in fact, an example of a *non-relativistic* system where bosons condense into a dynamical state which steadily accelerates has recently been constructed [5]. This example suggests the disturbing possibility that our intuition is wrong, and only relativistic quantum theory can provide a proper speed limit.

There are many important reasons, both theoretical and experimental, to investigate information propagation bounds in interacting particle systems. It turns out that such bounds lead directly to important, general results concerning the clustering of correlations in equilibrium states [2]. Lieb-Robinson bounds facilitate the simulatability of strongly interacting quantum systems—the mere existence of a Lieb-Robinson bound for a quantum system can be used to develop general, efficient, numerical procedures to simulate the dynamics of lattice models [6]. From a more practical perspective, new experiments allow one to explore the non-equilibrium dynamics of ultracold strongly correlated quantum particles—bosonic, fermionic, or mixtures thereof—in optical lattices with unprecedented control [7, 8]. In such experiments, it is important to understand how the particles move: For example, when studying instances of anomalous expansion, it is far from clear *a priori* whether it is possible to identify a meaningful speed of sound at all.

The original Lieb-Robinson bound already applies in a very general setting, namely, to any low-dimensional quantum spin system, and to any fermionic system confined to a lattice. It is therefore tempting to extend the original argument to other settings, in particular, to systems of interacting bosons; unfortunately, all attempts to do so have run into insuperable difficulties for systems with nonlinear interactions, including the Bose-Hubbard model. The reason for the failure of the original Lieb-Robinson argument is fundamentally connected to the unboundedness of the creation operator for bosons: The Lieb-Robinson velocity depends on the norm of the interaction, which is unbounded for, e.g., bosons hopping on a lattice, and examples without a speed limit can be constructed [5].

In this Letter, we show how these difficulties can be overcome by considering the right question concerning the propagation of information. Our approach allows us to determine Lieb-Robinson type bounds for the maximal speed at which information can propagate through systems of interacting particles in a very general scenario: In particular, it applies to systems of interacting bosons, as well as to fermions, spins, anyons, or mixtures thereof, both on lattices and in the continuum. Moreover, it can also be applied beyond Hamiltonian evolution, such as to systems evolving under some local dissipative dynamics.

The type of system we have in mind is exemplified by the *Bose-Hubbard model*, a model of bosons hopping on an arbitrary lattice G of any finite dimension and interacting via an on-site repulsion,

$$\hat{H}_{\mathrm{BH}} = -\tau \sum_{\langle j,k\rangle} (\hat{b}_j^\dagger \hat{b}_k + \mathrm{h.c.}) + \frac{U}{2} \sum_j \hat{n}_j(\hat{n}_j - 1) - \mu \sum_j \hat{n}_j,$$

where the first summation is over neighboring sites on the lattice, $\hat{b}_j$ is the boson annihilation operator for site

j, and $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ is the number operator. The natural distance in the lattice will be denoted by $d(\cdot,\cdot)$, e.g., $d(j,k) = |j-k|$ for a one-dimensional chain. While we will, for clarity, focus our discussion on the Bose-Hubbard model, our arguments directly generalize to models of the form

$$\hat{H} = -\tau \sum_{s=1}^{S} \sum_{\langle j,k \rangle} (\hat{b}_{s,j}^\dagger \hat{b}_{s,k} + \text{h.c.}) + f(\{\hat{n}_{1,j}, \ldots, \hat{n}_{S,j}\}_{j\in G}) \ , \tag{1}$$

where the $\hat{b}_{s,j}$ are annihilation operators for bosons, fermions, or even anyons of species $s = 1,\ldots,S$ at site j, and $\hat{n}_{s,j} = \hat{b}_{s,j}^\dagger \hat{b}_{s,j}$; the species could for instance refer to an internal spin degree of freedom. The interaction between the particles is characterized by f which can be an arbitrary function of the local densities, and may involve higher moments of the particle number, or even non-local interactions. Moreover, our argument also applies to time-dependent Hamiltonians of this form, as long as the tunneling amplitude is bounded.

The scenario we consider is described by the Bose-Hubbard model on a lattice G, where in the initial state all sites are empty (i.e., $\langle \hat{n}_j \rangle = 0$) except for the sites in a region R which can be in an arbitrary initial state with finite average particle number. Note that the region R may very well encompass the major part of the lattice. What we are interested in is how fast these bosons will travel into the empty part $G\backslash R$ of the lattice, as a function of the distance $d(\cdot,\cdot)$ on the underlying graph. In particular, we would like to find a "speed of sound" for the bosons, that is, a velocity v such that for any region S in $G\backslash R$ with $d(S,R) \geq l$ [i.e.: $d(s,r) \geq l \ \forall s \in S, \ r \in R$], and for all times t for which $vt < l$, the expectation value of any observable $\hat{O}_S$ on S is equal to the expectation value of the vacuum, up to a correction which decays exponentially away from the light cone, $e^{\gamma(vt-l)}$.

To start, we consider the Bose-Hubbard model $\hat{H}_{BH}$ and focus on measurements of the local particle number operators $\hat{n}_j$. This corresponds to looking for bosons at the initially empty sites, and thus captures the most natural notion of particles propagating into a region. Let us denote the initial state by $\rho(0)$, which evolves according to

$$\dot{\rho}(t) = -i[\hat{H}_{\rm BH}, \rho(t)]$$

for $t \geq 0$. As we are interested in the speed at which particles in the Bose-Hubbard model propagate, let us try to understand how the local particle densities

$$\alpha_j(t) = \mathrm{tr}(\hat{n}_j \rho(t)), \quad j \in G \ ,$$

evolve under $\hat{H}_{\rm BH}$. To this end, we derive a bound on the rate at which $\alpha_j(\cdot)$ changes, which in turn leads to a bound on the velocity at which particles can propagate through the system. It holds that

$$\begin{aligned} \dot{\alpha}_j(t) &= -i \, \mathrm{tr}\big(\hat{n}_j \, [\hat{H}_{\rm BH}, \rho(t)]\big) \\ &= -i \, \mathrm{tr}\big([\hat{n}_j, \hat{H}_{\rm BH}] \, \rho(t)\big) \\ &= 2\tau \sum_{\langle j,k \rangle} \mathrm{Im}\Big[\mathrm{tr}\big(\hat{b}_k^\dagger \hat{b}_j \rho(t)\big)\Big] \ , \end{aligned} \tag{2}$$

where the summation runs over all sites k neighboring j, $d(j,k) = 1$. Since we are only interested in an upper bound on this rate of change, we now consider $|\dot{\alpha}_j(t)|$ and apply the triangle inequality to obtain

$$|\dot{\alpha}_j(t)| \leq 2\tau \sum_{\langle j,k \rangle} \left| \mathrm{tr}(\hat{b}_k^\dagger \hat{b}_j \rho(t)) \right| \ . \tag{3}$$

To bound this term we use the operator Cauchy-Schwarz inequality, viewing

$$\mathrm{tr}(\hat{b}_k^\dagger \hat{b}_j \rho(t)) = \langle \hat{b}_k \rho^{1/2}(t), \hat{b}_j \rho^{1/2}(t) \rangle$$

as a Hilbert-Schmidt scalar product of $\hat{b}_j \rho^{1/2}(t)$ and $\hat{b}_k \rho^{1/2}(t)$, where $\rho^{1/2}(t)$ is the matrix square root of $\rho(t)$. This gives rise to

$$\left| \mathrm{tr}(\hat{b}_k^\dagger \hat{b}_j \rho(t)) \right| \leq \Big(\mathrm{tr}(\hat{b}_k^\dagger \hat{b}_k \rho(t)) \mathrm{tr}(\hat{b}_j^\dagger \hat{b}_j \rho(t)) \Big)^{1/2} .$$

Combining this with (3), we obtain a set of coupled differential inequalities

$$|\dot{\alpha}_j(t)| \leq 2\tau \sum_{\langle j,k \rangle} \left(\alpha_j(t) \alpha_k(t) \right)^{1/2} \ , \tag{4}$$

which, using $\sqrt{xy} \leq (x+y)/2$, yields the linearized system

$$|\dot{\alpha}_j(t)| \leq \tau \Big(\mathcal{D} \, \alpha_j(t) + \sum_{\langle j,k \rangle} \alpha_k(t) \Big) \ ,$$

where $\mathcal{D}$ is the maximal vertex degree of the interaction graph.

We are interested in the *worst-case* growth of $\alpha_j(t)$ as t progresses. This will occur when we have equality in the above expression (i.e., the derivative is as large as possible), and thus a bound $\gamma_k(t) \geq \alpha_k(t)$ is given by the solution of the linear system of differential equations

$$\dot{\gamma}_j(t) = \tau \Big(\mathcal{D} \, \gamma_j(t) + \sum_{\langle j,k \rangle} \gamma_k(t) \Big)$$

which fulfills $\gamma_j(0) = \alpha_j(0)$. This solution has the form

$$\vec{\gamma}(t) = e^{\mathcal{D}\tau t} e^{\tau M t} \, \vec{\gamma}(0),$$

where M is the adjacency matrix of the lattice, i.e., $M_{j,k} = 1$ if $d(j,k) = 1$ and 0 otherwise, and $\vec{\gamma} := (\gamma_k)_{k\in L}$. This yields an upper bound

$$\vec{\alpha}(t) \leq e^{\mathcal{D}\tau t} e^{\tau M t} \, \vec{\alpha}(0)$$

for the expected particle number at time t for any site, with $\vec{\alpha} := (\alpha_k)_{k\in L}$.

In order to understand how quickly particles propagate from the initially occupied region R into a region S with $d(R,S) \geq l$, we need to consider the off-diagonal block of $e^{\mathcal{D}\tau t}e^{\tau M t}$ corresponding to those two regions. Thus, in order to obtain a light cone with an exponential decay $\exp(vt-l)$ outside it, we need to understand how rapidly the off-diagonal elements of the banded matrix M grow under exponentiation $e^{\tau M t}$. This can be done by applying Theorem 6 from Ref. [9], which yields for the (i,j)-th element of $\exp(\tau M t)$ the bound

$$[\exp(\tau M t)]_{i,j} \leq C e^{v_0 t - d(i,j)}$$

with velocity $v_0 = \chi\Delta\tau$, where $\chi \approx 3.59$ is the solution of $\chi \ln\chi = \chi + 1$, $\Delta = \frac{1}{2}\|M\|_{\mathrm{op}}$ depends on the lattice dimension, and $C = 2\chi^2/(\chi-1) \approx 10$. Together with the prefactor $\exp(\mathcal{D}\tau t)$, this gives a Lieb-Robinson velocity $v = v_0 + \mathcal{D}\tau$ [10]. For the scenario of an empty lattice with particles initially placed in a region R, this implies that for any j with $d(j,R) \geq l$,

$$\alpha_j(t) \leq C e^{vt-l} \sum_{k\in R} \alpha_k(0) = C N_0\, e^{vt-l}\ , \qquad (5)$$

i.e., up to an exponentially small tail, the particles propagate with a speed no faster than v, independent of their initial state. Here, $N_0 = \sum_{k\in R}\alpha_k(0) = \langle \hat{N}\rangle$ is the total number of particles in the system (i.e., the expectation value of the total particle number operator $\hat{N} = \sum_j \hat{n}_j$). Note that while this (unsurprisingly) means that the strength of the signal observed may depend on the number of bosons initially put into the system, the maximum propagation speed v does not depend on N_0.

Having understood how to obtain a bound on the propagation speed of particles, we now turn to more general observables. First, let us show how we can bound the higher moments of the particle number operator. For $p \geq 1$,

$$\begin{aligned}
\alpha_j^{(p)}(t) &= \mathrm{tr}\big(\hat{n}_j^p \rho(t)\big) \\
&= \sum_N \mathrm{tr}\big(\hat{n}_j \hat{n}_j^{p-1} P_N \rho(t) P_N\big) \\
&\leq \sum_N \mathrm{tr}\big(\hat{n}_j N^{p-1} P_N \rho(t) P_N\big) \\
&\overset{(5)}{\leq} \sum_N N^{p-1}\left(C N e^{vt-l}\right)\mathrm{tr}(\rho(t)) \\
&= C\,\langle \hat{N}^p\rangle\, e^{vt-l}\ ,
\end{aligned} \qquad (6)$$

where P_N projects onto the subspace with a total of N particles, and we have used that Eq. (5) applies to each subspace with fixed particle number independently as the Hamiltonian commutes with P_N. Here, $\langle \hat{N}^p\rangle$ denotes the (time-independent) expectation value of the p-th moment of the total particle number operator. This proves a Lieb-Robinson bound for the higher moments of the particle number operator.

Let us now turn our attention towards arbitrary local observables $\hat{A}_j$. Any such observable can be written as $\hat{A}_j = \sum_{p,q} c_{p,q} (\hat{b}_j^\dagger)^p \hat{b}_j^q$, and we have thus that

$$\begin{aligned}
\left|\mathrm{tr}(\hat{A}_j\rho(t))\right| &\leq \sum_{p,q} |c_{p,q}|\,\left|\mathrm{tr}[(\hat{b}_j^\dagger)^p \hat{b}_j^q \rho(t)]\right| \\
&\leq \sum_{p,q} |c_{p,q}|\,\left(\mathrm{tr}\big[(\hat{b}_j^\dagger)^p \hat{b}_j^p \rho(t)\big]\,\mathrm{tr}\big[(\hat{b}_j^\dagger)^q \hat{b}_j^q \rho(t)\big]\right)^{1/2}\ .
\end{aligned}$$

In turn, for $p > 0$

$$\begin{aligned}
\mathrm{tr}\big[(\hat{b}_j^\dagger)^p \hat{b}_j^p \rho(t)\big] &= \mathrm{tr}\big[\hat{n}_j(\hat{n}_j - 1)\cdots(\hat{n}_j - p + 1)\rho(t)\big] \\
&= \sum_{r=1}^{p} d_{r,p}\,\alpha_j^{(r)}(t) \\
&\leq \tilde{C}_p e^{vt-l}
\end{aligned}$$

by virtue of Eq. (5), for some constant $\tilde{C}_p$. If $p = 0$, we trivially have $\mathrm{tr}[\rho(t)] = 1$. Together, this yields a bound

$$\left|\mathrm{tr}(\hat{A}_j\rho(t))\right| \leq C' e^{vt-l}$$

if $c_{0,q} = c_{p,0} = 0$ for all p and q, and

$$\left|\mathrm{tr}(\hat{A}_j\rho(t))\right| \leq C' e^{(vt-l)/2}$$

otherwise, where we have assumed that $\sum |c_{p,q}|$ is finite, and used that w.l.o.g. $c_{0,0} = 0$. In both cases, this means that outside the lightcone given by $vt = l$, $\mathrm{tr}(\hat{A}_j\rho(t))$ decays exponentially; however, the decay is on double the length scale in the latter case.

Finally, observables acting on more than one site can be bounded analogously to the local case: Any two-site operator acting on sites j, k can be written as the sum of terms $\hat{A}_j\hat{A}_k$, and

$$\left|\mathrm{tr}(\hat{A}_j\hat{A}_k\rho(t))\right| \leq \sqrt{\mathrm{tr}(\hat{A}_j^\dagger\hat{A}_j\rho(t))\mathrm{tr}(\hat{A}_k\hat{A}_k^\dagger\rho(t))}\ .$$

The terms on the r.h.s. are local observables which can be bounded as before by $\exp(vt-l)$, yielding the same exponential bound for two-site—and recursively for many-site—observables. (Note that there exist cases where terms which are bounded by $\exp[(vt-l)/2]$ only appear, and in addition one of the $\hat{A}$'s above could be the identity. Thus, bounds of the form $\exp((vt-l)/\kappa)$ can occur, where κ can grow exponentially in the block size. This, however, still implies that the signal is exponentially small outside the light cone.)

While we have illustrated our arguments for the Bose-Hubbard model, they generalize straightforwardly to the more general class of models described by (1). First, it is clear that we can replace the on-site replusion and chemical potential in the Bose-Hubbard model by any type of interaction (even a non-local one) which only depends on the particle numbers, since any such term vanishes in the commutator $[\hat{n}_j, \hat{H}]$ in Eq. (2). Second, for systems that contain several types of bosons the same arguments apply: Such systems can be modelled using multiple copies of the original graph, each of which supports the hopping of one individual boson species, and one obtains independent differential inequalities for the particle densities $\alpha_{j,s}(t) = \mathrm{tr}[\hat{n}_{j,s}\rho(t)]$ for each species.

Beyond general bosonic models, our arguments also apply to fermions and mixtures of bosons and fermions [11], and in fact even to anyonic systems. Again, in a first step one can decouple the individual species of particles (which mutually commute) to hop on independent graphs. Then, it is easy to check that our arguments work independently of the statistics of the particles, since $[\hat{n}_j, \hat{H}]$ in Eq. (2) evaluates to the same expression in terms of the fermionic (anyonic) creation and annihilation operators. Even better, fermionic and anyonic systems yield stronger bounds for the higher moments, and thus for the scenario of general local observables: In Eq. (6), $\hat{n}_j^{p-1}$ can be bounded by 1 instead of $\hat{N}^{p-1}$, which yields a bound $\alpha_j^{(p)}(t) \leq CN_0\, e^{vt-l}$ on the higher moments. Corresponding results also follow for spin systems, as these can be described as hardcore bosons.

Our arguments work not only for unitary theories, but also for certain types of dissipative (Markovian) models, extending [12] to bosonic systems. For instance, in the practically relevant case of a bosonic system with particle losses, we have that

$$\dot{\rho}(t) = -i\,[\hat{H}_{\mathrm{BH}}, \rho] - \lambda \sum_j \left(\{\hat{b}_j^\dagger \hat{b}_j, \rho(t)\} - 2\hat{b}_j \rho(t) \hat{b}_j^\dagger \right) .$$

Therefore,

$$\dot{\alpha}_j(t) = -i\,\mathrm{tr}([\hat{n}_j, \hat{H}_{\mathrm{BH}}]\rho(t)) - \lambda\, \mathrm{tr}\big(\hat{n}_j \rho(t)\big) ,$$

which shows that the contribution from the dissipative term to $\dot{\alpha}_j$ is negative; thus, tighter differential inequalities and thus a lower speed of sound than in the Hamiltonian case can be obtained.

To conclude, we have proven that there is a speed limit for the propagation of information in a system of interacting particles. This result is particularly relevant for the case of bosons on a lattice, as bosonic systems cannot be assessed using the established techniques of Lieb-Robinson bounds due to the unboundedness of the bosonic hopping operator. Our argument applies equally to bosonic, fermionic, anyonic, and spin systems, as well as mixtures thereof, with arbitrary interaction terms between the particles, and can be generalized to also address systems with dissipation.

The key point that allowed us to make statements about the propagation of information in bosonic systems beyond Lieb-Robinson bounds was first to focus on a subset of observables relevant to detecting the propagation of particles, namely the number of particles present at each site, and second to devise a closed system of inequalities bounding the evolution of their expectation values. This allowed us to reduce the problem of characterizing the full dynamics of the system, which takes place in a superexponentially large Fock space, to simply keeping track of the dynamics of a relatively small number of parameters. This considerably reduced the complexity of the problem and gave rise to an exactly solvable worst-case bound.

The idea of studying information propagation by restricting to a specific set of observables and investigating the resulting worst-case differential equation can also be applied to the study of continous systems. This can be done either by taking an appropriate continuum limit of a lattice model, or by directly considering a corresponding differential equation for the particle density which is continuous in space.

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$$\dot{\beta}_j(t) \leq \tau \sum_{\langle j,k \rangle} \beta_k(t)$$

with initial conditions $\beta(0) = \sqrt{\alpha(0)}$. The worst-case solution of this system is $\vec{\beta}(t) = e^{\tau M t}\,\vec{\beta}(0)$, which, using the previous estimate of $e^{\tau M t}$, yields a velocity v_0. In order to obtain bounds on $\alpha_j(t)$, we need to square this bound. On the one hand, this implies that the correlations outside the light cone decay as $e^{2(vt-l)}$; however, it also yields an unfavorable dependence of the prefactor on the initial conditions, $\big(\sum_j \sqrt{\alpha_j(0)}\,\big)^2$, which can diverge for a fixed number of bosons as the region R grows.

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