# Representations of multidimensional linear process bridges

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**Abstract**

We derive bridges from general multidimensional linear non time-homogeneous processes using only the transition densities of the original process giving their integral representations (in terms of a standard Wiener process) and so-called anticipative representations. We derive a stochastic differential equation satisfied by the integral representation and we prove a usual conditioning property for general multidimensional linear process bridges. We specialize our results for the one-dimensional case; especially, we study one-dimensional Ornstein-Uhlenbeck bridges.

## 1 Introduction

In this paper we deal with deriving bridges from general multidimensional linear processes giving their integral representations (in terms of a standard Wiener process) and so-called anticipative representations. Our results are also specialized for the one-dimensional case. A bridge process is a stochastic process that is pinned to some fixed point at a future time point. Important examples are provided by Wiener bridges, Bessel bridges and general Markovian bridges, which have been extensively studied and find numerous applications. See, for example, Karlin and Taylor [23, Chapter 15], Fitzsimmons, Pitman and Yor [14], Privault and Zambrini [28], Delyon and Hu [10], Gasbarra, Sottinen and Valkeila [15], Goldys and Maslowski [16], Chaumont and Uribe Bravo [8] and Baudoin and Nguyen-Ngoc [5]. Recently, Hoyle, Hughston and Macrina [18] studied so-called Lévy random bridges, that is Lévy processes conditioned to have a prespecified marginal law at the endpoint of the bridge (see also the Ph.D. dissertation of Hoyle [17]). Bichard [7] considered so-called bridged Wiener sheets, that is Wiener sheets which are forced to take some values along specified curves.

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In what follows first we give a motivation for our multidimensional results by presenting different representations of the one-dimensional Ornstein-Uhlenbeck bridges, and then we briefly summarize the structure of the paper.

**Motivation: representations of one-dimensional Ornstein-Uhlenbeck bridges**

Let $(B_t)_{t\geqslant 0}$ be a standard Wiener process and for $q \neq 0$, $\sigma \neq 0$ let us consider the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}Z_t = q\, Z_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t, & t \geqslant 0, \\ Z_0 = 0. \end{cases} \tag{1.1}$$

It is known that there exists a strong solution of this SDE, namely

$$Z_t = \sigma \int_0^t \mathrm{e}^{q(t-s)} \,\mathrm{d}B_s, \qquad t \geqslant 0, \tag{1.2}$$

and strong uniqueness for the SDE (1.1) holds. The process $(Z_t)_{t\geqslant 0}$ is called a one-dimensional Ornstein-Uhlenbeck process (OU-process). It is a time-homogeneous Gauss-Markov process with transition densities

$$p_t^Z(x,y) = \frac{1}{\sqrt{2\pi\sigma^2\kappa_q(t)}} \exp\left\{ -\frac{(y-\mathrm{e}^{qt}x)^2}{2\sigma^2\kappa_q(t)} \right\}, \qquad t>0, \quad x,y\in\mathbb{R}, \tag{1.3}$$

where we set

$$\kappa_q(t) := \frac{\mathrm{e}^{2qt}-1}{2q} = \frac{\mathrm{e}^{qt}}{q}\sinh(qt), \qquad t \geqslant 0. \tag{1.4}$$

For $a,b\in\mathbb{R}$ and $T>0$, by an Ornstein-Uhlenbeck bridge from $a$ to $b$ over the time interval $[0,T]$ derived from $Z$ we understand a Markov process $(U_t)_{t\in[0,T]}$ with initial distribution $P(U_0=a)=1$, with $P(U_T=b)=1$ and with transition densities

$$p_{s,t}^U(x,y) = \frac{p_{t-s}^Z(x,y)\, p_{T-t}^Z(y,b)}{p_{T-s}^Z(x,b)}, \quad x,y\in\mathbb{R}, \quad 0\leqslant s<t<T. \tag{1.5}$$

We also note that $U_t$ converges almost surely to $b$ as $t\uparrow T$, see, e.g., Fitzsimmons, Pitman and Yor [14, Proposition 1]. For the construction of bridges derived from a general time-homogeneous Markov process using only its transition densities, see, e.g., Barczy and Pap [4] and Chaumont and Uribe Bravo [8]. Standard calculations yield that for $x,y\in\mathbb{R}$ and $0\leqslant s<t<T$,

$$\frac{p_{t-s}^Z(x,y)\, p_{T-t}^Z(y,0)}{p_{T-s}^Z(x,0)} = \frac{1}{\sqrt{2\pi\sigma(s,t)}} \exp\left\{ -\frac{\left(y - \frac{\sinh(q(T-t))}{\sinh(q(T-s))}\,x\right)^2}{2\sigma(s,t)} \right\}, \tag{1.6}$$

which is a Gauss density (as a function of $y$) with mean $\frac{\sinh(q(T-t))}{\sinh(q(T-s))}\,x$ and variance $\sigma(s,t)$, where for all $0\leqslant s\leqslant t<T$,

$$\sigma(s,t) := \sigma^2\,\frac{\kappa_q(T-t)\kappa_q(t-s)}{\kappa_q(T-s)} = \frac{\sigma^2}{q}\,\frac{\sinh(q(T-t))\sinh(q(t-s))}{\sinh(q(T-s))}. \tag{1.7}$$

Note that if $\sigma = 0$ then for any $q \in \mathbb{R}$ the unique (deterministic) solution of (1.1) is $Z_t = 0$ for all $t \geqslant 0$ (which coincides with its own bridge from 0 to 0). On the other hand, if $q = 0$ and $\sigma \neq 0$, the unique strong solution of the SDE (1.1) is the Wiener process $Z_t = \sigma B_t$, $t \geqslant 0$, and it is well known that the Wiener bridge $(\widetilde{U}_t)_{t\in[0,T]}$ from 0 to 0 over $[0,T]$ derived from $Z = \sigma B$ admits the (stochastic) integral representation

$$\widetilde{U}_t = \sigma \int_0^t \frac{T-t}{T-s} \, \mathrm{d}B_s, \qquad t \in [0,T), \tag{1.8}$$

see, e.g., Section 5.6.B in Karatzas and Shreve [22]. Moreover, one easily verifies that $(\widetilde{U}_t)_{t\in[0,T]}$ is a Markov process with transition densities

$$p^{\widetilde{U}}_{s,t}(x,y) = \frac{1}{\sqrt{2\pi\widetilde{\sigma}(s,t)}} \exp\left\{ -\frac{\left(y - \frac{T-t}{T-s}\,x\right)^2}{2\widetilde{\sigma}(s,t)} \right\}, \quad x,y \in \mathbb{R}, \quad 0 \leqslant s < t < T, \tag{1.9}$$

where $\widetilde{\sigma}(s,t) := \sigma^2 \frac{(T-t)(t-s)}{T-s}$ for all $0 \leqslant s < t < T$, and that (1.5) is satisfied with $b = 0$, $U$ being replaced by $\widetilde{U}$ and

$$p^Z_t(x,y) = \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left\{ -\frac{(y-x)^2}{2t\sigma^2} \right\}, \quad x,y \in \mathbb{R}, \quad t > 0.$$

Comparing (1.6) with (1.9), it is quite reasonable that an integral representation for the Ornstein-Uhlenbeck bridge from 0 to 0 over $[0,T]$ derived from the process $Z$ given by the SDE (1.1) should have the form

$$U_t = \sigma \int_0^t \frac{\sinh(q(T-t))}{\sinh(q(T-s))} \, \mathrm{d}B_s, \qquad t \in [0,T),$$

and in fact this is made precise in the sequel. We will further consider general multivariate linear process bridges.

Besides the integral representation (1.8) of the Wiener bridge $(\widetilde{U}_t)_{t\in[0,T]}$ from 0 to 0 over $[0,T]$, one can find two equivalent representations in the literature. Namely, by Section 5.6.B in Karatzas and Shreve [22],

$$\begin{cases} \mathrm{d}\widetilde{U}_t = -\frac{1}{T-t}\widetilde{U}_t \, \mathrm{d}t + \mathrm{d}B_t, \qquad t \in [0,T), \\ \widetilde{U}_0 = 0, \end{cases} \tag{1.10}$$

and

$$\widehat{U}_t = B_t - \frac{t}{T} B_T, \qquad t \in [0,T]. \tag{1.11}$$

The representation (1.8) with $\sigma = 1$ is just a strong solution of the SDE (1.10). So, the equations (1.8) with $\sigma = 1$ and (1.10) define the same process $(\widetilde{U}_t)_{t\in[0,T]}$. However, the equation (1.11) does not define the same process as the equations (1.8) with $\sigma = 1$ and (1.10). The equality between representations (1.8) with $\sigma = 1$, (1.10) and (1.11) is only an equality in law, i.e., they determine the same probability measure on $(C([0,T]), \mathcal{B}(C([0,T])))$, where $C([0,T])$ denotes the set of all real-valued continuous functions on $[0,T]$ and $\mathcal{B}(C([0,T]))$ is the Borel $\sigma$-algebra on it. The fact that the processes $\widetilde{U}$ and $\widehat{U}$ are different follows from the fact that the process $\widetilde{U}$ is adapted to the filtration generated by $B$, while the process $\widehat{U}$ is not. Indeed, to construct $\widehat{U}$

we need the random variable $B_T$. One can call (1.11) a non-adapted, anticipative representation of a Wiener bridge. The attribute anticipative indicates that for the definition of $\widehat{U}_t$ we use the random variable $B_T$, where the time point $T$ is after the time point $t$.

A similar anticipative representation of an Ornstein-Uhlenbeck bridge derived from the SDE (1.1) can be found on page 378 in Donati-Martin [11] and in Lemma 1 in Papież and Sandison [27]. Donati-Martin gave an anticipative representation of an Ornstein-Uhlenbeck bridge from $a = 0$ to $b = 0$ derived from the SDE (1.1) with $q < 0$ and $\sigma = 1$, while Papież and Sandison formulated their lemma in case of arbitrary starting point $a$ and ending point $b$, but only for special values of $q$ and $\sigma$, but their proof is also valid for all $q \neq 0$ and $\sigma \neq 0$ (see our Remark 3.1).

Moreover, concerning the relationship between Wiener processes and Wiener bridges, by Problem 5.6.13 in Karatzas and Shreve [22], if $T > 0$ is fixed and $(B_t)_{t\geqslant 0}$ is a standard Wiener process (starting from 0), then for all $n \in \mathbb{N}$, $0 < t_1 < \ldots < t_n < T$, the conditional distribution of $(B_{t_1}, \ldots, B_{t_n})$ given $B_T = 0$ equals the distribution of $(\widetilde{U}_{t_1}, \ldots, \widetilde{U}_{t_n})$, where $\widetilde{U}$ is given by (1.8) with $\sigma = 1$ or by (1.10).

Finally, we note that the transition densities $p^U_{s,t}(x, y)$, $x, y \in \mathbb{R}$, $0 \leqslant s < t < T$, of the process bridge $(U_t)_{t\in[0,T]}$ can be derived using Doob's $h$-transform (see Doob [12]). In Section 2 we briefly study this approach for general multivariate linear process bridges.

**Structure of the paper**

In Section 2 we derive multidimensional linear process bridges from a multidimensional linear non time-homogeneous process $Z$ given by the SDE (2.1) using only the transition densities of $Z$, see Theorem 2.1 and Definition 2.1. We also give an integral and a so-called anticipative representation of the derived bridge, see formulae (2.11) and (2.13), respectively. We derive an SDE satisfied by this integral representation, see Theorem 2.2, and in Proposition 2.1 we prove a usual conditioning property for general multidimensional linear process bridges. In Remark 2.1 we point out that the integral representation and anticipative representation of the bridge are quite different. To shed more light on the different behavior of the different bridge representations, in a companion paper we will study sample path deviations of the Wiener process and the Ornstein-Uhlenbeck process from its bridges, see Barczy and Kern [3]. In Remark 2.2 we study that the SDE derived for the integral representation can be considered as a consequence of Proposition 3 in Delyon and Hu [10]. We use the expression 'can be considered' since the definition of bridges given in Delyon and Hu [10] and in the present paper are different. We have a different approach coming from the possibility that in our special case we are able to explicitly calculate the transition densities of the bridge from which we deduce an integral representation and finally end up with the same SDE of Proposition 3 in Delyon and Hu [10] such that this integral representation is a strong solution of the above mentioned SDE. We also note that the SDE of Proposition 3 in Delyon and Hu [10] contains the solution of a deterministic differential equation (see Remark 2.2) which solution always remains abstract, while in our special case we have an explicit solution via evolution matrices (see Section 2). Theorem 2 of Delyon and Hu [10] gives anticipative representations of general multidimensional conditioned diffusions, in Remark 2.3 we compare our results for anticipative representations of process bridges in the multidimensional case with the corresponding results of Delyon and Hu [10]. Concerning anticipative representations of one-dimensional Gauss bridges see the recent paper of Gasbarra, Sottinen and Valkeila [15]. Remark 2.4 is devoted to studying the construction of a bridge by subtracting from a process its conditional expectation given the process at a prescribed time point (the endpoint of the bridge).

In Section 3 we formulate our multidimensional results in case of dimension one which includes also the study of usual Ornstein-Uhlenbeck bridges. We note that not all of the results are immediate consequences of the multidimensional ones and in case of dimension one we can give an illuminating explanation for the anticipative representation motivated by Lemma 1 in Papież and Sandison [27], see Remark 3.1. Moreover, in Remark 3.6 we discuss the connections between Propositions 4 and 9 in Gasbarra, Sottinen and Valkeila [15] and our Theorems 3.3 and 3.1 (anticipative and integral representation of the bridge in case of dimension one).

Section 4 (Appendix) is devoted to the proofs of our results and contains also a supplement for our assumption on Kalman type matrices (introduced in Section 2).

# 2 Multidimensional linear process bridges

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of positive integers, real numbers and non-negative real numbers, respectively. For all $n, m \in \mathbb{N}$, let $\mathbb{R}^{n\times m}$ and $I_n$ denote the set of $n \times m$ matrices with real entries and the $n \times n$ identity matrix, respectively.

For all $d, p \in \mathbb{N}$, let us consider a general $d$-dimensional linear process given by the linear SDE

$$\text{(2.1)} \qquad \mathrm{d}\mathbf{Z}_t = \big(Q(t)\mathbf{Z}_t + \mathbf{r}(t)\big)\,\mathrm{d}t + \Sigma(t)\,\mathrm{d}\mathbf{B}_t, \qquad t \geqslant 0,$$

with continuous functions $Q : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$, $\Sigma : \mathbb{R}_+ \to \mathbb{R}^{d\times p}$ and $\mathbf{r} : \mathbb{R}_+ \to \mathbb{R}^d$, where $(\mathbf{B}_t)_{t\geqslant 0}$ is a $p$-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ satisfying the usual conditions (the filtration being constructed by the help of $\mathbf{B}$), i.e., $(\Omega, \mathcal{F}, P)$ is complete, $(\mathcal{F}_t)_{t\geqslant 0}$ is right continuous, $\mathcal{F}_0$ contains all the $P$-null sets in $\mathcal{F}$ and $\mathcal{F}_\infty = \mathcal{F}$, where $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geqslant 0}\mathcal{F}_t\right)$, see, e.g., Karatzas and Shreve [22, Section 5.2.A]. It is known that there exists a strong solution of the SDE (2.1), namely

$$\text{(2.2)} \qquad \mathbf{Z}_t = \Phi(t)\left[\mathbf{Z}_0 + \int_0^t \Phi^{-1}(s)\mathbf{r}(s)\,\mathrm{d}s + \int_0^t \Phi^{-1}(s)\Sigma(s)\,\mathrm{d}\mathbf{B}_s\right], \qquad t \geqslant 0,$$

where $\Phi$ is a solution to the deterministic matrix differential equation $\Phi'(t) = Q(t)\Phi(t)$, $t \geqslant 0$, with $\Phi(0) = I_d$ and strong uniqueness for the SDE (2.1) holds, see, e.g., Karatzas and Shreve [22, Section 5.6]. The unique solution of the above matrix differential equation can be given as $\Phi(t) = E(t, 0)$, $t \geqslant 0$, in terms of the evolution matrices

$$E(t, s) = I_d + \int_s^t Q(t_1)\,\mathrm{d}t_1 + \sum_{k=2}^{\infty}\int_s^t\int_s^{t_1}\cdots\int_s^{t_{k-1}} Q(t_1)\cdots Q(t_k)\,\mathrm{d}t_k\mathrm{d}t_{k-1}\cdots\mathrm{d}t_1$$

for $s, t \geqslant 0$. Indeed, by Theorem 1.8.2 in Conti [9], the general $d$-dimensional solution $\mathbf{y}(t) : \mathbb{R}_+ \to \mathbb{R}^d$ of $\mathbf{y}'(t) = Q(t)\mathbf{y}(t)$, $t \geqslant 0$, is represented by $\mathbf{y}(t) = E(t, s)\mathbf{y}(s)$ for all $s, t \geqslant 0$, which shows that $\Phi(t) = E(t, 0)$, $t \geqslant 0$. Note that, since $Q$ is continuous, there exists an $L > 0$ such that $\|Q(u)\| \leqslant L$ for all $u \in [\min(s, t), \max(s, t)]$, $s, t \geqslant 0$ (with some fixed matrix norm $\|.\|$ on $\mathbb{R}^{d\times d}$), and hence one easily calculates $\|E(t,s)\| \leqslant e^{L|t-s|}$. Note also that if $Q(t) = Q \in \mathbb{R}^{d\times d}$, $t \geqslant 0$, is constant then $E(t,s) = e^{(t-s)Q}$ for $t, s \geqslant 0$, and hence $\Phi(t) = e^{tQ}$, $t \geqslant 0$.

We will make frequent use of the following properties of evolution matrices stated as equations (1.9.2) and (1.9.3) in Conti [9]. For all $r, s, t \geqslant 0$ we have

$$E(t,s)E(s,r) = E(t,r), \tag{2.3}$$

$$E(t,t) = I_d, \qquad E(t,s)^{-1} = E(s,t), \tag{2.4}$$

$$\partial_1 E(t,s) = Q(t)E(t,s), \qquad \partial_2 E(t,s) = -E(t,s)Q(s). \tag{2.5}$$

The unique strong solution of the SDE (2.1) can now be written as

$$\mathbf{Z}_t = E(t,0)\mathbf{Z}_0 + \int_0^t E(t,s)\mathbf{r}(s)\,\mathrm{d}s + \int_0^t E(t,s)\Sigma(s)\,\mathrm{d}\mathbf{B}_s, \qquad t \geqslant 0.$$

Here and in what follows we assume that $\mathbf{Z}_0$ has a Gauss distribution independent of the Wiener process $(\mathbf{B}_t)_{t\geqslant 0}$. Then we may define the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ such that $\sigma\{\mathbf{Z}_0, \mathbf{B}_s : 0 \leqslant s \leqslant t\} \subset \mathcal{F}_t$ for all $t \geqslant 0$, see, e.g., Karatzas and Shreve [22, Section 5.2.A].

We will call the process $(\mathbf{Z}_t)_{t\geqslant 0}$ a $d$-dimensional linear process.

One can easily derive that for $0 \leqslant s \leqslant t$ we have

$$\mathbf{Z}_t = E(t,s)\mathbf{Z}_s + \int_s^t E(t,u)\mathbf{r}(u)\,\mathrm{d}u + \int_s^t E(t,u)\Sigma(u)\,\mathrm{d}\mathbf{B}_u. \tag{2.6}$$

Hence, given $\mathbf{Z}_s = \mathbf{x}$, the distribution of $\mathbf{Z}_t$ does not depend on $(\mathbf{Z}_u)_{u\in[0,s)}$ and thus $(\mathbf{Z}_t)_{t\geqslant 0}$ is a Gauss-Markov process (see, e.g., Karatzas and Shreve [22, Problem 5.6.2]). For any $0 \leqslant s \leqslant t$ and $\mathbf{x} \in \mathbb{R}^d$ let us define

$$\mathbf{m}_{\mathbf{x}}^{+}(s,t) := \mathbf{x} + \int_s^t E(s,u)\mathbf{r}(u)\,\mathrm{d}u \quad \text{and} \quad \mathbf{m}_{\mathbf{x}}^{-}(s,t) := \mathbf{x} - \int_s^t E(t,u)\mathbf{r}(u)\,\mathrm{d}u.$$

Then for any $\mathbf{x} \in \mathbb{R}^d$ and $0 \leqslant s < t$ the conditional distribution of $\mathbf{Z}_t$ given $\mathbf{Z}_s = \mathbf{x}$ is Gauss with mean

$$\mathbf{m}_{\mathbf{x}}(s,t) := E(t,s)\mathbf{m}_{\mathbf{x}}^{+}(s,t) = E(t,s)\mathbf{x} + \int_s^t E(t,u)\mathbf{r}(u)\mathrm{d}u,$$

and with covariance matrix of Kalman type (see Kalman [21])

$$\kappa(s,t) := \int_s^t E(t,u)\Sigma(u)\Sigma(u)^\top E(t,u)^\top\,\mathrm{d}u.$$

The matrices $\kappa(s,t)$ are symmetric and positive semi-definite for all $0 \leqslant s < t$, and in what follows we put the following assumption:

$$\kappa(s,t) \text{ is positive definite for all } 0 \leqslant s < t. \tag{2.7}$$

From control theory of linear systems we owe sufficient conditions for positive definiteness of the Kalman matrices (see, e.g., Theorems 7.7.1 - 7.7.3 in Conti [9]) which we present in the Appendix, see Proposition 4.1.

Hence the transition densities of the Gauss-Markov process $(\mathbf{Z}_t)_{t\geqslant 0}$ read as

$$p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^d \det \kappa(s,t)}} \exp\left\{-\frac{1}{2}\langle \kappa(s,t)^{-1}(\mathbf{y}-\mathbf{m}_{\mathbf{x}}(s,t)), \mathbf{y}-\mathbf{m}_{\mathbf{x}}(s,t)\rangle\right\} \tag{2.8}$$

for all $0 \leqslant s < t$ and $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$. Our aim is to derive a process bridge from $\mathbf{Z}$, namely, we will consider a bridge from $\mathbf{a}$ to $\mathbf{b}$ over the time interval $[0,T]$, where $\mathbf{a},\mathbf{b} \in \mathbb{R}^d$ and $T > 0$. Generalizing the formula (2.7) in Fitzsimmons, Pitman and Yor [14] to multidimensional non time-homogeneous Markov processes, for fixed $T > 0$ we are looking for a Markov process $(\mathbf{U}_t)_{t\in[0,T]}$ with initial distribution $P(\mathbf{U}_0 = \mathbf{a}) = 1$ and with transition densities

$$p^{\mathbf{U}}_{s,t}(\mathbf{x},\mathbf{y}) = \frac{p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\, p^{\mathbf{Z}}_{t,T}(\mathbf{y},\mathbf{b})}{p^{\mathbf{Z}}_{s,T}(\mathbf{x},\mathbf{b})}, \quad \mathbf{x},\mathbf{y}\in\mathbb{R}^d, \quad 0 \leqslant s < t < T, \tag{2.9}$$

provided that such a process exists. To properly speak of $(\mathbf{U}_t)_{t\in[0,T]}$ as a process bridge, we shall study the limit behavior of $\mathbf{U}_t$ as $t \uparrow T$, namely, we shall show that $\mathbf{U}_t \to \mathbf{b} =: \mathbf{U}_T$ almost surely and also in $L^2$ as $t \uparrow T$ (see Theorem 2.1).

Our approach can also be seen in the context of Doob's $h$-transform (see Doob [12]) as follows. For bounded Borel-measurable functions $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ one can define a family of operators $(P_{s,t})_{0\leqslant s<t}$ by

$$P_{s,t}f(s,\mathbf{x}) := \int_{\mathbb{R}^d} f(t,\mathbf{y})\, p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$$

for $0 \leqslant s < t$ and $\mathbf{x} \in \mathbb{R}^d$. Then

$$|P_{s,t}f(s,\mathbf{x})| \leqslant \int_{\mathbb{R}^d} |f(t,\mathbf{y})|\, p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \leqslant \sup_{\mathbf{y}\in\mathbb{R}^d} |f(t,\mathbf{y})| < \infty,$$

$$P_{s,t}f(s,\mathbf{Z}_s) = \mathsf{E}(f(t,\mathbf{Z}_t)\mid \mathbf{Z}_s) \qquad P\text{-a.s.},$$

and the family $(P_{s,t})_{0\leqslant s<t}$ forms a hemigroup of transition operators for the Markov process $\mathbf{Z}$. Indeed, for $0 \leqslant s < r < t$ and $\mathbf{x} \in \mathbb{R}^d$ we observe

$$\begin{aligned}
P_{s,r}P_{r,t}f(s,\mathbf{x}) &= \int_{\mathbb{R}^d} P_{r,t}f(r,\mathbf{y})\, p^{\mathbf{Z}}_{s,r}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,\mathbf{z})\, p^{\mathbf{Z}}_{r,t}(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{z}\, p^{\mathbf{Z}}_{s,r}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d} f(t,\mathbf{z}) \int_{\mathbb{R}^d} p^{\mathbf{Z}}_{s,r}(\mathbf{x},\mathbf{y})\, p^{\mathbf{Z}}_{r,t}(\mathbf{y},\mathbf{z})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z} \\
&= \int_{\mathbb{R}^d} f(t,\mathbf{z})\, p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{z})\,\mathrm{d}\mathbf{z} = P_{s,t}f(s,\mathbf{x}).
\end{aligned}$$

For fixed $T > 0$ and $\mathbf{b} \in \mathbb{R}^d$ we now define the function

$$h : [0,T) \times \mathbb{R}^d \to \mathbb{R}_+ \quad \text{by} \quad h(t,\mathbf{x}) = p^{\mathbf{Z}}_{t,T}(\mathbf{x},\mathbf{b}), \qquad t \in [0,T), \ \ \mathbf{x} \in \mathbb{R}^d.$$

By (2.8), $h$ is positive and bounded on $[0,t] \times \mathbb{R}^d$ for every $0 < t < T$. Indeed, (2.7) yields that

$$\inf_{s\in[0,t]} \det \kappa(s,T) > 0, \qquad t \in [0,T),$$

and hence

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} |h(s,x)| \leqslant \left( (2\pi)^d \inf_{s\in[0,t]} \det \kappa(s,T) \right)^{-1/2} < \infty, \qquad t \in [0,T).$$

This yields that $P_{s,t}h(s,\mathbf{x})$ is defined for all $0 \leqslant s < t < T$ and $\mathbf{x} \in \mathbb{R}^d$, although it can happen that $h$ is not bounded on $[0,T) \times \mathbb{R}^d$ (as it is in the case of $\mathbf{Z}$ being a one-dimensional standard Wiener process). Then $h$ is space-time harmonic for the Markov process $\mathbf{Z}$ in the sense that

$$P_{s,t}h(s,\mathbf{x}) = \int_{\mathbb{R}^d} h(t,\mathbf{y})\, p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^d} p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\, p^{\mathbf{Z}}_{t,T}(\mathbf{y},\mathbf{b})\,\mathrm{d}\mathbf{y} = p^{\mathbf{Z}}_{s,T}(\mathbf{x},\mathbf{b}) = h(s,\mathbf{x})$$

for $0 \leqslant s < t < T$ and $\mathbf{x} \in \mathbb{R}^d$. Now a generalization of Doob's $h$-transform approach (see Doob [12] gives a new operator hemigroup

$$\widetilde{P}_{s,t}f = \frac{1}{h}\, P_{s,t}(hf), \quad 0 \leqslant s < t < T$$

with

$$\begin{aligned}\widetilde{P}_{s,t}f(s,\mathbf{x}) &= \frac{1}{h(s,\mathbf{x})}\, P_{s,t}(hf)(s,\mathbf{x}) = \frac{1}{h(s,\mathbf{x})} \int_{\mathbb{R}^d} h(t,\mathbf{y}) f(t,\mathbf{y})\, p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^d} f(t,\mathbf{y})\, \frac{p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\, p^{\mathbf{Z}}_{t,T}(\mathbf{y},\mathbf{b})}{p^{\mathbf{Z}}_{s,T}(\mathbf{x},\mathbf{b})}\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^d} f(t,\mathbf{y})\, p^{\mathbf{U}}_{s,t}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y},\end{aligned}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a bounded Borel-measurable function and $\mathbf{x} \in \mathbb{R}^d$, i.e., the transition operators $(\widetilde{P}_{s,t})_{0\leqslant s<t<T}$ belong to a new Markov process $(\mathbf{U}_t)_{0\leqslant t<T}$, the desired process bridge, with transition densities $(p^{\mathbf{U}}_{s,t})_{0\leqslant s<t<T}$ given by (2.9).

For $T > 0$, $0 \leqslant s < t < T$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, let us define

$$\begin{aligned}\Gamma(s,t) &:= E(s,t)\kappa(s,t) = \int_s^t E(s,u)\Sigma(u)\Sigma(u)^\top E(t,u)^\top\,\mathrm{d}u, \\ \Sigma(s,t) &:= \Gamma(t,T)\Gamma(s,T)^{-1}\Gamma(s,t),\end{aligned}$$

and

$$\mathbf{n}_{\mathbf{a},\mathbf{b}}(s,t) := \Gamma(t,T)\Gamma(s,T)^{-1}\mathbf{m}^+_{\mathbf{a}}(s,t) + \Gamma(s,t)^\top \big(\Gamma(s,T)^\top\big)^{-1}\mathbf{m}^-_{\mathbf{b}}(t,T). \tag{2.10}$$

In what follows we prove the existence of a Markov process $(\mathbf{U}_t)_{t\in[0,T]}$ with initial distribution $P(\mathbf{U}_0 = \mathbf{a}) = 1$ and with transition densities $p^{\mathbf{U}}_{s,t}$ given in (2.9) such that $\mathbf{U}_t \to \mathbf{b} =: \mathbf{U}_T$ almost surely and also in $L^2$ as $t \uparrow T$. First we present an auxiliary lemma.

**2.1 Lemma.** *Let us suppose that condition* (2.7) *holds. Let* $\mathbf{b} \in \mathbb{R}^d$ *and* $T > 0$ *be fixed. Then for all* $0 \leqslant s < t < T$ *and* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ *we have*

$$\begin{aligned}&\frac{p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\, p^{\mathbf{Z}}_{t,T}(\mathbf{y},\mathbf{b})}{p^{\mathbf{Z}}_{s,T}(\mathbf{x},\mathbf{b})} \\ &\quad = \frac{1}{\sqrt{(2\pi)^d \det \Sigma(s,t)}} \exp\left\{ -\frac{1}{2} \left\langle \Sigma(s,t)^{-1}\big(\mathbf{y} - \mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\big), \mathbf{y} - \mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t) \right\rangle \right\},\end{aligned}$$

*which is a Gauss density with mean vector* $\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)$ *and with covariance matrix* $\Sigma(s,t)$.

The proof of Lemma 2.1 can be found in the Appendix.

**2.1 Theorem.** *Let us suppose that condition* (2.7) *holds. For fixed* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ *and* $T > 0$*, let the process* $(\mathbf{U}_t)_{t\in[0,T)}$ *be given by*

$$\mathbf{U}_t := \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) + \Gamma(t,T)\int_0^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u, \quad t \in [0,T). \tag{2.11}$$

*Then for any* $t \in [0,T)$ *the distribution of* $\mathbf{U}_t$ *is Gauss with mean* $\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t)$ *and covariance matrix* $\Sigma(0,t)$*. Especially,* $\mathbf{U}_t \to \mathbf{b}$ *almost surely (and hence in probability) and in* $L^2$ *as* $t \uparrow T$*. Hence the process* $(\mathbf{U}_t)_{t\in[0,T)}$ *can be extended to an almost surely (and hence stochastically) and* $L^2$*-continuous process* $(\mathbf{U}_t)_{t\in[0,T]}$ *with* $\mathbf{U}_0 = \mathbf{a}$ *and* $\mathbf{U}_T = \mathbf{b}$*. Moreover,* $(\mathbf{U}_t)_{t\in[0,T]}$ *is a Gauss-Markov process and for any* $\mathbf{x} \in \mathbb{R}^d$ *and* $0 \leqslant s < t < T$ *the transition density* $\mathbb{R}^d \ni \mathbf{y} \mapsto p^{\mathbf{U}}_{s,t}(\mathbf{x},\mathbf{y})$ *of* $\mathbf{U}_t$ *given* $\mathbf{U}_s = \mathbf{x}$ *is given by*

$$p^{\mathbf{U}}_{s,t}(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma(s,t)}} \exp\left\{-\frac{1}{2}\left\langle \Sigma(s,t)^{-1}\big(\mathbf{y} - \mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\big), \mathbf{y} - \mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\right\rangle\right\},$$

*which coincides with the density given in Lemma 2.1.*

The proof of Theorem 2.1 can be found in the Appendix.

**2.1 Definition.** *Let* $(\mathbf{Z}_t)_{t\geqslant 0}$ *be the d-dimensional linear process given by the SDE* (2.1) *with an initial Gauss random variable* $\mathbf{Z}_0$ *independent of* $(\mathbf{B}_t)_{t\geqslant 0}$ *and let us assume that condition* (2.7) *holds. For fixed* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ *and* $T > 0$*, the process* $(\mathbf{U}_t)_{t\in[0,T]}$ *defined in Theorem 2.1 is called a linear process bridge from* $\mathbf{a}$ *to* $\mathbf{b}$ *over* $[0,T]$ *derived from* $\mathbf{Z}$*. More generally, we call any almost surely continuous (Gauss) process on the time interval* $[0,T]$ *having the same finite-dimensional distributions as* $(\mathbf{U}_t)_{t\in[0,T]}$ *a multidimensional linear process bridge from* $\mathbf{a}$ *to* $\mathbf{b}$ *over* $[0,T]$ *derived from* $\mathbf{Z}$*.*

Formula (2.11) can be considered as an integral representation of the linear process bridge $\mathbf{U}$.

In the next theorem we present an SDE satisfied by the linear process bridge $\mathbf{U}$.

**2.2 Theorem.** *Let us suppose that condition* (2.7) *holds. The process* $(\mathbf{U}_t)_{t\in[0,T)}$ *defined by* (2.11) *is a strong solution of the linear SDE*

$$\begin{aligned} \mathrm{d}\mathbf{U}_t = &\Big[\big(Q(t) - \Sigma(t)\Sigma(t)^\top E(T,t)^\top \Gamma(t,T)^{-1}\big)\mathbf{U}_t \\ &+ \Sigma(t)\Sigma(t)^\top \big(\Gamma(t,T)^\top\big)^{-1}\mathbf{m}^-_{\mathbf{b}}(t,T) + \mathbf{r}(t)\Big]\mathrm{d}t + \Sigma(t)\,\mathrm{d}\mathbf{B}_t \end{aligned} \tag{2.12}$$

*for* $t \in [0,T)$ *and with initial condition* $\mathbf{U}_0 = \mathbf{a}$*, and strong uniqueness for the SDE* (2.12) *holds.*

The proof of Theorem 2.2 can be found in the Appendix.

Now we turn to give alternative representations of the bridge. The next theorem is about the existence of a so-called anticipative representation of the bridge which is a weak solution to the bridge SDE (2.12).

**2.3 Theorem.** *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ and $T > 0$ be fixed. Let $(\mathbf{Z}_t)_{t \geqslant 0}$ be the linear process given by the SDE* (2.1) *with initial condition $\mathbf{Z}_0 = \mathbf{0}$ and let us assume that condition* (2.7) *holds. Then the process $(\mathbf{Y}_t)_{t \in [0,T]}$ given by*

$$\mathbf{Y}_t := \Gamma(t,T)\Gamma(0,T)^{-1}\mathbf{a} + \mathbf{Z}_t - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}(\mathbf{Z}_T - \mathbf{b}), \quad t \in [0,T], \tag{2.13}$$

*equals in law the linear process bridge $(\mathbf{U}_t)_{t \in [0,T]}$ from $\mathbf{a}$ to $\mathbf{b}$ over $[0,T]$ derived from $\mathbf{Z}$.*

The proof of Theorem 2.3 can be found in the Appendix.

Next we present a usual conditioning property for multidimensional linear process bridges.

**2.1 Proposition.** *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ and $T > 0$ be fixed. Let $(\mathbf{Z}_t)_{t \geqslant 0}$ be the $d$-dimensional linear process given by the SDE* (2.1) *with initial condition $\mathbf{Z}_0 = \mathbf{a}$ and let us assume that condition* (2.7) *holds. Let $n \in \mathbb{N}$ and $0 < t_1 < t_2 < \ldots < t_n < T$. Then the conditional distribution of $(\mathbf{Z}_{t_1}^\top, \ldots, \mathbf{Z}_{t_n}^\top)^\top$ given $\mathbf{Z}_T = \mathbf{b}$ equals the distribution of $(\mathbf{U}_{t_1}^\top, \ldots, \mathbf{U}_{t_n}^\top)^\top$, where $(\mathbf{U}_t)_{t \in [0,T]}$ is the linear process bridge from $\mathbf{a}$ to $\mathbf{b}$ over $[0,T]$ derived from $(\mathbf{Z}_t)_{t \geqslant 0}$.*

The proof of Proposition 2.1 can be found in the Appendix. One can also realize that in case of time-homogeneity Proposition 2.1 is a consequence of Proposition 1 in Fitzsimmons, Pitman and Yor [14]. To be more precise, restricting considerations to our situation of a $d$-dimensional linear process $(\mathbf{Z}_t)_{t \geqslant 0}$ given by the SDE (2.1) with initial condition $\mathbf{Z}_0 = \mathbf{a}$, Proposition 1 in Fitzsimmons, Pitman and Yor [14] states that if $\mathbf{Z}$ is time-homogeneous (with transition densities $p_t^{\mathbf{Z}}(\mathbf{x}, \mathbf{y}) := p_{s,s+t}^{\mathbf{Z}}(\mathbf{x}, \mathbf{y})$ for all $s, t \geqslant 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$), then for fixed $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, $T > 0$ there exists a unique probability measure $\widetilde{\mathsf{P}}_{\mathbf{a},\mathbf{b}}^T$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}_{T-})$ such that $(\mathbf{Z}_t)_{t \in [0,T)}$ under $\widetilde{\mathsf{P}}_{\mathbf{a},\mathbf{b}}^T$ is a (non time-homogeneous) Markov process with transition densities given by (2.9), where $\widetilde{\Omega}$ is the set of all real-valued càdlàg functions on $[0, \infty)$, $(\widetilde{\mathcal{F}}_t)_{t \geqslant 0}$ is the natural (uncompleted) filtration of the coordinate process $(\mathbf{Z}_t)_{t \geqslant 0}$ on $\widetilde{\Omega}$ (which we also denote by $\mathbf{Z}$ for simplicity) and $\widetilde{\mathcal{F}}_{T-} := \sigma\Big(\bigcup_{t \in [0,T)} \widetilde{\mathcal{F}}_t\Big)$. Moreover, by Proposition 1 in Fitzsimmons, Pitman and Yor [14], $(\widetilde{\mathsf{P}}_{\mathbf{a},\mathbf{b}}^T)_{\mathbf{b} \in \mathbb{R}^d}$ is a regular version of the family of conditional distributions $\widetilde{\mathsf{P}}(\cdot \,|\, \mathbf{Z}_T = \mathbf{b})$, $\mathbf{b} \in \mathbb{R}^d$, where $\widetilde{\mathsf{P}}$ denotes the law of $(\mathbf{Z}_t)_{t \geqslant 0}$. Hence for any $n \in \mathbb{N}$, $0 < t_1 < \cdots t_n < T$ and any $A \in \mathcal{B}(\mathbb{R}^{nd})$, by Theorem 2.1, we get

$$\widetilde{\mathsf{P}}\big((\mathbf{Z}_{t_1}^\top, \ldots, \mathbf{Z}_{t_n}^\top)^\top \in A \,|\, \mathbf{Z}_T = \mathbf{b}\big) = \widetilde{\mathsf{P}}_{\mathbf{a},\mathbf{b}}^T\big((\mathbf{Z}_{t_1}^\top, \ldots, \mathbf{Z}_{t_n}^\top)^\top \in A\big) = \mathsf{P}\big((\mathbf{U}_{t_1}^\top, \ldots, \mathbf{U}_{t_n}^\top)^\top \in A\big).$$

The next remark shows that the integral and anticipative representation of the bridge are quite different.

**2.1 Remark.** Note that the process $(\mathbf{Y}_t)_{t \in [0,T]}$ defined in (2.13) is only a weak solution of the SDE (2.12), since in contrast to the bridge $(\mathbf{U}_t)_{t \in [0,T]}$ it is not adapted to the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ of the underlying Wiener process $\mathbf{B}$. This can be easily seen by the definition of $\mathbf{Y}_t$ which requires the knowledge of $\mathbf{Z}_T$ at any time point $t \in (0,T)$. Nevertheless we have $\mathbf{Y}_t$ and $\mathbf{Z}_T$ are independent for any $t \in [0,T]$, since by part (a) of Lemma 4.4,

$$\begin{aligned}
\operatorname{Cov}(\mathbf{Y}_t, \mathbf{Z}_T) &= \operatorname{Cov}(\mathbf{Z}_t, \mathbf{Z}_T) - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1} \operatorname{Cov}(\mathbf{Z}_T, \mathbf{Z}_T) \\
&= \Gamma(0,t)^\top E(T,0)^\top - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1} \Gamma(0,T)^\top E(T,0)^\top = 0 \in \mathbb{R}^{d \times d},
\end{aligned}$$

and the random vector $(\mathbf{Y}_t^\top, \mathbf{Z}_T^\top)^\top$ has a Gauss distribution.

In the next remark we compare the SDE (2.12) derived for the integral representation (2.11) of the bridge $\mathbf{U}$ with the corresponding result of Delyon and Hu [10].

**2.2 Remark.** In this remark we discuss the connections between Proposition 3 in Delyon and Hu [10] and our Theorem 2.2. Let $p, d \in \mathbb{N}$ and let us consider the SDE

$$\begin{cases} \mathrm{d}\mathbf{Z}_t = (A(t)\mathbf{Z}_t + \mathbf{g}(t) + \sigma(t)\mathbf{h}(t, \mathbf{Z}_t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}\mathbf{B}_t, \qquad t \geqslant 0, \\ \mathbf{Z}_0 = \mathbf{a}, \end{cases} \tag{2.14}$$

where $\mathbf{a} \in \mathbb{R}^d$, $A : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$, $\mathbf{g} : \mathbb{R}_+ \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}^{d\times p}$ are continuous functions, $\mathbf{h} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^p$ is a locally bounded function such that it is locally Lipschitz with respect to its second variable uniformly with respect to its first variable, i.e., for any $R > 0$, there exists a constant $C_R > 0$ such that for any $(t, \mathbf{x}), (t, \mathbf{y}) \in \mathbb{R}_+ \times \mathbb{R}^d$ with $\|\mathbf{x}\| \leqslant R$, $\|\mathbf{y}\| \leqslant R$ we have

$$\|\mathbf{h}(t, \mathbf{x}) - \mathbf{h}(t, \mathbf{y})\| \leqslant C_R\|\mathbf{x} - \mathbf{y}\|, \quad \forall\, t \geqslant 0.$$

Moreover, we assume that the SDE (2.14) has a (unique) strong solution (the question of uniqueness is not important here, Delyon and Hu [10] suppose only that there exists a strong solution), $\mathbf{h}$ is continuous with respect to its first variable, and that $\sigma$ admits a measurable left pseudo-inverse, denoted by $\sigma^+ := (\sigma^\top\sigma)^{-1}\sigma^\top$, which is left continuous (here for simplicity we suppose also that the symmetric matrix $\sigma^\top\sigma$ is positive definite). Let us denote by $P_t$, $t \geqslant 0$, the unique solution of the deterministic matrix differential equation $P_t' = A(t)P_t$, $t \geqslant 0$, with initial condition $P_0 = I_d$. With the special choices $\mathbf{h}(t) := \mathbf{0}$, $t \geqslant 0$, $\mathbf{g}(t) := \mathbf{r}(t)$, $t \geqslant 0$, $A(t) := Q(t)$, $t \geqslant 0$, $\sigma(t) := \Sigma(t)$, $t \geqslant 0$, we get the SDE (2.14) is the same as the SDE (2.1) with initial condition $\mathbf{Z}_0 = \mathbf{a}$. Further, we have $P_t = E(t, 0)$, $t \geqslant 0$, and, by (2.3) and (2.4),

$$\begin{aligned} M_t &:= \int_t^T P_u^{-1}\sigma(u)\sigma(u)^\top(P_u^{-1})^\top\,\mathrm{d}u = \int_t^T E(u,0)^{-1}\Sigma(u)\Sigma(u)^\top(E(u,0)^{-1})^\top\,\mathrm{d}u \\ &= \int_t^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top\,\mathrm{d}u = E(0,t)\Gamma(t,T)E(0,T)^\top \\ &= E(0,t)E(t,T)\kappa(t,T)E(0,T)^\top = E(0,T)\kappa(t,T)E(0,T)^\top, \qquad t \in [0,T]. \end{aligned}$$

Hence, by our assumption (2.7) on $\kappa$, $M_t$ is positive definite for all $t \in [0, T)$. Since $Q$, $\mathbf{r}$ and $\Sigma$ are continuous, if we suppose also that $\Sigma$ has a left continuous (measurable) left pseudo-inverse, which is guaranteed by assuming that $\Sigma\Sigma^\top$ is positive definite, then, by Proposition 3 in Delyon and Hu [10], the bridge $(\mathbf{U}_t)_{t\in[0,T]}$ from $\mathbf{a}$ to $\mathbf{b}$ over $[0, T]$ derived from $\mathbf{Z}$ given by the SDE (3.1) (with initial condition $\mathbf{Z}_0 = \mathbf{a}$) is a strong solution of the SDE

(2.15)
$$\begin{cases} \mathrm{d}\mathbf{U}_t = A(t)\mathbf{U}_t\mathrm{d}t + \mathbf{g}(t)\mathrm{d}t + \sigma(t)\sigma(t)^\top(P_t^{-1})^\top M_t^{-1}\Big(P_t^{-1}(\mathsf{E}\mathbf{Z}_t - \mathbf{U}_t) - P_T^{-1}(\mathsf{E}\mathbf{Z}_T - \mathbf{b})\Big)\,\mathrm{d}t \\ \qquad\qquad + \sigma(t)\,\mathrm{d}\mathbf{B}_t, \qquad t \in [0, T), \\ \mathbf{U}_0 = \mathbf{a}. \end{cases}$$

To be a little bit more precise, the definition of a bridge in Delyon and Hu [10] is different from our definition: they define a bridge as in Qian and Zheng [29], Lyons and Zheng [25], i.e., via

Radon-Nycodim derivatives (detailed below), and using their Theorem 2 and Proposition 3 the bridge process is not a strong solution of the SDE (2.15), but equals in law to this strong solution. We also note that the results of Qian and Zheng [29] and Lyons and Zheng [25] are valid for time-homogeneous diffusions, while Delyon and Hu [10] consider time inhomogeneous diffusions. Further, Qian and Zheng [29] refer to their Section 2.1 on conditional processes as a set of folklore facts for which they could not find a reference. In what follows we briefly describe a possible *heuristic* approach for time inhomogeneous diffusions which is a counterpart of the approach presented in Qian and Zheng [29, Section 2.1] for time homogeneous diffusions. Let $\mathsf{P}^T_{\mathbf{a},\mathbf{b}}$ be the probability measure on $\mathcal{F}_{T-} = \sigma\left(\bigcup_{t\in[0,T)} \mathcal{F}_t\right)$ defined by

$$\mathsf{P}^T_{\mathbf{a},\mathbf{b}}(F) := \int_F \frac{p^{\mathbf{Z}}_{0,t}(\mathbf{a},\mathbf{Z}_t)p^{\mathbf{Z}}_{t,T}(\mathbf{Z}_t,\mathbf{b})}{p^{\mathbf{Z}}_{0,T}(\mathbf{a},\mathbf{b})}\,\mathrm{d}\mathsf{P}, \qquad F\in\mathcal{F}_t, \quad t\in[0,T),$$

i.e., the Radon–Nycodim derivative of $\mathsf{P}^T_{\mathbf{a},\mathbf{b}}$ with respect to $\mathsf{P}^{\mathbf{Z}_t}_{\mathbf{a}}$, considering them as probability measures on $(\Omega,\mathcal{F}_t)$, is given by

$$\frac{\mathrm{d}\mathsf{P}^T_{\mathbf{a},\mathbf{b}}}{\mathrm{d}\mathsf{P}^{\mathbf{Z}_t}_{\mathbf{a}}} = \frac{p^{\mathbf{Z}}_{t,T}(\mathbf{Z}_t,\mathbf{b})}{p^{\mathbf{Z}}_{0,T}(\mathbf{a},\mathbf{b})},$$

where $\mathsf{P}^{\mathbf{Z}_t}_{\mathbf{a}}$ is given by

$$\mathsf{P}^{\mathbf{Z}_t}_{\mathbf{a}}(F) := \int_F p^{\mathbf{Z}}_{0,t}(\mathbf{a},\mathbf{Z}_t)\,\mathrm{d}\mathsf{P}, \qquad F\in\mathcal{F}_t.$$

Using that $\mathbf{Z}$ is almost surely (left) continuous we get $\mathcal{F}_{T-} = \mathcal{F}_T$ (see, e.g., Karatzas and Shreve [22, Problem 2.7.6 and Corollary 2.7.8]), and hence $\mathsf{P}^T_{\mathbf{a},\mathbf{b}}$ is a probability measure also on $\mathcal{F}_T$. A possible generalization of Lemma 2.1 in Qian and Zheng [29] for time inhomogeneous diffusions sounds as follows: under the probability measure $\mathsf{P}^T_{\mathbf{a},\mathbf{b}}$, the process $(\mathbf{Z})_{t\in[0,T]}$ has a.s. continuous sample paths and admits transition densities given in (2.9), i.e.,

$$\mathsf{P}^T_{\mathbf{a},\mathbf{b}}(Z_t\in B\mid Z_s) = \int_B p^{\mathbf{U}}_{s,t}(\mathbf{Z}_s,\mathbf{y})\,\mathrm{d}y, \qquad B\in\mathcal{B}(\mathbb{R}), \quad 0\leqslant s<t<T,$$

where $\mathcal{B}(\mathbb{R})$ denotes the set of Borel sets in $\mathbb{R}$. Moreover, by the introduction in Delyon and Hu [10], $\mathsf{P}^T_{\mathbf{a},\mathbf{b}}$ equals the law of the weak solution of the SDE

$$\mathrm{d}\mathbf{V}_t = \Big(Q(t)\mathbf{V}_t + \mathbf{r}(t) + \sigma(t)\sigma(t)^\top\mathbf{L}(t,\mathbf{V}_t)\Big)\mathrm{d}t + \Sigma(t)\,\mathrm{d}\mathbf{B}_t, \qquad t\in[0,T),$$

where

$$\mathbf{L}(t,\mathbf{x}) := \nabla_{\mathbf{x}}\ln p^{\mathbf{Z}}_{t,T}(\mathbf{x},\mathbf{b}), \qquad t\in[0,T), \quad \mathbf{x}\in\mathbb{R}^d,$$

with the notation

$$\nabla_{\mathbf{x}}f(\mathbf{x}) := \left(\frac{\partial f}{\partial x_1}(\mathbf{x}),\ldots,\frac{\partial f}{\partial x_d}(\mathbf{x})\right)^\top, \qquad \mathbf{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$$

for a differentiable function $f:\mathbb{R}^d\to\mathbb{R}$. Note that the above SDE is the SDE (2.1) with a modified drift function. However, we can not address a rigorous proof of this approach and in fact this is one of the motivations for our different approach in the present paper.

The right hand side of the SDE (2.15) takes the form

$$Q(t)\mathbf{U}_t\mathrm{d}t + \mathbf{r}(t)\mathrm{d}t + \Sigma(t)\,\mathrm{d}\mathbf{B}_t + \Sigma(t)\Sigma(t)^\top (E(t,0)^{-1})^\top (E(0,T)^\top)^{-1}\kappa(t,T)^{-1}E(0,T)^{-1} \times \Big(E(t,0)^{-1}(\mathbf{m}_\mathbf{a}(0,t) - \mathbf{U}_t) - E(T,0)^{-1}(\mathbf{m}_\mathbf{a}(0,T) - \mathbf{b})\Big)\,\mathrm{d}t.$$

Here the coefficient of $\mathrm{d}t$ is equal to

$$\begin{aligned}
&Q(t)\mathbf{U}_t + \mathbf{r}(t) + \Sigma(t)\Sigma(t)^\top E(0,t)^\top E(T,0)^\top \kappa(t,T)^{-1}E(T,0) \\
&\qquad\times \Bigg[E(0,t)\left(E(t,0)\mathbf{a} + \int_0^t E(t,u)q(u)\,\mathrm{d}u - \mathbf{U}_t\right) \\
&\qquad\qquad - E(0,T)\left(E(T,0)\mathbf{a} + \int_0^T E(T,u)q(u)\,\mathrm{d}u - \mathbf{b}\right)\Bigg] \\
&= \Sigma(t)\Sigma(t)^\top E(0,t)^\top E(T,0)^\top \kappa(t,T)^{-1}E(T,0) \\
&\quad \times \left(\int_0^t E(0,u)q(u)\,\mathrm{d}u - E(0,t)\mathbf{U}_t - \int_0^T E(0,u)q(u)\,\mathrm{d}u + E(0,T)\mathbf{b}\right) + Q(t)\mathbf{U}_t + \mathbf{r}(t) \\
&= -\Sigma(t)\Sigma(t)^\top E(T,t)^\top \kappa(t,T)^{-1}E(T,t)\mathbf{U}_t \\
&\quad - \Sigma(t)\Sigma(t)^\top E(T,t)^\top \kappa(t,T)^{-1}E(T,0)\int_t^T E(0,u)q(u)\,\mathrm{d}u \\
&\quad + \Sigma(t)\Sigma(t)^\top E(T,t)^\top \kappa(t,T)^{-1}\mathbf{b} + Q(t)\mathbf{U}_t + \mathbf{r}(t) \\
&= -\Sigma(t)\Sigma(t)^\top E(T,t)^\top \Gamma(t,T)^{-1}\mathbf{U}_t + \Sigma(t)\Sigma(t)^\top (\Gamma(t,T)^\top)^{-1}\mathbf{m}_\mathbf{b}^-(t,T) + Q(t)\mathbf{U}_t + \mathbf{r}(t).
\end{aligned}$$

This implies that the SDE (2.15) with our special choices is the same as the SDE (2.12) in Theorem 2.2. But we emphasize again that the definition of a bridge in Delyon and Hu [10] is different from our definition and also the proofs of Proposition 3 in Delyon and Hu [10] and our Theorem 2.2 are different. Hence our Theorem 2.2 is not an immediate consequence of Proposition 3 in Delyon and Hu [10].

In the next remark we compare the anticipative representation (2.13) of the bridge $\mathbf{U}$ with the corresponding result in Delyon and Hu [10].

**2.3 Remark.** In this remark we discuss the connections between Theorem 2 in Delyon and Hu [10] and our Theorem 2.3. Let us consider the SDE (2.14) in Remark 2.2. With the special choices $\mathbf{h}(t) := \mathbf{0}$, $t \geqslant 0$, $\mathbf{g}(t) := \mathbf{r}(t)$, $t \geqslant 0$, $A(t) := Q(t)$, $t \geqslant 0$, $\sigma(t) := \Sigma(t)$, $t \geqslant 0$, we get the SDE (2.14) is the same as the SDE (2.1) with initial condition $\mathbf{Z}_0 = \mathbf{a}$. Theorem 2 in Delyon and Hu [10] implies that the linear process bridge from $\mathbf{a}$ to $\mathbf{b}$ over $[0,T]$ derived from $\mathbf{Z}^*$, which is given by the SDE (2.1) with initial condition $\mathbf{Z}_0^* = \mathbf{a}$, equals in law the process

$$\mathbf{Y}_t^* = R^*(t,T)R^*(T,T)^{-1}\mathbf{b} + \Big(\mathbf{Z}_t^* - R^*(t,T)R^*(T,T)^{-1}\mathbf{Z}_T^*\Big), \quad t \in [0,T],$$

where $R^*(s,t) := \operatorname{Cov}(\mathbf{Z}_s^*, \mathbf{Z}_t^*)$, $s,t \geqslant 0$, is the covariance function of $\mathbf{Z}^*$. Since $\mathbf{Z}_t^* = E(t,0)\mathbf{a} + \mathbf{Z}_t$, $t \geqslant 0$, where $(\mathbf{Z}_t)_{t\geqslant 0}$ is given by the SDE (2.1) with initial condition $\mathbf{Z}_0 = \mathbf{0}$, by Lemma 4.4, we get for all $t \in [0,T]$,

$$
\begin{aligned}
\mathbf{Y}_t^* &= (E(T,0)\Gamma(0,t))^\top \big((E(T,0)\Gamma(0,T))^\top\big)^{-1}\mathbf{b} \\
&\quad + \Big[E(t,0)\mathbf{a} + \mathbf{Z}_t - (E(T,0)\Gamma(0,t))^\top \big((E(T,0)\Gamma(0,T))^\top\big)^{-1}(E(T,0)\mathbf{a} + \mathbf{Z}_T)\Big] \\
&= \Gamma(0,t)^\top(\Gamma(0,T)^\top)^{-1}\mathbf{b} + \Big(E(t,0) - \Gamma(0,t)^\top(\Gamma(0,T)^\top)^{-1}E(T,0)\Big)\mathbf{a} \\
&\quad + \mathbf{Z}_t - \Gamma(0,t)^\top(\Gamma(0,T)^\top)^{-1}\mathbf{Z}_T \\
&= \Gamma(t,T)\Gamma(0,T)^{-1}\mathbf{a} + \mathbf{Z}_t - \Gamma(0,t)^\top(\Gamma(0,T)^\top)^{-1}(\mathbf{Z}_T - \mathbf{b}) = Y_t,
\end{aligned}
$$

where the last but one equality follows by part (c) of Lemma 4.2. The proof of our Theorem 2.3 is the very same as the corresponding part of the proof of Theorem 2 in Delyon and Hu [10].

**2.4 Remark.** With the notations of Remark 2.3 one may define a bridge from $\mathbf{0}$ to $\mathbf{0}$ over $[0,T]$ derived from $\mathbf{Z}^*$ by $\mathbf{Z}_t^* - \mathsf{E}(\mathbf{Z}_t^* \,|\, \mathbf{Z}_T^*)$, $t \in [0,T]$, subtracting from $\mathbf{Z}^*$ its conditional expectation given the process at the endpoint of the bridge. Then, by Theorem 2 in Chapter II, §13 of Shiryaev [30] and our assumption (2.7), it is known that the conditional distribution of $\mathbf{Z}_t^*$ given $\mathbf{Z}_T^* = \mathbf{x}$ is normal with mean $\mathsf{E}\mathbf{Z}_t^* + R^*(t,T)R^*(T,T)^{-1}(x - \mathsf{E}\mathbf{Z}_T^*)$, $t \in [0,T]$. Hence we have

$$
\begin{aligned}
\mathbf{Z}_t^* - \mathsf{E}(\mathbf{Z}_t^* \,|\, \mathbf{Z}_T^*) &= \mathbf{Z}_t^* - \mathsf{E}\mathbf{Z}_t^* - R^*(t,T)R^*(T,T)^{-1}(\mathbf{Z}_T^* - \mathsf{E}\mathbf{Z}_T^*) \\
&= \mathbf{Z}_t^* - R^*(t,T)R^*(T,T)^{-1}\mathbf{Z}_T^* - \mathsf{E}(\mathbf{Z}_t^* - R^*(t,T)R^*(T,T)^{-1}\mathbf{Z}_T^*) \\
&= \mathbf{Y}_t^* - \mathsf{E}\mathbf{Y}_t^* = \mathbf{Y}_t - \mathsf{E}\mathbf{Y}_t, \qquad t \in [0,T],
\end{aligned}
$$

which is nothing else but the centered anticipative representation of the bridge from $\mathbf{0}$ to $\mathbf{0}$. Thus in general this definition of the bridge has a different mean function than the bridge given by Definition 2.1.

In case of dimension 1, we will also study the connections between Proposition 4 in Gasbarra, Sottinen and Valkeila [15] and our Theorem 2.3. The reason for restricting ourselves to the case dimension one is that Gasbarra, Sottinen and Valkeila [15] consider only one-dimensional processes.

# 3 One-dimensional linear process bridges

Let us consider a general one-dimensional linear process given by the linear SDE

$$
\mathrm{d}Z_t = \big(q(t)\,Z_t + r(t)\big)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, \qquad t \geqslant 0, \tag{3.1}
$$

with continuous functions $q : \mathbb{R}_+ \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \to \mathbb{R}$ and $r : \mathbb{R}_+ \to \mathbb{R}$, where $(B_t)_{t\geqslant 0}$ is a standard Wiener process. By Section 5.6 in Karatzas and Shreve [22], it is known that there exists a strong solution of the SDE (3.1), namely

$$
Z_t = \mathrm{e}^{\bar{q}(t)}\left(Z_0 + \int_0^t \mathrm{e}^{-\bar{q}(s)} r(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\bar{q}(s)}\sigma(s)\,\mathrm{d}B_s\right), \qquad t \geqslant 0, \tag{3.2}
$$

with $\bar{q}(t) := \int_0^t q(u)\,\mathrm{d}u$, $t \geqslant 0$, and strong uniqueness for the SDE (3.1) holds. In what follows, we assume that $Z_0$ has a Gauss distribution independent of $(B_t)_{t\geqslant 0}$. We call the process $(Z_t)_{t\geqslant 0}$ a one-dimensional linear process. One can easily derive that for $0 \leqslant s < t$ we have

$$Z_t = \mathrm{e}^{\bar{q}(t)-\bar{q}(s)} Z_s + \int_s^t \mathrm{e}^{\bar{q}(t)-\bar{q}(u)} r(u)\,\mathrm{d}u + \int_s^t \mathrm{e}^{\bar{q}(t)-\bar{q}(u)} \sigma(u)\,\mathrm{d}B_u. \tag{3.3}$$

Hence, given $Z_s = x$, the distribution of $Z_t$ does not depend on $(Z_r)_{r\in[0,s)}$ which yields that $(Z_t)_{t\geqslant 0}$ is a Markov process. Moreover, for any $x \in \mathbb{R}$ and $0 \leqslant s < t$ the conditional distribution of $Z_t$ given $Z_s = x$ is Gauss with mean

$$m_x(s,t) := \mathrm{e}^{\bar{q}(t)-\bar{q}(s)} x + \int_s^t \mathrm{e}^{\bar{q}(t)-\bar{q}(u)} r(u)\,\mathrm{d}u,$$

and with variance

$$\gamma(s,t) := \int_s^t \mathrm{e}^{2(\bar{q}(t)-\bar{q}(u))} \sigma^2(u)\,\mathrm{d}u < \infty.$$

In what follows we put the following assumption

$$\sigma(t) \neq 0 \qquad \text{for all } t \geqslant 0. \tag{3.4}$$

This yields that the variance $\gamma(s,t)$ is positive for all $0 \leqslant s < t$ (which corresponds to condition (2.7) in dimension one). Hence $(Z_t)_{t\geqslant 0}$ is a Gauss-Markov process with transition densities

$$p^Z_{s,t}(x,y) = \frac{1}{\sqrt{2\pi\gamma(s,t)}} \exp\left\{ -\frac{(y - m_x(s,t))^2}{2\gamma(s,t)} \right\}, \qquad 0 \leqslant s < t, \quad x, y \in \mathbb{R}. \tag{3.5}$$

For all $a, b \in \mathbb{R}$ and $0 \leqslant s \leqslant t < T$, let us introduce the notations

$$n_{a,b}(s,t) := \frac{\gamma(s,t)}{\gamma(s,T)} \mathrm{e}^{\bar{q}(T)-\bar{q}(t)} \left( b - \int_t^T \mathrm{e}^{\bar{q}(T)-\bar{q}(u)} r(u)\,\mathrm{d}u \right) + \frac{\gamma(t,T)}{\gamma(s,T)} m_a(s,t), \tag{3.6}$$

and

$$\sigma(s,t) := \frac{\gamma(s,t)\,\gamma(t,T)}{\gamma(s,T)}. \tag{3.7}$$

Theorem 2.1 has the following consequence.

**3.1 Theorem.** *Let us suppose that condition* (3.4) *holds. For fixed $a, b \in \mathbb{R}$ and $T > 0$, let the process $(U_t)_{t\in[0,T)}$ be given by*

$$U_t := n_{a,b}(0,t) + \int_0^t \frac{\gamma(t,T)}{\gamma(s,T)} \mathrm{e}^{\bar{q}(t)-\bar{q}(s)} \sigma(s)\,\mathrm{d}B_s, \qquad t \in [0,T). \tag{3.8}$$

*Then for any $t \in [0,T)$ the distribution of $U_t$ is Gauss with mean $n_{a,b}(0,t)$ and with variance $\sigma(0,t)$. Especially, $U_t \to b$ almost surely (and hence in probability) and in $L^2$ as $t \uparrow T$. Hence the process $(U_t)_{t\in[0,T)}$ can be extended to an almost surely (and hence stochastically) and $L^2$-continuous process $(U_t)_{t\in[0,T]}$ with $U_0 = a$ and $U_T = b$. Moreover, $(U_t)_{t\in[0,T]}$ is a Gauss-Markov process and for any $x \in \mathbb{R}$ and $0 \leqslant s < t < T$ the transition density $\mathbb{R} \ni y \mapsto p^U_{s,t}(x,y)$ of $U_t$ given $U_s = x$ is given by*

$$p^U_{s,t}(x,y) = \frac{1}{\sqrt{2\pi\sigma(s,t)}} \exp\left\{ -\frac{(y - n_{x,b}(s,t))^2}{2\sigma(s,t)} \right\}, \qquad y \in \mathbb{R}.$$

The proof of Theorem 3.1 can be found in the Appendix.

For completeness we formulate the definition of a one-dimensional linear process bridge, which definition is a special case of the multidimensional one (see Definition 2.1).

**3.1 Definition.** *Let* $(Z_t)_{t\geqslant 0}$ *be the one-dimensional linear process given by the SDE* (3.1) *with an initial Gauss random variable* $Z_0$ *independent of* $(B_t)_{t\geqslant 0}$ *and let us assume that condition* (3.4) *holds. For fixed* $a, b \in \mathbb{R}$ *and* $T > 0$, *the process* $(U_t)_{t\in[0,T]}$ *defined in Theorem 3.1 is called a linear process bridge from* $a$ *to* $b$ *over* $[0,T]$ *derived from* $Z$. *More generally, we call any almost surely continuous (Gauss) process on the time interval* $[0,T]$ *having the same finite-dimensional distributions as* $(U_t)_{t\in[0,T]}$ *a linear process bridge from* $a$ *to* $b$ *over* $[0,T]$ *derived from* $Z$.

Theorem 2.2 has the following consequence.

**3.2 Theorem.** *Let us suppose that condition* (3.4) *holds. The process* $(U_t)_{t\in[0,T)}$ *defined by* (3.8) *is a unique strong solution of the linear SDE*

$$
\begin{aligned}
\mathrm{d}U_t = &\left(q(t) - \frac{\mathrm{e}^{2(\bar{q}(T)-\bar{q}(t))}}{\gamma(t,T)}\sigma^2(t)\right) U_t \,\mathrm{d}t \\
&+ \left(r(t) + \frac{\mathrm{e}^{\bar{q}(T)-\bar{q}(t)}}{\gamma(t,T)}\left(b - \int_t^T \mathrm{e}^{\bar{q}(T)-\bar{q}(u)} r(u)\,\mathrm{d}u\right)\sigma^2(t)\right)\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t
\end{aligned}
\tag{3.9}
$$

*for* $t \in [0,T)$ *and with initial condition* $U_0 = a$, *and strong uniqueness for the SDE* (3.9) *holds.*

As a consequence of Theorem 2.3 we give an anticipative representation of the linear process bridge introduced in Theorem 3.1 and Definition 3.1.

**3.3 Theorem.** *Let* $(Z_t)_{t\geqslant 0}$ *be a linear process given by the SDE* (3.1) *with initial condition* $Z_0 = 0$ *and let us suppose that condition* (3.4) *holds. Then the process* $(Y_t)_{t\in[0,T]}$ *given by*

$$
Y_t := a\frac{\widetilde{R}(t,T)}{\widetilde{R}(0,T)} + Z_t - \frac{\widetilde{R}(0,t)}{\widetilde{R}(0,T)}(Z_T - b), \quad t \in [0,T],
$$

*equals in law the linear process bridge* $(U_t)_{t\in[0,T]}$ *from* $a$ *to* $b$ *over* $[0,T]$ *derived from the process* $Z$, *where*

$$
\widetilde{R}(s,t) := \gamma(s,t)\mathrm{e}^{\bar{q}(s)-\bar{q}(t)}, \qquad 0 \leqslant s \leqslant t \leqslant T.
$$

*Moreover,*

$$
\widetilde{R}(s,t) = \mathrm{e}^{\bar{q}(s)-\bar{q}(t)} R(t,t) - \mathrm{e}^{\bar{q}(t)-\bar{q}(s)} R(s,s), \qquad 0 \leqslant s \leqslant t \leqslant T,
$$

*where* $R$ *denotes the covariance function of* $Z$, *and*

$$
\begin{aligned}
Y_t = &\, a\left(\mathrm{e}^{\bar{q}(t)} - \mathrm{e}^{2\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)}\right) + b\mathrm{e}^{\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)} \\
&+ \left(Z_t - \mathrm{e}^{\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)} Z_T\right), \qquad t \in [0,T].
\end{aligned}
\tag{3.10}
$$

We remark that the process $(Y_t)_{t\in[0,T]}$ in Theorem 3.3 can be written also in the form

$$Y_t = a\left(\mathrm{e}^{\bar{q}(t)} - \mathrm{e}^{\bar{q}(T)}\frac{R(t,T)}{R(T,T)}\right) + b\frac{R(t,T)}{R(T,T)} + \left(Z_t - \frac{R(t,T)}{R(T,T)}Z_T\right), \quad t\in[0,T]. \tag{3.11}$$

We also note that in Remark 3.1 we will give an illuminating explanation for the representation (3.10).

As a consequence of Proposition 2.1 now we present a usual conditioning property for one-dimensional linear processes.

**3.1 Proposition.** *Let $a, b\in\mathbb{R}$ and $T>0$ be fixed. Let $(Z_t)_{t\geqslant 0}$ be the one-dimensional linear process given by the SDE (3.1) with initial condition $Z_0 = a$ and let us assume that condition (3.4) holds. Let $n\in\mathbb{N}$ and $0<t_1<t_2<\ldots<t_n<T$. Then the conditional distribution of $(Z_{t_1},\ldots,Z_{t_n})$ given $Z_T = b$ equals the distribution of $(U_{t_1},\ldots,U_{t_n})$, where $(U_t)_{t\in[0,T]}$ is the linear process bridge from $a$ to $b$ over $[0,T]$ derived from $(Z_t)_{t\geqslant 0}$.*

Next we give an illuminating explanation for the representation (3.10) in Theorem 3.3 (see Remark 3.1), but preparatory we present a generalization of Lamperti transformation (see, e.g., Karlin and Taylor [23, page 218]) for one-dimensional linear processes. This generalization may be known, but we were not able to find any reference, its proof can be found in the Appendix.

**3.2 Proposition.** *Let $(B^*_t)_{t\geqslant 0}$ be a standard Wiener process starting from 0 and*

$$Z^*_t := m_0(0,t) + \mathrm{e}^{\bar{q}(t)}B^*(\mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)), \quad t\geqslant 0.$$

*Then $(Z^*_t)_{t\geqslant 0}$ is a weak solution of the SDE (3.1) with initial condition $Z^*_0 = 0$.*

**3.1 Remark.** Using Proposition 3.2 one can give an illuminating explanation for the representation (3.10) in Theorem 3.3. By Problem 5.6.14 in Karatzas and Shreve [22], the process $(\widehat{U}_t)_{t\in[0,T]}$ defined by

$$\widehat{U}_t := a\frac{T-t}{T} + b\frac{t}{T} + \left(\widehat{B}_t - \frac{t}{T}\widehat{B}_T\right), \quad t\in[0,T],$$

equals in law the Wiener bridge from $a$ to $b$ over $[0,T]$, where $(\widehat{B}_t)_{t\geqslant 0}$ is a standard Wiener process. Motivated by Lemma 1 in Papież and Sandison [27] and Proposition 3.2, first we will do the time change $[0,T]\ni t\mapsto \mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)$, the rescaling with coefficient $\mathrm{e}^{\bar{q}(t)}$, and then the translation with $m_0(0,t)$ for the process $(\widehat{U}_t)_{t\in[0,T]}$. Namely, we consider the process

$$\begin{aligned} U^*_t := m_0(0,t) + \mathrm{e}^{\bar{q}(t)}\Bigg(& a\frac{\mathrm{e}^{-2\bar{q}(T)}\gamma(0,T) - \mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)}{\mathrm{e}^{-2\bar{q}(T)}\gamma(0,T)} + b\frac{\mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)}{\mathrm{e}^{-2\bar{q}(T)}\gamma(0,T)} \\ &+\widehat{B}(\mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)) - \frac{\mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)}{\mathrm{e}^{-2\bar{q}(T)}\gamma(0,T)}\widehat{B}(\mathrm{e}^{-2\bar{q}(T)}\gamma(0,T))\Bigg), \quad t\in[0,T]. \end{aligned}$$

Then for all $t\in[0,T]$ we have

$$\begin{aligned} U^*_t = m_0(0,t) &+ a\left(\mathrm{e}^{\bar{q}(t)} - \mathrm{e}^{2\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)}\right) + b\mathrm{e}^{\bar{q}(T)}\mathrm{e}^{\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)} \\ &+ \mathrm{e}^{\bar{q}(t)}\widehat{B}(\mathrm{e}^{-2\bar{q}(t)}\gamma(0,t)) - \mathrm{e}^{\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)}\mathrm{e}^{\bar{q}(T)}\widehat{B}(\mathrm{e}^{-2\bar{q}(T)}\gamma(0,T)) = \end{aligned}$$

$$= a\left(\mathrm{e}^{\bar{q}(t)} - \mathrm{e}^{2\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)}\right) + \left(\mathrm{e}^{\bar{q}(T)}b + m_0(0,T)\right)\mathrm{e}^{\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)}$$

$$+ \left(Z_t^* - \mathrm{e}^{\bar{q}(T)-\bar{q}(t)}\frac{\gamma(0,t)}{\gamma(0,T)}Z_T^*\right),$$

where, using Proposition 3.2, $(Z_t^*)_{t\geqslant 0}$ equals in law the one-dimensional linear process given by the SDE (3.1) with initial condition $Z_0 = 0$. By Theorem 3.3, the process $(U_t^*)_{t\in[0,T]}$ equals in law the one-dimensional linear process bridge $(U_t)_{t\in[0,T]}$ from $a$ to $\mathrm{e}^{\bar{q}(T)}b + m_0(0,T)$ over $[0,T]$ derived from $Z$ given by the SDE (3.1) with initial condition $Z_0 = 0$. Roughly speaking, we have to apply the same time change, rescaling and translation to the Wiener bridge from $a$ to $b$ over $[0,T]$ in order to get the linear process bridge from $a$ to $\mathrm{e}^{\bar{q}(T)}b + m_0(0,T)$ over $[0,T]$ (derived from $Z$ given by the SDE (3.1) with initial condition $Z_0 = 0$) what we apply to a Wiener process in order to get the linear process $Z$.

Especially, concerning Wiener bridges and Ornstein-Uhlenbeck bridges, we have to apply the same time change and rescaling to the Wiener bridge from $a$ to $b$ over $[0,T]$ in order to get the Ornstein-Uhlenbeck bridge from $a$ to $\mathrm{e}^{qT}b$ over $[0,T]$ (derived from $Z$ given by the SDE (1.1)) what we apply to a Wiener process in order to get the Ornstein-Uhlenbeck process $Z$. We note that the original definition of an Ornstein-Uhlenbeck bridge of Papież and Sandison is different from ours, they define the bridge as a probability measure on the space of continuous functions $f:[0,T]\to\mathbb{R}$ such that $f(0) = a$ and $f(T) = \mathrm{e}^{qT}b$.

Next we formulate special cases of the presented one-dimensional results.

**3.2 Remark.** Note that in case of $q(t) = q \neq 0$, $t \geqslant 0$, and $\sigma(t) = \sigma \neq 0$, $t \geqslant 0$, (for any continuous deterministic forcing term $r$) the variance $\sigma(s,t)$ defined by (3.7) gives back (1.7). □

Theorem 3.1 has the following consequence.

**3.3 Remark.** Note that in case of $q(t) = q = 0$, $\sigma(t) = \sigma \neq 0$, $r(t) = 0$, $t \geqslant 0$, and $a = 0 = b$, we recover the Wiener bridge $(\widetilde{U}_t)_{t\in[0,T]}$ from $0$ to $0$ stated in (1.8). Moreover, in case of $q(t) = q \neq 0$, $\sigma(t) = \sigma \neq 0$, and $r(t) = 0$, $t \geqslant 0$, the linear process bridge (Ornstein-Uhlenbeck bridge) $(U_t)_{t\in[0,T]}$ from $a$ to $b$ over $[0,T]$ defined in (3.8) has the form

$$U_t = a\,\frac{\sinh(q(T-t))}{\sinh(qT)} + b\,\frac{\sinh(qt)}{\sinh(qT)} + \sigma\int_0^t \frac{\sinh(q(T-t))}{\sinh(q(T-s))}\,\mathrm{d}B_s, \qquad t\in[0,T), \tag{3.12}$$

and admits transition densities

$$p_{s,t}^U(x,y) = \frac{1}{\sqrt{2\pi\sigma(s,t)}}\exp\left\{-\frac{\left(y - \frac{\sinh(q(t-s))}{\sinh(q(T-s))}\,b - \frac{\sinh(q(T-t))}{\sinh(q(T-s))}\,x\right)^2}{2\sigma(s,t)}\right\}$$

for all $0 \leqslant s < t < T$ and $x, y \in \mathbb{R}$, where $\sigma(s,t)$ is given by (1.7).

Theorem 3.2 has the following consequence.

**3.4 Remark.** Note that in case of $q(t) = q \neq 0$, $\sigma(t) = \sigma \neq 0$ and $r(t) = 0$, $t \geqslant 0$, the SDE (3.9) has the form

$$
\begin{cases} \mathrm{d}U_t = q\left(-\coth(q(T-t))\,U_t + \frac{b}{\sinh(q(T-t))}\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t, & t \in [0,T), \\ U_0 = a. \end{cases} \tag{3.13}
$$

Note also that both the SDE (3.13) and the integral representation (3.12) are invariant under a change of sign for the parameter $q$. Hence the Ornstein-Uhlenbeck bridges derived from the SDE (1.1) with $q$ and $-q$ are (almost surely) pathwise identical.

Theorem 3.3 has the following consequence.

**3.5 Remark.** We consider a special case of Theorem 3.3, namely, let us suppose that $r(t) = 0$, $t \geqslant 0$, and that there exist real numbers $q \neq 0$ and $\sigma \neq 0$ such that $q(t) = q$, $t \geqslant 0$, and $\sigma(t) = \sigma$, $t \geqslant 0$. Then $\bar{q}(t) = qt$, $t \geqslant 0$, and

$$
\begin{aligned}
\widetilde{R}(s,t) = \gamma(s,t)\mathrm{e}^{\bar{q}(s)-\bar{q}(t)} &= \sigma^2\mathrm{e}^{q(s-t)}\int_s^t \mathrm{e}^{2q(t-u)}\,\mathrm{d}u = \sigma^2\mathrm{e}^{q(s-t)}\frac{1}{2q}(\mathrm{e}^{2q(t-s)}-1) \\
&= \frac{\sigma^2}{2q}(\mathrm{e}^{q(t-s)} - \mathrm{e}^{-q(t-s)}) = \frac{\sigma^2}{q}\sinh(q(t-s)), \quad 0 \leqslant s \leqslant t \leqslant T,
\end{aligned} \tag{3.14}
$$

and

$$
R(s,t) = \operatorname{Cov}(Z_s, Z_t) = \frac{\sigma^2}{2q}\mathrm{e}^{q(s+t)}(1-\mathrm{e}^{-2qs}) = \frac{\sigma^2}{q}\mathrm{e}^{qt}\sinh(qs), \quad 0 \leqslant s \leqslant t.
$$

An easy calculation shows that for all $t \in [0,T]$,

$$
\begin{aligned}
\frac{\widetilde{R}(0,t)}{\widetilde{R}(0,T)} &= \mathrm{e}^{q(T-t)}\frac{R(t,t)}{R(T,T)} = \frac{R(t,T)}{R(T,T)}, \\
\frac{\widetilde{R}(t,T)}{\widetilde{R}(0,T)} &= \mathrm{e}^{qt} - \mathrm{e}^{2qT-qt}\frac{R(t,t)}{R(T,T)} = \frac{\sigma^2}{2q}\frac{\mathrm{e}^{qt+2qT}(1-\mathrm{e}^{-2qT}) - \mathrm{e}^{2qT+qt}(1-\mathrm{e}^{-2qt})}{R(T,T)} \\
&= \frac{\sigma^2}{2q}\frac{\mathrm{e}^{2qT-qt} - \mathrm{e}^{qt}}{R(T,T)} = \frac{R(T-t,T)}{R(T,T)}.
\end{aligned}
$$

Hence the process $(Y_t)_{t\in[0,T]}$ introduced in Theorem 3.3 (with our special choices of $q$, $r$ and $\sigma$) has the form

$$
Y_t = a\frac{R(T-t,T)}{R(T,T)} + b\frac{R(t,T)}{R(T,T)} + \left(Z_t - \frac{R(t,T)}{R(T,T)}Z_T\right), \quad t \in [0,T].
$$

Moreover, by (3.14),

$$
Y_t = a\frac{\sinh(q(T-t))}{\sinh(qT)} + b\frac{\sinh(qt)}{\sinh(qT)} + \left(Z_t - \frac{\sinh(qt)}{\sinh(qT)}Z_T\right), \quad t \in [0,T]. \tag{3.15}
$$

Finally, we remark that in case of $q(t) = q \neq 0$, $\sigma(t) = \sigma \neq 0$, $t \geqslant 0$ and $r(t) = 0$, $t \geqslant 0$ with the special choices $q = -\sqrt{k\gamma}/2$ and $\sigma = k/4$, where $k > 0$ and $\gamma > 0$, our Theorem 3.3 is the same as Lemma 1 in Papież and Sandison [27].

In the next remark we discuss the connections between Propositions 4 and 9 in Gasbarra, Sottinen and Valkeila [15] and our Theorems 3.3 and 3.1 (anticipative and integral representation of the bridge in case of dimension one), respectively.

**3.6 Remark.** It turns out that our Theorem 3.3 can be considered as a consequence of Proposition 4 in Gasbarra, Sottinen and Valkeila [15]. Namely, by Theorem 3.3, the process $(Y_t)_{t\in[0,T]}$ given by (3.11) equals in law the one-dimensional linear process bridge from $a$ to $b$ over $[0,T]$ derived from the process $Z$ given by the SDE (3.1) with initial condition $Z_0 = 0$. By Proposition 4 in Gasbarra, Sottinen and Valkeila [15], if $(Z^*_t)_{t\geqslant 0}$ is given by the SDE

$$\begin{cases} \mathrm{d}Z^*_t = (q(t)Z^*_t + r(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, & t \geqslant 0, \\ Z^*_0 = a, \end{cases}$$

then the bridge form $a$ to $b$ over $[0,T]$ derived from $Z^*$ (defined as a Gauss process in the sense of Definition 2 in Gasbarra, Sottinen and Valkeila [15]) admits the representation

$$b\frac{R^*(t,T)}{R^*(T,T)} + \left(Z^*_t - \frac{R^*(t,T)}{R^*(T,T)} Z^*_T\right), \qquad t \in [0,T],$$

where $R^*(s,t) := \operatorname{Cov}(Z^*_s, Z^*_t,)\ s,t \geqslant 0,$ is the covariance function of $Z^*$. Since $Z^*_t = \mathrm{e}^{\bar{q}(t)}a + Z_t$, $t \geqslant 0$, and $R^*(s,t) = R(s,t)$, $s,t \geqslant 0$ (where $R$ denotes the covariance function of $Z$), we have for all $t \in [0,T]$,

$$\begin{aligned} & b\frac{R^*(t,T)}{R^*(T,T)} + Z^*_t - \frac{R^*(t,T)}{R^*(T,T)} Z^*_T \\ &\qquad = a\left(\mathrm{e}^{\bar{q}(t)} - \frac{R(t,T)}{R(T,T)}\mathrm{e}^{\bar{q}(T)}\right) + b\frac{R(t,T)}{R(T,T)} + \left(Z_t - \frac{R(t,T)}{R(T,T)} Z_T\right), \end{aligned}$$

as desired. But we emphasize that the definition of the bridge in Definition 2 in Gasbarra, Sottinen and Valkeila [15] is via a properly defined conditional probability measure on the given probability space, and hence it differs from our definition of a bridge. Hence in this respect our Theorem 3.3 is not an immediate consequence of Proposition 4 in Gasbarra, Sottinen and Valkeila [15].

Further, if $(Z_t)_{t\geqslant 0}$ is given by the SDE (3.1) with $r(t) = 0$, $t \geqslant 0$, and with initial condition $Z_0 = 0$, then the process

$$M_t := a + \mathrm{e}^{-\bar{q}(t)} Z_t = a + \int_0^t \mathrm{e}^{-\bar{q}(s)}\sigma(s)\,\mathrm{d}B_s, \quad t \geqslant 0, \tag{3.16}$$

is a continuous Gauss martingale (with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ introduced in Section 2). By Proposition 9 in Gasbarra, Sottinen and Valkeila [15], the anticipative representation of the bridge from $a$ to $\mathrm{e}^{-\bar{q}(T)}b$ over $[0,T]$ derived from $M$ is given by

$$\widetilde{Y}_t := \mathrm{e}^{-\bar{q}(T)}b\,\frac{\langle M\rangle_t}{\langle M\rangle_T} + \left(M_t - \frac{\langle M\rangle_t}{\langle M\rangle_T}\,M_T\right), \quad t \in [0,T], \tag{3.17}$$

where $\langle M\rangle$ denotes the quadratic variation process of $M$. Since

$$\langle M\rangle_t = \int_0^t \mathrm{e}^{-2\bar{q}(s)}\sigma^2(s)\ \mathrm{d}\langle B\rangle_s = \int_0^t \mathrm{e}^{-2\bar{q}(s)}\sigma^2(s)\ \mathrm{d}s = \mathrm{e}^{-2\bar{q}(t)}\gamma(0,t), \qquad t \in [0,T], \tag{3.18}$$

(3.17) yields for $t \in [0,T]$,

$$\widetilde{Y}_t = \mathrm{e}^{\bar{q}(T)-2\bar{q}(t)} b \frac{\gamma(0,t)}{\gamma(0,T)} + \mathrm{e}^{-\bar{q}(t)} Z_t - \mathrm{e}^{\bar{q}(T)-2\bar{q}(t)} \frac{\gamma(0,t)}{\gamma(0,T)} Z_T + a \left( 1 - \mathrm{e}^{2(\bar{q}(T)-\bar{q}(t))} \frac{\gamma(0,t)}{\gamma(0,T)} \right). \tag{3.19}$$

This implies that $(\mathrm{e}^{\bar{q}(t)}\widetilde{Y}_t)_{t\in[0,T]}$ coincides with (3.10), the anticipative representation of the bridge from $a$ to $b$ over $[0,T]$ derived from $Z$.

Moreover, motivated by equation (3.2) in Proposition 9 of Gasbarra, Sottinen and Valkeila [15] (in case of $Z_0 = 0$ and $r(t) = 0$, $t \geqslant 0$) a non-anticipative (integral) representation of the bridge from $a$ to $\mathrm{e}^{-\bar{q}(T)}b$ over $[0,T]$ derived from $M$ is given by

$$\widetilde{U}_t := a + \left(\mathrm{e}^{-\bar{q}(T)}b - a\right) \frac{\langle M\rangle_t}{\langle M\rangle_T} + \int_0^t \frac{\langle M\rangle_{T,t}}{\langle M\rangle_{T,s}} \,\mathrm{d}M_s, \quad t \in [0,T], \tag{3.20}$$

where, similarly to (3.18), $\langle M\rangle_{T,t}$, $t \in [0,T]$, is given by

$$\langle M\rangle_{T,t} := \langle M\rangle_T - \langle M\rangle_t = \int_t^T \mathrm{e}^{-2\bar{q}(s)}\sigma^2(s)\,\mathrm{d}\langle B\rangle_s = \int_t^T \mathrm{e}^{-2\bar{q}(s)}\sigma^2(s)\,\mathrm{d}s = \mathrm{e}^{-2\bar{q}(T)}\gamma(t,T).$$

Hence (3.20), (3.16) and (3.18) yield for $t \in [0,T]$

$$\widetilde{U}_t = a \left( 1 - \mathrm{e}^{2(\bar{q}(T)-\bar{q}(t))} \frac{\gamma(0,t)}{\gamma(0,T)} \right) + \mathrm{e}^{\bar{q}(T)-2\bar{q}(t)} b \frac{\gamma(0,t)}{\gamma(0,T)} + \int_0^t \frac{\gamma(t,T)}{\gamma(s,T)} \mathrm{e}^{-\bar{q}(s)}\sigma(s)\,\mathrm{d}B_s,$$

which implies that $(\mathrm{e}^{\bar{q}(t)}\widetilde{U}_t)_{t\in[0,T]}$ coincides with (3.8), the integral representation of the bridge from $a$ to $b$ over $[0,T]$ derived from $Z$, since from the definition of $(\gamma(s,t))_{0\leqslant s\leqslant t}$ one can easily derive that

$$\mathrm{e}^{\bar{q}(t)} - \mathrm{e}^{2\bar{q}(T)-\bar{q}(t)} \frac{\gamma(0,t)}{\gamma(0,T)} = \mathrm{e}^{\bar{q}(t)} \frac{\gamma(t,T)}{\gamma(0,T)}, \qquad t \in [0,T).$$

# 4 Appendix

First we give sufficient conditions for positive definiteness of the Kalman matrices introduced in Section 2, see, e.g., Theorems 7.7.1-7.7.3 in Conti [9].

**4.1 Proposition.** *Let $0 \leqslant s < t$ be given. Then $\kappa(s,t)$ is positive definite if one of the following conditions is satisfied:*

*(a) there exists $t_0 \in (s,t)$ such that $\Sigma(t_0)$ has full rank $d$ (for which $p \geqslant d$ is required).*

*(b) there exist $t_0 \in (s,t)$, an open neighborhood $I_0$ around $t_0$ and some $k \in \mathbb{N}$ such that $\Sigma \in \mathcal{C}^{(k)}_{d\times p}(I_0)$, $Q \in \mathcal{C}^{(k-1)}_{d\times d}(I_0)$ and the controllability matrix $\left[\Sigma(t_0), \Delta\Sigma(t_0), \ldots, \Delta^k\Sigma(t_0)\right]$ has full rank $d$, where $\Delta$ is the operator $\Delta\Sigma(t) = \Sigma'(t) - Q(t)\Sigma(t)$, $t \in I_0$ and for all $n, m \in \mathbb{N}$, $\mathcal{C}^{(k)}_{n\times m}(I_0)$ denotes the set of $k$-times continuously differentiable functions on $I_0$ with values in $\mathbb{R}^{n\times m}$ (for which $(k+1)p \geqslant d$ is required).*

*(c) there exist $t_0 \in (s,t)$, an open neighborhood $I_0$ around $t_0$ and some $k \in \mathbb{N}$ such that $\Sigma \in \mathcal{C}^{(\infty)}_{d\times p}(I_0)$, $Q \in \mathcal{C}^{(\infty)}_{d\times d}(I_0)$ and the controllability matrix $\left[\Sigma(t_0), \Delta\Sigma(t_0), \ldots, \Delta^k\Sigma(t_0)\right]$ has full rank $d$, where $\mathcal{C}^{(\infty)}_{n\times m}(I_0)$ denotes the set of infinitely differentiable functions on $I_0$ with values in $\mathbb{R}^{n\times m}$ (for which $(k+1)p \geqslant d$ is required).*

Next we present two lemmata which will be used several times in the proofs later on.

**4.1 Lemma.** *Let us suppose that condition* (2.7) *holds. For fixed $T > 0$ and all $0 \leqslant s < t < T$,*

$$\Sigma(s,t)^{-1} = \kappa(s,t)^{-1} + E(T,t)^\top \kappa(t,T)^{-1} E(T,t). \tag{4.1}$$

*Especially, $\Sigma(s,t)$ is symmetric and positive definite for all $0 \leqslant s < t < T$.*

**Proof.** By assumption (2.7), $\kappa(s,t)$ is symmetric and positive definite for all $0 \leqslant s < t$, which implies that the right-hand side of (4.1), and thus also its inverse, is symmetric and positive definite. For $0 \leqslant s < t < T$ we calculate

$$\begin{aligned}
&\kappa(s,t)^{-1} + E(T,t)^\top \kappa(t,T)^{-1} E(T,t) \\
&\quad = \kappa(s,t)^{-1} \left( I_d + \kappa(s,t) E(T,t)^\top \kappa(t,T)^{-1} E(T,t) \right) \\
&\quad = \kappa(s,t)^{-1} \Bigg( I_d + \int_s^t E(t,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top \, \mathrm{d}u \\
&\qquad\qquad\qquad \times \left( \int_t^T E(t,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top \, \mathrm{d}u \right)^{-1} \Bigg) \\
&\quad = \kappa(s,t)^{-1} \int_s^T E(t,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top \, \mathrm{d}u \, \Gamma(t,T)^{-1} \\
&\quad = \Gamma(s,t)^{-1}\Gamma(s,T)\Gamma(t,T)^{-1} = \Sigma(s,t)^{-1},
\end{aligned}$$

which concludes the proof. □

**4.2 Lemma.** *Let us suppose that condition* (2.7) *holds. For fixed $T > 0$ and all $0 \leqslant s < t < T$ we have*

*(a)* $\kappa(t,T)^{-1} - \kappa(s,T)^{-1} = \Gamma(s,T)^{-1}\Gamma(s,t)\big(\Gamma(t,T)^\top\big)^{-1}$,

*(b)* $\Sigma(s,t) = \Gamma(t,T) \int_s^t \Gamma(u,T)^{-1}\Sigma(u)\Sigma(u)^\top \big(\Gamma(u,T)^\top\big)^{-1} \, \mathrm{d}u \, \Gamma(t,T)^\top$,

*(c)* $E(t,0) - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1} E(T,0) = \Gamma(t,T)\Gamma(0,T)^{-1}$,

*(d)* $\Gamma(s,T)^\top E(t,s)^\top - E(T,s)\Gamma(s,t) = \Gamma(t,T)^\top$.

**Proof.** Since $\kappa(t,T)$ is symmetric we calculate

$$\begin{aligned}
\kappa(t,T)^{-1} - \kappa(s,T)^{-1} &= \big(\kappa(t,T)^\top\big)^{-1} - \kappa(s,T)^{-1} \\
&= E(t,T)^\top \big(\Gamma(t,T)^\top\big)^{-1} - \Gamma(s,T)^{-1} E(s,T) \\
&= \Gamma(s,T)^{-1} \left( \Gamma(s,T) E(t,T)^\top - E(s,T)\Gamma(t,T)^\top \right) \left( \Gamma(t,T)^\top \right)^{-1},
\end{aligned}$$

where the middle factor coincides with

$$\int_s^T E(s,u)\Sigma(u)\Sigma(u)^\top E(t,u)^\top \, \mathrm{d}u - \int_t^T E(s,u)\Sigma(u)\Sigma(u)^\top E(t,u)^\top \, \mathrm{d}u = \Gamma(s,t),$$

which proves (a). In a similar manner one can prove (c) and (d). To prove (b), note that the function $(0,T) \ni u \mapsto \kappa(u,T) \in \mathbb{R}^{d\times d}$ is a differentiable curve in the set of symmetric and positive definite $(d\times d)$-matrices with

$$-\partial_1\kappa(u,T) = -\frac{\mathrm{d}}{\mathrm{d}u}\int_u^T E(T,v)\Sigma(v)\Sigma(v)^\top E(T,v)^\top \mathrm{d}v = E(T,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top,\ u\in(0,T).$$

Using (a) and the fact that $\partial_1\big(\kappa(u,T)^{-1}\big) = -\kappa(u,T)^{-1}\big(\partial_1\kappa(u,T)\big)\kappa(u,T)^{-1}$, $u\in(0,T)$, see, e.g., formula (3.2) on page 73 in Baker [2], we calculate

$$\begin{aligned}
&\int_s^t \Gamma(u,T)^{-1}\Sigma(u)\Sigma(u)^\top\big(\Gamma(u,T)^\top\big)^{-1}\,\mathrm{d}u\\
&\quad= \int_s^t \kappa(u,T)^{-1}E(T,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top\kappa(u,T)^{-1}\,\mathrm{d}u\\
&\quad= -\int_s^t \kappa(u,T)^{-1}\big(\partial_1\kappa(u,T)\big)\kappa(u,T)^{-1}\,\mathrm{d}u\\
&\quad= \kappa(t,T)^{-1}-\kappa(s,T)^{-1} = \Gamma(s,T)^{-1}\Gamma(s,t)\big(\Gamma(t,T)^\top\big)^{-1},
\end{aligned}$$

from which the assertion of part (b) easily follows. □

**Proof of Lemma 2.1.** Since $\det\Sigma(s,t) = \det\kappa(s,t)\det\kappa(t,T)\big(\det\kappa(s,T)\big)^{-1}$, for $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$ and $0\leqslant s<t<T$ we obtain, by (2.8),

$$\frac{p^{\mathbf{Z}}_{s,t}(\mathbf{x},\mathbf{y})\,p^{\mathbf{Z}}_{t,T}(\mathbf{y},\mathbf{b})}{p^{\mathbf{Z}}_{s,T}(\mathbf{x},\mathbf{b})} = \frac{1}{\sqrt{(2\pi)^d\det\Sigma(s,t)}}\exp\left\{-\frac{1}{2}\psi_{s,t}(\mathbf{x},\mathbf{y})\right\},$$

where using (4.1) and that

$$\mathbf{b}-\mathbf{m}_{\mathbf{y}}(t,T) = -E(T,t)\mathbf{y}+\mathbf{m}^-_{\mathbf{b}}(t,T),$$

we calculate

$$\begin{aligned}
\psi_{s,t}(\mathbf{x},\mathbf{y}) &= \big\langle\kappa(s,t)^{-1}\big(\mathbf{y}-\mathbf{m}_{\mathbf{x}}(s,t)\big),\mathbf{y}-\mathbf{m}_{\mathbf{x}}(s,t)\big\rangle\\
&\quad+\big\langle\kappa(t,T)^{-1}\big(\mathbf{b}-\mathbf{m}_{\mathbf{y}}(t,T)\big),\mathbf{b}-\mathbf{m}_{\mathbf{y}}(t,T)\big\rangle\\
&\quad-\big\langle\kappa(s,T)^{-1}\big(\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big),\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big\rangle\\
&= \big\langle\big(\kappa(s,t)^{-1}+E(T,t)^\top\kappa(t,T)^{-1}E(T,t)\big)\,\mathbf{y},\mathbf{y}\big\rangle\\
&\quad+\big\langle\kappa(s,t)^{-1}\mathbf{m}_{\mathbf{x}}(s,t),\mathbf{m}_{\mathbf{x}}(s,t)\big\rangle-2\big\langle\kappa(s,t)^{-1}\mathbf{y},\mathbf{m}_{\mathbf{x}}(s,t)\big\rangle\\
&\quad+\big\langle\kappa(t,T)^{-1}\mathbf{m}^-_{\mathbf{b}}(t,T),\mathbf{m}^-_{\mathbf{b}}(t,T)\big\rangle-2\big\langle\kappa(t,T)^{-1}E(T,t)\mathbf{y},\mathbf{m}^-_{\mathbf{b}}(t,T)\big\rangle\\
&\quad-\big\langle\kappa(s,T)^{-1}\big(\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big),\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big\rangle\\
&= \big\langle\Sigma(s,t)^{-1}\mathbf{y},\mathbf{y}\big\rangle\\
&\quad-2\big\langle\Sigma(s,t)^{-1}\mathbf{y},\Sigma(s,t)\kappa(s,t)^{-1}\mathbf{m}_{\mathbf{x}}(s,t)+\Sigma(s,t)E(T,t)^\top\kappa(t,T)^{-1}\mathbf{m}^-_{\mathbf{b}}(t,T)\big\rangle\\
&\quad+\big\langle\kappa(s,t)^{-1}\mathbf{m}_{\mathbf{x}}(s,t),\mathbf{m}_{\mathbf{x}}(s,t)\big\rangle+\big\langle\kappa(t,T)^{-1}\mathbf{m}^-_{\mathbf{b}}(t,T),\mathbf{m}^-_{\mathbf{b}}(t,T)\big\rangle\\
&\quad-\big\langle\kappa(s,T)^{-1}\big(\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big),\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big\rangle.
\end{aligned}$$

Since

$$\begin{aligned}\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t) &= \Gamma(t,T)\Gamma(s,T)^{-1}\mathbf{m}_{\mathbf{x}}^{+}(s,t) + \Sigma(s,t)^{\top}\big(\Gamma(t,T)^{\top}\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\ &= \Sigma(s,t)\kappa(s,t)^{-1}\mathbf{m}_{\mathbf{x}}(s,t) + \Sigma(s,t)E(T,t)^{\top}\kappa(t,T)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T),\end{aligned}$$

we have

$$\begin{aligned}\psi_{s,t}(\mathbf{x},\mathbf{y}) =& \left\langle \Sigma(s,t)^{-1}\mathbf{y},\mathbf{y}\right\rangle - 2\left\langle \Sigma(s,t)^{-1}\mathbf{y},\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\right\rangle + \left\langle \kappa(s,t)^{-1}\mathbf{m}_{\mathbf{x}}(s,t),\mathbf{m}_{\mathbf{x}}(s,t)\right\rangle\\ &+ \left\langle \kappa(t,T)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T),\mathbf{m}_{\mathbf{b}}^{-}(t,T)\right\rangle - \left\langle \kappa(s,T)^{-1}\big(\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big),\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\right\rangle.\end{aligned}$$

Hence, in order to show that

$$\psi_{s,t}(\mathbf{x},\mathbf{y}) = \left\langle \Sigma(s,t)^{-1}\big(\mathbf{y}-\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\big),\mathbf{y}-\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\right\rangle,$$

it remains to prove that

$$\begin{aligned}&\left\langle \Sigma(s,t)^{-1}\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t),\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)\right\rangle\\ &\quad= \left\langle \kappa(s,t)^{-1}\mathbf{m}_{\mathbf{x}}(s,t),\mathbf{m}_{\mathbf{x}}(s,t)\right\rangle + \left\langle \kappa(t,T)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T),\mathbf{m}_{\mathbf{b}}^{-}(t,T)\right\rangle\\ &\quad\quad - \left\langle \kappa(s,T)^{-1}\big(\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\big),\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T)\right\rangle.\end{aligned}$$

The right-hand side of the above equation, due to

$$\mathbf{b}-\mathbf{m}_{\mathbf{x}}(s,T) = \mathbf{b} - E(T,s)\mathbf{x} - \int_s^T E(T,u)\mathbf{r}(u)\,\mathrm{d}u = \mathbf{m}_{\mathbf{b}}^{-}(t,T) - E(T,s)\mathbf{m}_{\mathbf{x}}^{+}(s,t), \tag{4.2}$$

coincides with

$$\begin{aligned}&\left\langle \big(E(t,s)^{\top}\Gamma(s,t)^{-1} - E(T,s)^{\top}\Gamma(s,T)^{-1}\big)\mathbf{m}_{\mathbf{x}}^{+}(s,t),\mathbf{m}_{\mathbf{x}}^{+}(s,t)\right\rangle\\ &\quad+ \left\langle \big(\kappa(t,T)^{-1}-\kappa(s,T)^{-1}\big)\mathbf{m}_{\mathbf{b}}^{-}(t,T),\mathbf{m}_{\mathbf{b}}^{-}(t,T)\right\rangle + 2\left\langle \Gamma(s,T)^{-1}\mathbf{m}_{\mathbf{x}}^{+}(s,t),\mathbf{m}_{\mathbf{b}}^{-}(t,T)\right\rangle.\end{aligned}$$

Hence it remains to show the following three identities:

$$\begin{aligned}&E(t,s)^{\top}\Gamma(s,t)^{-1} - E(T,s)^{\top}\Gamma(s,T)^{-1}\\ &\quad= \big(\Gamma(s,T)^{\top}\big)^{-1}\Gamma(t,T)^{\top}\Sigma(s,t)^{-1}\Gamma(t,T)\Gamma(s,T)^{-1}\end{aligned} \tag{4.3}$$

$$\kappa(t,T)^{-1} - \kappa(s,T)^{-1} = \Gamma(s,T)^{-1}\Gamma(s,t)\Sigma(s,t)^{-1}\Gamma(s,t)^{\top}\big(\Gamma(s,T)^{\top}\big)^{-1}, \tag{4.4}$$

$$\Gamma(s,T)^{-1} = \Gamma(s,T)^{-1}\Gamma(s,t)\Sigma(s,t)^{-1}\Gamma(t,T)\Gamma(s,T)^{-1}. \tag{4.5}$$

The validity of (4.5) is obvious, by the definition of $\Sigma(s,t)$. Using again the definition of $\Sigma(s,t)$, the right-hand side of (4.4) coincides with

$$\begin{aligned}\Gamma(t,T)^{-1}\Gamma(s,t)^{\top}\big(\Gamma(s,T)^{\top}\big)^{-1} &= \Gamma(t,T)^{-1}\Gamma(s,t)^{\top}\big(\Gamma(s,T)^{\top}\big)^{-1}\\ &= \big(\kappa(t,T)^{-1}-\kappa(s,T)^{-1}\big)^{\top} = \kappa(t,T)^{-1}-\kappa(s,T)^{-1},\end{aligned}$$

where the last but one equality follows from part (a) of Lemma 4.2. Moreover, multiplying (4.3) with $\Gamma(s,t)^\top$ from the left and $\Gamma(s,T)$ from the right, we easily obtain that (4.3) is equivalent to

$$E(t,s)\Gamma(s,T) - \Gamma(s,t)^\top E(T,s)^\top = \Gamma(t,T),$$

and this equality holds true, by part (d) of Lemma 4.2. □

For proving almost surely continuity of the linear process bridge at the endpoint $T$, we recall a strong law of large numbers for continuous square integrable multivariate martingales with deterministic quadratic variation process due to Dzhaparidze and Spreij [13, Corollary 2]; see also Koval' [24, Corollary 1]. We note that the above mentioned citations are about continuous square integrable martingales with time interval $[0,\infty)$, but they are also valid for continuous square integrable martingales with time interval $[0,T)$, $T \in (0,\infty)$, with appropriate modifications in the conditions, see as follows (the proof of Koval' [24, Corollary 1] can be easily formulated for the time interval $[0,T)$, $T\in(0,\infty)$).

**4.1 Theorem.** *Let $T \in (0,\infty]$ be fixed and let $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\in[0,T)},P)$ be a filtered probability space satisfying the usual conditions, i.e., $(\Omega,\mathcal{G},P)$ is complete, the filtration $(\mathcal{G}_t)_{t\in[0,T)}$ is right continuous, $\mathcal{G}_0$ contains all the $P$-null sets in $\mathcal{G}$ and $\mathcal{G}_{T-}=\mathcal{G}$, where $\mathcal{G}_{T-} := \sigma\left(\bigcup_{t\in[0,T)}\mathcal{G}_t\right)$. Let $(\mathbf{M}_t)_{t\in[0,T)}$ be an $\mathbb{R}^d$-valued continuous square integrable martingale with respect to the filtration $(\mathcal{G}_t)_{t\in[0,T)}$ such that $P(\mathbf{M}_0=\mathbf{0})=1$. (The square integrability means that $\mathsf{E}(m_t^{(i)})^2<\infty$, $t\in[0,T)$, $i=1,\dots,d$, where $\mathbf{M}_t := (m_t^{(1)},\dots,m_t^{(d)})$, $t\in[0,T)$.) Further, we assume that the quadratic variation process $(\langle\mathbf{M}\rangle_t)_{t\in[0,T)}$ is deterministic (which yields that $(\langle\mathbf{M}\rangle_t)_{i,j}=\mathsf{E}(m_t^{(i)}m_t^{(j)})$, $t\in[0,T)$, $i,j=1,\dots,d$). If there exists some $t_0\in[0,T)$ such that $\langle\mathbf{M}\rangle_{t_0}$ is positive definite and $\lim_{t\uparrow T}\langle\mathbf{M}\rangle_t^{-1}=0\in\mathbb{R}^{d\times d}$, then $P(\lim_{t\uparrow T}\langle\mathbf{M}\rangle_t^{-1}\mathbf{M}_t=\mathbf{0})=1$.*

Using Theorem 4.1 we formulate an auxiliary result which we will use for proving almost surely continuity of the linear process bridge at the endpoint $T$, see the proof of Theorem 2.1.

**4.3 Lemma.** *Let us assume that condition (2.7) holds. Let $T\in(0,\infty)$ be fixed and let $(\mathbf{B}_t)_{t\geqslant0}$ be an $p$-dimensional standard Wiener process on a filtered probability space $(\Omega,\mathcal{A},(\mathcal{A}_t)_{t\in[0,T)},P)$ satisfying the usual conditions, constructed by the help of the standard Wiener process $\mathbf{B}$ (see, e.g., Karatzas and Shreve [22, Section 5.2.A]). The process $(\mathbf{S}_t)_{t\in[0,T]}$ defined by*

$$\mathbf{S}_t := \begin{cases} \Gamma(t,T)\int_0^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u & \text{if } t\in[0,T),\\ \mathbf{0} & \text{if } t=T,\end{cases} \tag{4.6}$$

*is a centered Gauss process with almost surely continuous paths.*

**Proof.** In (4.6) the stochastic integral can be understood as a usual stochastic integral see, e.g., Ash and Gardner [1, Theorem 5.1.4]. For this we only have to check that for all $0\leqslant t<T$,

$$\int_0^t \left((\Gamma(u,T)^{-1}\Sigma(u))_{i,j}\right)^2\mathrm{d}u<\infty, \quad i=1,\dots,d,\ \ j=1,\dots,p,$$

which follows directly from part (b) of Lemma 4.2. Due to the fact that the integrand in the stochastic integral of (4.6) is deterministic, by Bauer [6, Lemma 48.2], $(\mathbf{S}_t)_{t\in[0,T]}$ is a centered

Gauss process. To prove almost sure continuity, we follow the method of the proof of Lemma 5.6.9 in Karatzas and Sherve [22]. For all $t \in [0,T)$, let

$$\mathbf{M}_t := \int_0^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u = \begin{pmatrix} \sum_{k=1}^p \int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{1,k}\,\mathrm{d}B_u^{(k)} \\ \vdots \\ \sum_{k=1}^p \int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{d,k}\,\mathrm{d}B_u^{(k)} \end{pmatrix},$$

where $\mathbf{B}_t := (B_t^{(1)}, \ldots, B_t^{(p)})$, $t \geqslant 0$. Then, by Proposition 3.2.10 in Karatzas and Shreve [22], $(\mathbf{M}_t)_{t\in[0,T)}$ is a continuous, square integrable martingale with respect to the filtration $(\mathcal{A}_t)_{t\in[0,T)}$ and with quadratic variation process $(\langle \mathbf{M}\rangle_t)_{t\in[0,T)}$,

$$\begin{aligned}
(\langle \mathbf{M}\rangle_t)_{i,j} &= \mathsf{E}\left(\sum_{k=1}^p \int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{i,k}\,\mathrm{d}B_u^{(k)} \sum_{\ell=1}^p \int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{j,\ell}\,\mathrm{d}B_u^{(\ell)}\right) \\
&= \sum_{k=1}^p \mathsf{E}\left(\int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{i,k}\,\mathrm{d}B_u^{(k)} \int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{j,k}\,\mathrm{d}B_u^{(k)}\right) \\
&= \sum_{k=1}^p \int_0^t (\Gamma(u,T)^{-1}\Sigma(u))_{i,k}(\Gamma(u,T)^{-1}\Sigma(u))_{j,k}\,\mathrm{d}u \\
&= \int_0^t \left(\Gamma(u,T)^{-1}\Sigma(u)\Sigma(u)^\top(\Gamma(u,T)^{-1})^\top\right)_{i,j}\mathrm{d}u \\
&= \left(\Gamma(t,T)^{-1}\Sigma(0,t)(\Gamma(t,T)^\top)^{-1}\right)_{i,j}, \qquad t\in[0,T),\ \ 1\leqslant i,j\leqslant d,
\end{aligned}$$

where the last equality follows by part (b) of Lemma 4.2. Hence

$$\langle \mathbf{M}\rangle_t = \Gamma(t,T)^{-1}\Sigma(0,t)(\Gamma(t,T)^\top)^{-1}, \qquad t\in[0,T),$$

which shows that $\langle \mathbf{M}\rangle_t$ is symmetric and positive definite (and hence invertible) for all $t\in(0,T)$, by Lemma 4.1.

Now we check that $\lim_{t\uparrow T}\langle \mathbf{M}\rangle_t^{-1} = 0 \in \mathbb{R}^{d\times d}$. By the definition of $\Sigma(0,t)$,

$$\langle \mathbf{M}\rangle_t = \Gamma(0,T)^{-1}\Gamma(0,t)(\Gamma(t,T)^\top)^{-1}, \qquad t\in[0,T), \tag{4.7}$$

which yields that

$$\begin{aligned}
&\langle \mathbf{M}\rangle_t^{-1} = \Gamma(t,T)^\top\Gamma(0,t)^{-1}\Gamma(0,T) \\
&= \left(\int_t^T E(t,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top\,\mathrm{d}u\right)^\top \left(\int_0^t E(0,u)\Sigma(u)\Sigma(u)^\top E(t,u)^\top\,\mathrm{d}u\right)^{-1}\Gamma(0,T) \\
&= \left(E(t,0)\int_t^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top\,\mathrm{d}u\,E(T,0)^\top\right)^\top \\
&\quad\times\left(\int_0^t E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top\,\mathrm{d}u\,E(t,0)^\top\right)^{-1}\Gamma(0,T) =
\end{aligned}$$

$$= E(T,0)\left(\int_t^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top \,\mathrm{d}u\right)^\top E(t,0)^\top (E(t,0)^\top)^{-1}$$

$$\times \left(\int_0^t E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top \,\mathrm{d}u\right)^{-1} \Gamma(0,T), \qquad t \in (0,T).$$

Hence

$$\lim_{t\uparrow T} \langle \mathbf{M}\rangle_t^{-1} = E(T,0)\cdot \mathbf{0} \cdot \left(\int_0^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top \,\mathrm{d}u\right)^{-1} \Gamma(0,T) = 0 \in \mathbb{R}^{d\times d},$$

where the inverse matrix

$$\left(\int_0^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top \,\mathrm{d}u\right)^{-1}$$

exists, since

$$\int_0^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top \,\mathrm{d}u = E(0,T)\int_0^T E(T,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top \,\mathrm{d}u \; E(0,T)^\top$$

$$= E(0,T)\kappa(0,T)E(0,T)^\top,$$

and $E(0,T)$ and $\kappa(0,T)$ are invertible matrices (using also assumption (2.7)).

By Theorem 4.1, we get $P(\lim_{t\uparrow T}\langle \mathbf{M}\rangle_t^{-1}\mathbf{M}_t = \mathbf{0}) = 1$. Then

$$\mathbf{S}_t = \Gamma(t,T)\mathbf{M}_t = \Gamma(t,T)\langle \mathbf{M}\rangle_t \langle \mathbf{M}\rangle_t^{-1}\mathbf{M}_t, \qquad t\in(0,T).$$

By part (a) of Lemma 4.2,

$$\Gamma(0,T)^{-1}\Gamma(0,t)(\Gamma(t,T)^\top)^{-1} = \kappa(t,T)^{-1} - \kappa(0,T)^{-1}, \qquad 0 \leqslant t < T,$$

and hence, by (4.7),

$$\Gamma(t,T)\langle \mathbf{M}\rangle_t = \Gamma(t,T)\Gamma(0,T)^{-1}\Gamma(0,t)(\Gamma(t,T)^\top)^{-1} = \Gamma(t,T)(\kappa(t,T)^{-1} - \kappa(0,T)^{-1})$$

$$= E(t,T)\kappa(t,T)(\kappa(t,T)^{-1} - \kappa(0,T)^{-1}) = E(t,T) - E(t,T)\kappa(t,T)\kappa(0,T)^{-1}$$

$$= E(t,T) - E(t,T)\int_t^T E(T,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top \,\mathrm{d}u\, \kappa(0,T)^{-1}$$

$$= E(t,T) - E(t,T)E(T,0)\int_t^T E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top \,\mathrm{d}u\, E(T,0)^\top \kappa(0,T)^{-1},$$

which yields that $\lim_{t\uparrow T}\Gamma(t,T)\langle \mathbf{M}\rangle_t = I_d$. Hence we get $P(\lim_{t\uparrow T}\mathbf{S}_t = \mathbf{0}) = 1$. □

**Proof of Theorem 2.1.** Due to the fact that the integrand in the stochastic integral of (2.11) is deterministic, by Lemma 48.2 in Bauer [6], $(U_t)_{t\in[0,T)}$ is a Gauss process and the distribution of $\mathbf{U}_t$ is Gauss with mean $\mathsf{E}\mathbf{U}_t = \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t)$ and by part (b) of Lemma 4.2 we get

$$\operatorname{Cov}(\mathbf{U}_t,\mathbf{U}_t) = \Gamma(t,T)\int_0^t \Gamma(u,T)^{-1}\Sigma(u)\Sigma(u)^\top \big(\Gamma(u,T)^\top\big)^{-1} \,\mathrm{d}u\, \Gamma(t,T)^\top = \Sigma(0,t)$$

for all $t \in [0,T)$. Since the deterministic functions appearing in (2.11) are continuous on $[0,T)$, Theorem 5.1.5 in Ash and Gardner [1] implies that $(\mathbf{U}_t)_{t\in[0,T)}$ is stochastically and $L^2$-continuous. Moreover, since the covariance-function $[0,T) \ni t \mapsto \Sigma(0,t)$ is continuous with $\Sigma(0,0) = 0 \in \mathbb{R}^{d\times d}$ and $\Sigma(0,t) \to 0$ as $t \uparrow T$, the continuity theorem yields that $\mathbf{U}_t \to \mathbf{b} = \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)$ in distribution as $t \uparrow T$. Since the limit $\mathbf{b}$ is a constant, we get stochastic continuity of the extension $(\mathbf{U}_t)_{t\in[0,T]}$ with $\mathbf{U}_0 = \mathbf{a} = \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,0)$ and $\mathbf{U}_T = \mathbf{b} = \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)$. To prove the $L^2$-continuity it remains to check that $\mathbf{U}_t \to \mathbf{b}$ in $L^2$ as $t \uparrow T$. By part (b) of Lemma 4.2, for all $t \in [0,T)$ we get

$$
\begin{aligned}
\mathsf{E}\|\mathbf{U}_t - \mathbf{b}\|^2 &\leqslant \mathsf{E}\left[(\|\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)\| + \|\mathbf{U}_t - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t)\|)^2\right] \\
&\leqslant 2\|\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)\|^2 + 2\mathsf{E}\left\|\Gamma(t,T)\int_0^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u\right\|^2 \\
&= 2\|\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)\|^2 + 2\left(\sum_{i=1}^d\sum_{k=1}^p \int_0^t \Big(\Gamma(t,T)\Gamma(u,T)^{-1}\Sigma(u)\Big)_{i,k}^2\,\mathrm{d}u\right) \\
&= 2\|\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)\|^2 \\
&\quad + 2\,\mathrm{tr}\left(\int_0^t \Gamma(t,T)\Gamma(u,T)^{-1}\Sigma(u)\Sigma(u)^\top(\Gamma(u,T)^{-1})^\top\Gamma(t,T)^\top\,\mathrm{d}u\right) \\
&= 2\|\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)\|^2 + 2\,\mathrm{tr}(\Sigma(0,t)) \\
&= 2\|\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t) - \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,T)\|^2 + 2\,\mathrm{tr}\left(\Gamma(t,T)\Gamma(0,T)^{-1}\Gamma(0,t)\right) \to 0 \qquad \text{as } \ t\uparrow T,
\end{aligned}
$$

where $\mathrm{tr}(A)$ denotes the trace of a squared matrix $A$.

Further, Lemma 4.3 yields that $(\mathbf{U}_t)_{t\in[0,T]}$ is almost surely continuous.

Moreover, for all $0 \leqslant s \leqslant t < T$ we have

$$
\begin{aligned}
\mathbf{U}_t =&\Gamma(t,T)\Gamma(s,T)^{-1} \\
&\times\Bigg(\Gamma(s,T)\Gamma(0,T)^{-1}\mathbf{m}_{\mathbf{a}}^+(0,t) + \Gamma(s,T)\Gamma(t,T)^{-1}\Gamma(0,t)^\top\left(\Gamma(0,T)^\top\right)^{-1}\mathbf{m}_{\mathbf{b}}^-(t,T) \\
&\quad + \Gamma(s,T)\int_0^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u\Bigg) \\
=&\Gamma(t,T)\Gamma(s,T)^{-1}\Bigg(\mathbf{U}_s + \Gamma(s,T)\Gamma(0,T)^{-1}\int_s^t E(0,u)\mathbf{r}(u)\,\mathrm{d}u \\
&\qquad + \Gamma(0,s)^\top\left(\Gamma(0,T)^\top\right)^{-1}\int_s^t E(T,u)\mathbf{r}(u)\,\mathrm{d}u\Bigg) \\
&+ \left(\Gamma(0,t)^\top\left(\Gamma(0,T)^\top\right)^{-1} - \Gamma(t,T)\Gamma(s,T)^{-1}\Gamma(0,s)^\top\left(\Gamma(0,T)^\top\right)^{-1}\right)\mathbf{m}_{\mathbf{b}}^-(t,T) \\
&+ \Gamma(t,T)\int_s^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u,
\end{aligned}
$$

where for the derivation of the last equality we used that

$$\begin{aligned}
&\Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \mathbf{m}_{\mathbf{b}}^-(s,T) + \Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \int_s^t E(T,u)\mathbf{r}(u)\,\mathrm{d}u \\
&= \Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \left(\mathbf{b} - \int_s^T E(T,u)\mathbf{r}(u)\,\mathrm{d}u\right) + \Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \int_s^t E(T,u)\mathbf{r}(u)\,\mathrm{d}u \\
&= \Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \left(\mathbf{b} - \int_t^T E(T,u)\mathbf{r}(u)\,\mathrm{d}u\right) \\
&= \Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \mathbf{m}_{\mathbf{b}}^-(t,T).
\end{aligned}$$

Next we argue that for any $0 \leqslant s \leqslant u < t < T$ we have

$$\Gamma(s,T)\Gamma(0,T)^{-1}E(0,u) + \Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} E(T,u) = E(s,u). \tag{4.8}$$

Multiplying (4.8) by $E(u,T)\kappa(0,T)$ from the right, equivalently we have to show that

$$\Gamma(s,T) + \Gamma(0,s)^\top E(T,0)^\top = E(s,T)\kappa(0,T), \tag{4.9}$$

which holds by part (d) of Lemma 4.2. Furthermore, by part (a) of Lemma 4.2 and the symmetry of $\kappa^{-1}$, we have

$$\begin{aligned}
&\Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1} - \Gamma(t,T)\Gamma(s,T)^{-1}\Gamma(0,s)^\top \big(\Gamma(0,T)^\top\big)^{-1} \\
&\quad = \Gamma(t,T)\Big[\Big(\Gamma(0,T)^{-1}\Gamma(0,t)\big(\Gamma(t,T)^\top\big)^{-1}\Big)^\top \\
&\qquad\qquad - \Big(\Gamma(0,T)^{-1}\Gamma(0,s)\big(\Gamma(s,T)^\top\big)^{-1}\Big)^\top\Big] \\
&\quad = \Gamma(t,T)\Big[\big(\kappa(t,T)^{-1} - \kappa(0,T)^{-1}\big)^\top - \big(\kappa(s,T)^{-1} - \kappa(0,T)^{-1}\big)^\top\Big] \\
&\quad = \Gamma(t,T)\big(\kappa(t,T)^{-1} - \kappa(s,T)^{-1}\big) = \Gamma(s,t)^\top \big(\Gamma(s,T)^\top\big)^{-1}.
\end{aligned}$$

Putting all together, using (4.8) and the above, we get

$$\begin{aligned}
\mathbf{U}_t =& \Gamma(t,T)\Gamma(s,T)^{-1}\mathbf{m}_{\mathbf{U}_s}^+(s,t) + \Gamma(s,t)^\top \big(\Gamma(s,T)^\top\big)^{-1} \mathbf{m}_{\mathbf{b}}^-(t,T) \\
&+ \Gamma(t,T) \int_s^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u \\
=& \mathbf{n}_{\mathbf{U}_s,\mathbf{b}}(s,t) + \Gamma(t,T)\int_s^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u,
\end{aligned} \tag{4.10}$$

where the last integral is independent of $\mathbf{U}_s$ (by page 434 in Bauer [6]). Given $\mathbf{U}_s = \mathbf{x}$, the distribution of $\mathbf{U}_t$ does not depend on $(\mathbf{U}_r)_{r\in[0,s)}$ and hence $(\mathbf{U}_t)_{t\in[0,T]}$ is a Markov process. Moreover, for any $\mathbf{x} \in \mathbb{R}^d$ and $0 \leqslant s < t < T$ the conditional distribution of $\mathbf{U}_t$ given $\mathbf{U}_s = \mathbf{x}$ is Gauss with mean $\mathbf{n}_{\mathbf{x},\mathbf{b}}(s,t)$ and covariance matrix $\Sigma(s,t)$ by Lemma 4.2 (b), which coincides with the Gauss density given by Lemma 2.1. □

**Proof of Theorem 2.2.** Define the $d$-dimensional Ito-process $(\mathbf{V}_t)_{t\in[0,T)}$ by

$$\mathbf{V}_t := \int_0^t \Gamma(u,T)^{-1}\Sigma(u)\,\mathrm{d}\mathbf{B}_u, \qquad \text{i.e.} \qquad \mathrm{d}\mathbf{V}_t = \Gamma(t,T)^{-1}\Sigma(t)\,\mathrm{d}\mathbf{B}_t, \qquad t\in[0,T).$$

Further, let $F:[0,T)\times\mathbb{R}^d\to\mathbb{R}^d$ be given by $F(t,\mathbf{x}) := \mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t)+\Gamma(t,T)\mathbf{x}$, $t\in[0,T)$, $\mathbf{x}\in\mathbb{R}^d$. By (2.11), $\mathbf{U}_t = F(t,\mathbf{V}_t)$ for $t\in[0,T)$. Recall that for $0\leqslant s<t<T$ we have

$$\Gamma(s,t) = E(s,t)\int_s^t E(t,u)\Sigma(u)\Sigma(u)^\top E(t,u)^\top\,\mathrm{d}u = \int_s^t E(s,u)\Sigma(u)\Sigma(u)^\top E(s,u)^\top\,\mathrm{d}u\,E(t,s)^\top,$$

and hence using (2.5) we get

$$\begin{aligned}\partial_1\Gamma(t,T) =&Q(t)E(t,T)\int_t^T E(T,u)\Sigma(u)\Sigma(u)^\top E(T,u)^\top\,\mathrm{d}u\\ &- E(t,T)E(T,t)\Sigma(t)\Sigma(t)^\top E(T,t)^\top\\ =&Q(t)\Gamma(t,T)-\Sigma(t)\Sigma(t)^\top E(T,t)^\top,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\partial_2\Gamma(0,t) =&E(0,t)\Sigma(t)\Sigma(t)^\top E(0,t)^\top E(t,0)^\top\\ &+\int_0^t E(0,u)\Sigma(u)\Sigma(u)^\top E(0,u)^\top\,\mathrm{d}u\,E(t,0)^\top Q(t)^\top\\ =&\Gamma(0,t)Q(t)^\top + E(0,t)\Sigma(t)\Sigma(t)^\top.\end{aligned} \tag{4.12}$$

Further we calculate

$$\begin{aligned}\partial_1 F(t,\mathbf{x}) =&\big(\partial_1\Gamma(t,T)\big)\Gamma(0,T)^{-1}\mathbf{m}_{\mathbf{a}}^{+}(0,t)+\Gamma(t,T)\Gamma(0,T)^{-1}E(0,t)\mathbf{r}(t)\\ &+\big(\partial_2\Gamma(0,t)\big)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\ &+\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}E(T,t)\mathbf{r}(t)+\big(\partial_1\Gamma(t,T)\big)\mathbf{x},\end{aligned}$$

and $D_2F(t,\mathbf{x}) = \Gamma(t,T)$, independent of $\mathbf{x}\in\mathbb{R}^d$. Hence, by an application of the multivariate Ito-rule, we get

$$\mathrm{d}\mathbf{U}_t = \mathrm{d}F(t,\mathbf{V}_t) = \partial_1F(t,\mathbf{V}_t)\,\mathrm{d}t + D_2F(t,\mathbf{V}_t)\,\mathrm{d}\mathbf{V}_t = \partial_1F(t,\mathbf{V}_t)\,\mathrm{d}t+\Sigma(t)\,\mathrm{d}\mathbf{B}_t,$$

where, by (4.11) and (4.12), the coefficient of $\mathrm{d}t$ equals

$$\begin{aligned}\partial_1F(t,\mathbf{V}_t) =& \big(Q(t)-\Sigma(t)\Sigma(t)^\top E(T,t)^\top\Gamma(t,T)^{-1}\big)\mathbf{U}_t\\ &+\Sigma(t)\Sigma(t)^\top\Big(E(0,t)^\top\big(\Gamma(0,t)^\top\big)^{-1}+E(T,t)^\top\Gamma(t,T)^{-1}\Big)\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\ &+\Big(\Gamma(t,T)\Gamma(0,T)^{-1}E(0,t)+\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}E(T,t)\Big)\mathbf{r}(t),\end{aligned}$$

since

$$
\begin{aligned}
&-\Big(Q(t)-\Sigma(t)\Sigma(t)^\top E(T,t)^\top\Gamma(t,T)^{-1}\Big)\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t)+(\partial_1\Gamma(t,T))\Gamma(0,T)^{-1}\mathbf{m}_{\mathbf{a}}^{+}(0,t)\\
&\quad+(\partial_2\Gamma(0,t))^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\
&=-\partial_1\Gamma(t,T)\Gamma(t,T)^{-1}\Gamma(t,T)\Gamma(0,T)^{-1}\mathbf{m}_{\mathbf{a}}^{+}(0,t)\\
&\quad-\partial_1\Gamma(t,T)\Gamma(t,T)^{-1}\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\
&\quad+\partial_1\Gamma(t,T)\Gamma(0,T)^{-1}\mathbf{m}_{\mathbf{a}}^{+}(0,t)+(\partial_2\Gamma(0,t))^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\
&=\Big(-\partial_1\Gamma(t,T)\Gamma(t,T)^{-1}+(\partial_2\Gamma(0,t))^\top\big(\Gamma(0,t)^\top\big)^{-1}\Big)\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\
&=\Big(\Sigma(t)\Sigma(t)^\top E(T,t)^\top\Gamma(t,T)^{-1}-Q(t)\Gamma(t,T)\Gamma(t,T)^{-1}+Q(t)\Gamma(0,t)^\top\big(\Gamma(0,t)^\top\big)^{-1}\\
&\quad+\Sigma(t)\Sigma(t)^\top E(0,t)^\top\big(\Gamma(0,t)^\top\big)^{-1}\Big)\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\
&=\Sigma(t)\Sigma(t)^\top\Big(E(0,t)^\top\big(\Gamma(0,t)^\top\big)^{-1}+E(T,t)^\top\Gamma(t,T)^{-1}\Big)\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T).
\end{aligned}
$$

This expression can be simplified, since by (4.1) we have

$$
\begin{aligned}
&\Big(E(0,t)^\top\big(\Gamma(0,t)^\top\big)^{-1}+E(T,t)^\top\Gamma(t,T)^{-1}\Big)\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\\
&\quad=\Big(\kappa(0,t)^{-1}+E(T,t)^\top\kappa(t,T)^{-1}E(T,t)\Big)\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\\
&\quad=\Sigma(0,t)^{-1}\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\\
&\quad=\big(\Sigma(0,t)^\top\big)^{-1}\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}=\big(\Gamma(t,T)^\top\big)^{-1},
\end{aligned}
$$

and using (4.9) we further calculate

$$
\begin{aligned}
&\Gamma(t,T)\Gamma(0,T)^{-1}E(0,t)+\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}E(T,t)\\
&\quad=\Big(\Gamma(t,T)+\Gamma(0,t)^\top E(T,0)^\top\Big)\kappa(0,T)^{-1}E(T,t)\\
&\quad=E(t,T)E(T,t)=I_d.
\end{aligned}
$$

Putting things together, we have the SDE (2.12), as desired. Since the process $(\mathbf{U}_t)_{t\in[0,T)}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T)}$, it is a strong solution of the SDE (2.12). By Theorem 5.2.1 in Øksendal [26] or Theorem 2.32 in Chapter III in Jacod and Shiryaev [19], strong uniqueness holds for the SDE (2.12). □

The following lemma is about the covariance structure of the linear process $\mathbf{Z}$ and its bridge $\mathbf{U}$ (given in Definition 2.1). We use this lemma in the proofs of Theorem 2.3 and Proposition 2.1.

**4.4 Lemma.** *For fixed* $\mathbf{a},\mathbf{b}\in\mathbb{R}^d$ *and* $T>0$, *let* $(\mathbf{Z}_t)_{t\geqslant0}$ *be the d-dimensional linear process given by the SDE* (2.1) *with a Gauss initial random vector* $\mathbf{Z}_0$ *independent of the underlying Wiener process* $(\mathbf{B}_t)_{t\geqslant0}$. *Let us suppose that condition* (2.7) *holds and let* $(\mathbf{U}_t)_{t\in[0,T]}$ *be the linear process* *bridge from* $\mathbf{a}$ *to* $\mathbf{b}$ *over* $[0,T]$ *derived from* $\mathbf{Z}$ *(given by Theorem 2.1 and Definition 2.1). Then for* $0 \leqslant s \leqslant t$ *the covariance matrices of* $\mathbf{Z}$ *and* $\mathbf{U}$ *are given by*

*(a)* $\operatorname{Cov}(\mathbf{Z}_s, \mathbf{Z}_t) = \operatorname{Cov}(\mathbf{Z}_t, \mathbf{Z}_s)^\top = \big(E(t,0)\Gamma(0,s)\big)^\top$,

*(b)* $\operatorname{Cov}(\mathbf{U}_s, \mathbf{U}_t) = \operatorname{Cov}(\mathbf{U}_t, \mathbf{U}_s)^\top = \big(\Gamma(t,T)\Gamma(0,T)^{-1}\Gamma(0,s)\big)^\top$.

**Proof.** Since the stochastic integral in (2.6) is independent of $\mathbf{Z}_s$, by (2.3) we get

$$\begin{aligned}\operatorname{Cov}(\mathbf{Z}_s, \mathbf{Z}_t) &= \operatorname{Cov}(\mathbf{Z}_s, E(t,s)\mathbf{Z}_s) = \operatorname{Cov}(\mathbf{Z}_s, \mathbf{Z}_s)E(t,s)^\top \\ &= \kappa(0,s)^\top E(0,s)^\top E(t,0)^\top = \Gamma(0,s)^\top E(t,0)^\top,\end{aligned}$$

which proves (a). Similarly, using (4.10) and Theorem 2.1, we get

$$\begin{aligned}\operatorname{Cov}(\mathbf{U}_s, \mathbf{U}_t) &= \operatorname{Cov}(\mathbf{U}_s, \Gamma(t,T)\Gamma(s,T)^{-1}\mathbf{U}_s) = \operatorname{Cov}(\mathbf{U}_s, \mathbf{U}_s)\big(\Gamma(s,T)^{-1}\big)^\top \Gamma(t,T)^\top \\ &= \Sigma(0,s)^\top \big(\Gamma(s,T)^{-1}\big)^\top \Gamma(t,T)^\top = \big(\Gamma(t,T)\Gamma(0,T)^{-1}\Gamma(0,s)\big)^\top,\end{aligned}$$

where the last equality follows by the definition of $\Sigma(0,s)$. □

**Proof of Theorem 2.3.** Since the processes $\mathbf{Y}$ and $\mathbf{U}$ are almost surely continuous Gauss processes, using also that the law of a (continuous) stochastic process is determined by its finite-dimensional distributions (see, e.g., Kallenberg [20, Proposition 2.2]), it is enough to show the equality of their mean and covariance functions. Using the definition (2.13) we get

$$\mathsf{E}\mathbf{Y}_t = \Gamma(t,T)\Gamma(0,T)^{-1}\mathbf{a} + \mathbf{m_0}(0,t) + \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}(\mathbf{b} - \mathbf{m_0}(0,T)),$$

for all $t \in [0,T]$. Since

$$\begin{aligned}&\mathbf{m_0}(0,t) + \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}(\mathbf{b} - \mathbf{m_0}(0,T)) \\ &= \int_0^t E(t,u)\mathbf{r}(u)\,\mathrm{d}u + \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}\left(\mathbf{b} - \int_0^T E(T,u)\mathbf{r}(u)\,\mathrm{d}u\right) \\ &= E(t,0)\int_0^t E(0,u)\mathbf{r}(u)\,\mathrm{d}u - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1} E(T,0)\int_0^T E(0,u)\mathbf{r}(u)\,\mathrm{d}u \\ &\quad + \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}\mathbf{b} \\ &= \left(E(t,0) - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}E(T,0)\right)\int_0^t E(0,u)\mathbf{r}(u)\,\mathrm{d}u \\ &\quad + \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}\left(\mathbf{b} - \int_t^T E(T,u)\mathbf{r}(u)\,\mathrm{d}u\right) \\ &= \left(E(t,0) - \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}E(T,0)\right)\int_0^t E(0,u)\mathbf{r}(u)\,\mathrm{d}u + \Gamma(0,t)^\top \big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T),\end{aligned}$$

we get

$$
\begin{aligned}
\mathsf{E}\mathbf{Y}_t =&\Gamma(t,T)\Gamma(0,T)^{-1}\\
&\times\left(\mathbf{a}+\Gamma(0,T)\Gamma(t,T)^{-1}\left[E(t,0)-\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}E(T,0)\right]\int_0^t E(0,u)\mathbf{r}(u)\,\mathrm{d}u\right)\\
&+\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^{-}(t,T)\\
=&\mathbf{n}_{\mathbf{a},\mathbf{b}}(0,t)=\mathsf{E}\mathbf{U}_t,\quad t\in[0,T],
\end{aligned}
$$

where the last but one equality follows by part (c) of Lemma 4.2 together with (2.10), and the last equality by Theorem 2.1. Further, using part (a) of Lemma 4.4, for all $0\leqslant s\leqslant t\leqslant T$ we have

$$
\begin{aligned}
\operatorname{Cov}(\mathbf{Y}_s,\mathbf{Y}_t)=&\operatorname{Cov}\left(\mathbf{Z}_s-\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{Z}_T,\mathbf{Z}_t-\Gamma(0,t)^\top\big(\Gamma(0,T)^\top\big)^{-1}\mathbf{Z}_T\right)\\
=&\operatorname{Cov}(\mathbf{Z}_s,\mathbf{Z}_t)-\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\operatorname{Cov}(\mathbf{Z}_T,\mathbf{Z}_t)\\
&-\operatorname{Cov}(\mathbf{Z}_s,\mathbf{Z}_T)\Gamma(0,T)^{-1}\Gamma(0,t)\\
&+\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\operatorname{Cov}(\mathbf{Z}_T,\mathbf{Z}_T)\Gamma(0,T)^{-1}\Gamma(0,t)\\
=&\Gamma(0,s)^\top E(t,0)^\top-\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}E(T,0)\Gamma(0,t)\\
&-\Gamma(0,s)^\top E(T,0)^\top\Gamma(0,T)^{-1}\Gamma(0,t)\\
&+\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\kappa(0,T)\Gamma(0,T)^{-1}\Gamma(0,t),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\operatorname{Cov}(\mathbf{Y}_s,\mathbf{Y}_t)=&\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\big[\Gamma(0,T)^\top E(t,0)^\top-E(T,0)\Gamma(0,t)\\
&\qquad-\Gamma(0,T)^\top E(T,0)^\top\Gamma(0,T)^{-1}\Gamma(0,t)+\kappa(0,T)\Gamma(0,T)^{-1}\Gamma(0,t)\big]\\
=&\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\big[\Gamma(0,T)^\top E(t,0)^\top-E(T,0)\Gamma(0,t)\big]\\
=&\Gamma(0,s)^\top\big(\Gamma(0,T)^\top\big)^{-1}\Gamma(t,T)^\top,
\end{aligned}
$$

where the last equality follows by part (d) of Lemma 4.2. The last line of the above equation coincides with $\operatorname{Cov}(\mathbf{U}_s,\mathbf{U}_t)$ by part (b) of Lemma 4.4, as desired. □

**Proof of Proposition 2.1.** Since $(\mathbf{Z}_t)_{t\geqslant 0}$ is a Gauss process, by Shiryaev [30, Theorem 2, page 303], the conditional distribution of $\mathbf{Z}:=(\mathbf{Z}_{t_1}^\top,\dots,\mathbf{Z}_{t_n}^\top)^\top$ given $\mathbf{Z}_T=\mathbf{b}$ is known to be a $\mathbb{R}^{dn}$-dimensional Gauss distribution with mean vector

$$
\mathbf{m}:=\mathsf{E}\mathbf{Z}+\left[\bigotimes_{i=1}^n\operatorname{Cov}(\mathbf{Z}_{t_i},\mathbf{Z}_T)\right]\operatorname{Cov}(\mathbf{Z}_T,\mathbf{Z}_T)^{-1}\big(\mathbf{b}-\mathsf{E}\mathbf{Z}_T\big),
$$

and with covariance matrix

$$
C:=\operatorname{Cov}(\mathbf{Z},\mathbf{Z})-\left[\bigotimes_{i=1}^n\operatorname{Cov}(\mathbf{Z}_{t_i},\mathbf{Z}_T)\right]\operatorname{Cov}(\mathbf{Z}_T,\mathbf{Z}_T)^{-1}\left[\bigotimes_{i=1}^n\operatorname{Cov}(\mathbf{Z}_{t_i},\mathbf{Z}_T)\right]^\top,
$$

where, due to symmetry, the $(nd \times d)$-matrix $\bigotimes_{i=1}^n \operatorname{Cov}(\mathbf{Z}_{t_i}, \mathbf{Z}_T)$ is defined by

$$\bigotimes_{i=1}^n \operatorname{Cov}(\mathbf{Z}_{t_i}, \mathbf{Z}_T) := \big(\operatorname{Cov}(\mathbf{Z}_{t_1}, \mathbf{Z}_T), \dots, \operatorname{Cov}(\mathbf{Z}_{t_n}, \mathbf{Z}_T)\big)^\top.$$

Due to the fact that the integrand in the stochastic integral of (2.11) is deterministic, by Lemma 48.2 in Bauer [6], the process $(\mathbf{U}_t)_{t\in[0,T]}$ is a Gauss process, and hence $(\mathbf{U}_{t_1}^\top, \dots, \mathbf{U}_{t_n}^\top)^\top$ is a $nd$-dimensional Gauss distributed random variable. Then all we have to do is to check that the mean vector and the covariance matrix of $(\mathbf{U}_{t_1}^\top, \dots, \mathbf{U}_{t_n}^\top)^\top$ coincides with the mean vector and the covariance matrix given above. Using Theorem 2.1, due to the tensor calculus it is sufficient to prove that for all $0 \leqslant i \leqslant j \leqslant n$

$$\mathbf{m}_i := \mathsf{E}\mathbf{Z}_{t_i} + \operatorname{Cov}(\mathbf{Z}_{t_i}, \mathbf{Z}_T)\operatorname{Cov}(\mathbf{Z}_T, \mathbf{Z}_T)^{-1}\big(\mathbf{b} - \mathsf{E}\mathbf{Z}_T\big) = \mathsf{E}\mathbf{U}_{t_i} = \mathbf{n}_{\mathbf{a},\mathbf{b}}(0, t_i), \tag{4.13}$$

and

$$\begin{aligned} C_{ij} &:= \operatorname{Cov}(\mathbf{Z}_{t_i}, \mathbf{Z}_{t_j}) - \operatorname{Cov}(\mathbf{Z}_{t_i}, \mathbf{Z}_T)\operatorname{Cov}(\mathbf{Z}_T, \mathbf{Z}_T)^{-1}\operatorname{Cov}(\mathbf{Z}_{t_j}, \mathbf{Z}_T)^\top \\ &= \operatorname{Cov}(\mathbf{U}_{t_i}, \mathbf{U}_{t_j}) = \big(\Gamma(t_j, T)\Gamma(0, T)^{-1}\Gamma(0, t_i)\big)^\top, \end{aligned} \tag{4.14}$$

where the last equality follows from part (b) of Lemma 4.4. Using part (a) of Lemma 4.4 and (4.2) we calculate

$$\begin{aligned} \mathbf{m}_i &= \mathbf{m}_{\mathbf{a}}(0, t_i) + \Gamma(0, t_i)^\top E(T, 0)^\top\big(\Gamma(0, T)^\top E(T, 0)^\top\big)^{-1}\big(\mathbf{b} - \mathbf{m}_{\mathbf{a}}(0, T)\big) \\ &= E(t_i, 0)\mathbf{m}_{\mathbf{a}}^+(0, t_i) + \Gamma(0, t_i)^\top\big(\Gamma(0, T)^\top\big)^{-1}\big(\mathbf{m}_{\mathbf{b}}^-(t_i, T) - E(T, 0)\mathbf{m}_{\mathbf{a}}^+(0, t_i)\big) \\ &= \Gamma(t_i, T)\Gamma(0, T)^{-1}\mathbf{m}_{\mathbf{a}}^+(0, t_i) + \Gamma(0, t_i)^\top\big(\Gamma(0, T)^\top\big)^{-1}\mathbf{m}_{\mathbf{b}}^-(t_i, T), \end{aligned}$$

where the last equality follows by part (c) of Lemma 4.2. This shows (4.13) by using (2.10). Moreover, using part (a) of Lemma 4.4 we have

$$\begin{aligned} C_{ij} &= \Gamma(0, t_i)^\top E(t_j, 0)^\top - \Gamma(0, t_i)^\top E(T, 0)^\top\big(\Gamma(0, T)^\top E(T, 0)^\top\big)^{-1}E(T, 0)\Gamma(0, t_j) \\ &= \Gamma(0, t_i)^\top\big(\Gamma(0, T)^\top\big)^{-1}\big[\Gamma(0, T)^\top E(t_j, 0)^\top - E(T, 0)\Gamma(0, t_j)\big] \\ &= \big(\Gamma(t_j, T)\Gamma(0, T)^{-1}\Gamma(0, t_i)\big)^\top, \end{aligned}$$

where the last equality follows by part (d) of Lemma 4.2. This shows (4.14). □

**Proof of Theorem 3.1.** Using the notations of Section 2, we have

$$\begin{aligned} &\Phi(t) = E(t, 0) = \mathrm{e}^{\overline{q}(t)}, \qquad t \geqslant 0, \\ &E(t, s) = E(t, 0)E(0, s) = E(t, 0)E(s, 0)^{-1} = \mathrm{e}^{\overline{q}(t) - \overline{q}(s)}, \qquad t, s \geqslant 0, \end{aligned}$$

and hence for all $0 \leqslant s < t$, we get $\kappa(s, t) = \gamma(s, t)$ and

$$\Gamma(s, t) = \mathrm{e}^{\overline{q}(s) - \overline{q}(t)}\gamma(s, t),$$

$$\Sigma(s, t) = \mathrm{e}^{\overline{q}(t) - \overline{q}(T)}\gamma(t, T)\big(\mathrm{e}^{\overline{q}(s) - \overline{q}(T)}\gamma(s, T)\big)^{-1}\mathrm{e}^{\overline{q}(s) - \overline{q}(t)}\gamma(s, t) = \frac{\gamma(s, t)\gamma(t, T)}{\gamma(s, T)} = \sigma(s, t).$$

Then for all $0 \leqslant s < t < T$,

$$\begin{aligned}
&\Gamma(t,T)\Gamma(s,T)^{-1}m_a^+(s,t) + \Gamma(s,t)\Gamma(s,T)^{-1}m_b^-(t,T) \\
&= \frac{\mathrm{e}^{\overline{q}(t)-\overline{q}(T)}\gamma(t,T)}{\mathrm{e}^{\overline{q}(s)-\overline{q}(T)}\gamma(s,T)}\left(a + \int_s^t E(s,u)r(u)\,\mathrm{d}u\right) + \frac{\mathrm{e}^{\overline{q}(s)-\overline{q}(t)}\gamma(s,t)}{\mathrm{e}^{\overline{q}(s)-\overline{q}(T)}\gamma(s,T)}\left(b - \int_t^T E(T,u)r(u)\,\mathrm{d}u\right) \\
&= \mathrm{e}^{\overline{q}(t)-\overline{q}(s)}\frac{\gamma(t,T)}{\gamma(s,T)}\left(a + \int_s^t \mathrm{e}^{\overline{q}(s)-\overline{q}(u)}r(u)\,\mathrm{d}u\right) + \mathrm{e}^{\overline{q}(T)-\overline{q}(t)}\frac{\gamma(s,t)}{\gamma(s,T)}\left(b - \int_t^T \mathrm{e}^{\overline{q}(T)-\overline{q}(u)}r(u)\,\mathrm{d}u\right) \\
&= \frac{\gamma(t,T)}{\gamma(s,T)}m_a(s,t) + \mathrm{e}^{\overline{q}(T)-\overline{q}(t)}\frac{\gamma(s,t)}{\gamma(s,T)}\left(b - \int_t^T \mathrm{e}^{\overline{q}(T)-\overline{q}(u)}r(u)\,\mathrm{d}u\right) = n_a(s,t).
\end{aligned}$$

Hence Theorem 2.1 yields that for all $t \in [0,T)$,

$$\begin{aligned}
U_t &= n_{a,b}(0,t) + \mathrm{e}^{\overline{q}(t)-\overline{q}(T)}\gamma(t,T)\int_0^t \left(\mathrm{e}^{\overline{q}(u)-\overline{q}(T)}\gamma(u,T)\right)^{-1}\sigma(u)\,\mathrm{d}B_u \\
&= n_{a,b}(0,t) + \int_0^t \frac{\gamma(t,T)}{\gamma(u,T)}\mathrm{e}^{\overline{q}(t)-\overline{q}(u)}\sigma(u)\,\mathrm{d}B_u,
\end{aligned}$$

as desired. □

**Proof of Proposition 3.2.** It is enough to check that the process $(Z_t^*)_{t\geqslant 0}$ has the same finite dimensional distributions as

$$Z_t = m_0(0,t) + \int_0^t \mathrm{e}^{\overline{q}(t)-\overline{q}(s)}\sigma(s)\,\mathrm{d}B_s, \quad t \geqslant 0,$$

has. Indeed, $(Z_t)_{t\geqslant 0}$ is a strong solution of the SDE (3.1) and the law of a stochastic process is determined by its finite dimensional distributions (see, e.g., Kallenberg [20, Proposition 2.2]). Hence it is sufficient to check that $(Z_t^*)_{t\geqslant 0}$ is a Gauss process with the same expectation and covariance function that $Z$ has. Since $B^*$ is a Gauss process we have $Z^*$ is also a Gauss process. The expectation functions are the same, since $\mathsf{E}Z_t = \mathsf{E}Z_t^* = m_0(0,t)$, $t \geqslant 0$. Moreover, using (3.3) one can calculate

$$\operatorname{Cov}(Z_s, Z_t) = \mathrm{e}^{\overline{q}(s)+\overline{q}(t)}\int_0^{s\wedge t} \mathrm{e}^{-2\overline{q}(u)}\sigma^2(u)\,\mathrm{d}u, \quad s,t \geqslant 0.$$

One can also easily derive that

$$\begin{aligned}
\operatorname{Cov}(Z_s^*, Z_t^*) &= \mathrm{e}^{\overline{q}(s)+\overline{q}(t)}\operatorname{Cov}\left(B^*(\mathrm{e}^{-2\overline{q}(s)}\gamma(0,s)), B^*(\mathrm{e}^{-2\overline{q}(t)}\gamma(0,t))\right) \\
&= \mathrm{e}^{\overline{q}(s)+\overline{q}(t)}\min(\mathrm{e}^{-2\overline{q}(s)}\gamma(0,s), \mathrm{e}^{-2\overline{q}(t)}\gamma(0,t)), \quad s,t \geqslant 0.
\end{aligned}$$

Since the function

$$[0,\infty) \ni t \mapsto \mathrm{e}^{-2\overline{q}(t)}\gamma(0,t) = \int_0^t \mathrm{e}^{-2\overline{q}(u)}\sigma^2(u)\,\mathrm{d}u,$$

is monotone increasing, we have

$$\operatorname{Cov}(Z_s^*, Z_t^*) = \mathrm{e}^{\overline{q}(s)+\overline{q}(t)}\int_0^{s\wedge t} \mathrm{e}^{-2\overline{q}(u)}\sigma^2(u)\,\mathrm{d}u, \quad s,t \geqslant 0,$$

as desired. □

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