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A COMPLETE UNITARY SIMILARITY INVARIANT FOR UNICELLULAR MATRICES

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Abstract. Necessary and sufficient conditions for two $n \times n$ unicellular complex matrices to be unitarily equivalent are established.

1. Introduction

A fundamental problem in operator theory and matrix analysis — see for example [2, 9] — is the *unitary similarity problem*: under what necessary and sufficient conditions are two Hilbert space operators unitarily similar?

At this level of generality, the problem is not at all tractable in infinite dimensions. But for finite-dimensional Hilbert space, there is a classical and purely algebraic solution to this problem due to W. Specht [10]: matrices A and B are unitarily similar if and only if

$$\text{Trace}\,\omega(A, A^*) \;=\; \text{Trace}\,\omega(B, B^*)\,, \tag{1.1}$$

for every word ω in two noncommuting variables x and y. (A modern approach to Specht's theorem is given in [8].) But in many applications, the data one has about a particular matrix is not based on the trace of the matrix, but rather on some other analytical information: the spectrum or pseudospectrum, the numerical range or polynomial numerical hull, the singular values, a unitarily invariant norm, and so forth. None of these analytic invariants are known to determine a matrix up to unitary similarity, except perhaps in the most exceptional of circumstances.

Nevertheless, our concern in the present paper is with an invariant based on the norm of a matrix, considered as an operator on n-dimensional complex space $\mathbb{C}^n$.

Let M_n be the space of all $n \times n$ complex matrices and by U_n we denote the unitary group. Two elements $A, B \in \mathrm{M}_n$ are unitarily similar, which we denote by $A \sim B$, if there is a $U \in \mathrm{U}_n$ such that $B = U^*AU$. It is well known that there are no canonical choices for the representative of $A \in \mathrm{M}_n$ in the space $\mathrm{M}_n/\sim$ of equivalences classes under unitary similarity $\sim$. In this regard, unitary similarity departs substantially from similarity, where one has the Jordan canonical form.

Let $\langle \xi, \eta \rangle$ denote the canonical inner product of $\xi, \eta \in \mathbb{C}^n$, the vector space of complex n-tuples. The inner product induces norms $\|\xi\| = \langle \xi, \xi \rangle^{1/2}$ on $\mathbb{C}^n$ and $\|A\| = \max_{0 \neq \xi \in \mathbb{C}^n} \frac{\|A\xi\|}{\|\xi\|}$ on M_n such that $\|U^*AU\| = \|A\|$, for $U \in \mathrm{U}_n$.

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Let $\mathbb{C}[t]$ denote the ring of polynomials with complex coefficients. If $A \sim B$, then necessarily $\|f(A)\| = \|f(B)\|$ for all $f \in \mathbb{C}[t]$. Conversely, if $A, B \in \mathrm{M}_n$ are such that $\|f(A)\| = \|f(B)\|$ for all $f \in \mathbb{C}[t]$, then A and B yield to the same "matrix analysis." In particular:

(i) A and B have the same spectrum;

(ii) $A + z1$ and $B + z1$ have the same condition numbers, for all nonspectral z in the complex plane;

(iii) A and B have the same polynomial numerical hulls and, in particular, the same numerical range;

(iv) A and B have the same spectral sets;

(v) A and B have the same pseudospectrum.

Our first objective is determine cases in which the condition $\|f(A)\| = \|f(B)\|$ for all $f \in \mathbb{C}[t]$ is also sufficient for $A \sim B$. In general it will not be so, for if one takes any two nonzero projections (selfadjoint idempotents) P and Q, then one has $\|f(P)\| = \|f(Q)\|$ for all $f \in \mathbb{C}[t]$, independent of the ranks of P and Q. Therefore, the hypotheses $\|f(A)\| = \|f(B)\|$ for all $f \in \mathbb{C}[t]$ is relevant only for the analysis of nonnormal matrices.

We will show here for many upper triangular Toeplitz matrices R, the condition $\|f(A)\| = \|f(R)\|$ for all $f \in \mathbb{C}[t]$ is indeed sufficient for $A \sim R$: see Corollary 2.5. Yet, we believe that this is still a rather rare circumstance. And for many highly nonnormal matrices A and B, it is possible for $\|f(A)\| = \|f(B)\|$ to hold for all $f \in \mathbb{C}[t]$, yet $A \not\sim B$ (see, for example, Proposition 3.1). One reason that these polynomial/norm conditions are insufficient for unitarily similarity is because the condition $\|f(A)\| = \|f(B)\|$, for all $f \in \mathbb{C}[t]$, fails to capture analytically the action of a matrix A on its invariant subspaces, much in the way the norm of $f(P)$, for a projection P, does not tell us anything about the dimension of the nonzero eigenspace. (A fuller discussion is in Sections 3 and 4.2.)

The matrices with the simplest lattices of invariant subspaces are the *unicellular matrices* [7, §2.5]. Unicellular matrices A are at the opposite end of the scale from selfadjoint matrices: such A posses only one eigenvalue and the only matrices that commute with A and its adjoint A^* are those that are scalar multiples of the identity (in other words, A is *irreducible*). A necessary and sufficient condition for A to be unicellular is that the Jordan canonical form of A consist of exactly one Jordan block; thus, unicellular matrices are *nonderogatory.*

Our main result in this paper is Theorem 3.2, namely a necessary and sufficient condition for the unitary similarity of unicellular matrices.

2. Upper Triangular Toeplitz Matrices

In this section, let $Q, J(\lambda) \in \mathrm{M}_n$, where $\lambda \in \mathbb{C}$, denote the matrices

$$Q = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \quad \text{and} \quad J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \lambda & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & \cdots & \lambda \end{bmatrix}.$$

The main theorem of this section is:

Theorem 2.1. *If $A \in \mathrm{M}_n$ is any matrix for which $\|f(A)\| = \|f(Q)\|$, for all $f \in \mathbb{C}[t]$, then $A \sim Q$.*

The proof of Theorem 2.1 requires the following lemmas.

Lemma 2.2. $\sum_{k=1}^{\infty}(-1)^{k+1}Q^k = S$, *where* $S = J(0)$.

Proof. Clearly $Q = \sum_{k=1}^{\infty} S^k$. Thus, $1 + Q = \sum_{j=0}^{\infty} S^j = (1-S)^{-1}$, whence $1 = (1-S)(1+Q)$. That is, $S = 1 - (1+Q)^{-1} = \sum_{k=1}^{\infty}(-1)^{k+1}Q^k$. □

Lemma 2.3. *If*

$$A = \begin{bmatrix} 0 & 1 & a_{13} & \cdots & a_{1n} \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & a_{n-2,n} \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

has the property that $\left\|\sum_{k=1}^{\infty}(-1)^{k+1}A^k\right\| \leq 1$, *then* $A = Q$.

Proof. We proceed by induction on n. The base case is $n = 3$. In this case,

$$\sum_{k=1}^{\infty}(-1)^{k+1}A^k = \begin{bmatrix} 0 & 1 & a_{13}-1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $a_{13} - 1$ were nonzero, then the third column of the matrix above would have norm exceeding 1, which means that the matrix above would map a unit vector — namely, e_3 — to a vector of norm exceeding 1. But such a scenario is impossible, as the norm of the matrix above is at most 1. Hence, it must be that $a_{13} = 1$, implying that $A = Q$.

Assume now the statement holds in n-dimensional space and consider A, Q, and S (the nilpotent Jordan block) as acting on $\mathbb{C}^{n+1}$. Let $\tilde{A}$, $\tilde{Q}$, and $\tilde{S}$ denote the versions of A, Q, and S that act on $\mathbb{C}^n$, and let $e_1, \ldots, e_n$ denote the canonical orthonormal basis vectors in $\mathbb{C}^n$. Hence, as a partitioned matrix, A has the form

$$A = \left[\begin{array}{ccc|c} & \tilde{A} & & \eta \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \quad \text{where } \eta = e_n + \sum_{i=1}^{n-1} a_{i,n+1} e_i \in \mathbb{C}^n.$$

Because $1 \geq \left\| \sum_{k=1}^{\infty} (-1)^{k+1} A^k \right\| \geq \left\| \sum_{k=1}^{\infty} (-1)^{k+1} \tilde{A}^k \right\|$, the induction hypothesis yields $\tilde{A} = \tilde{Q}$. Hence, using Lemma 2.2, we obtain

$$\sum_{k=1}^{\infty} (-1)^{k+1} A^k = \left[\begin{array}{c|c} \tilde{S} & \sum_{k=1}^{\infty} (-1)^{k+1} \tilde{A}^{k-1} \eta \\ \hline 0 \quad \cdots \quad 0 & 0 \end{array} \right].$$

Similar to the case $n = 3$, the norm condition $1 \geq \left\| \sum_{k=1}^{\infty} (-1)^{k+1} A^k \right\|$ for the matrix above holds only if

$$0 = \left(\sum_{k=1}^{\infty} (-1)^{k+1} \tilde{A}^{k-1} \eta \right) e_i = ((\tilde{1} + \tilde{A})^{-1} \eta) e_i , \quad 1 \leq i \leq n-1 .$$

Therefore, using $\tilde{A} = \tilde{Q}$, we have $(\tilde{1} - \tilde{S})\eta = \lambda e_n$ for some complex number λ. Hence,

$$\eta = \lambda (\tilde{1} - \tilde{S})^{-1} e_n = \lambda (\tilde{1} + \tilde{S} + \tilde{S}^2 + \cdots + \tilde{S}^{n-2}) e_n = \lambda (e_n + e_{n-1} + \ldots + e_1) .$$

But on the other hand,

$$\eta = e_n + \sum_{i=1}^{n-1} a_{i,n+1} e_i ,$$

which implies that $\lambda = 1$ and $a_{i,n} = 1$ for all $1 \leq i \leq n-1$. Therefore, $A = Q$. □

We are now set to prove Theorem 2.1:

Proof. Recall that $A \in \mathrm{M}_n$ satisfies $\|f(A)\| = \|f(Q)\|$, for all $f \in \mathbb{C}[t]$. Because of the Spectral Radius Formula, namely $\operatorname{spr} R = \lim_{k \to \infty} \|R^k\|^{1/k}$ for every $R \in \mathrm{M}_n$, this condition on the norms of $f(A)$ implies that A is nilpotent.

Without loss of generality, A may be assumed to be in upper triangular form. Furthermore, using a diagonal unitary similarity transformation, the entries $a_{i,i+1}$ may assumed to be nonnegative, for all $1 \leq i \leq n-1$. Indeed, since $1 = \|Q^{n-1}\| = \|A^{n-1}\| = |a_{12} a_{23} \cdots a_{n-1,n}|$, each $a_{i,i+1}$ is nonzero; thus, we may $a_{i,i+1} > 0$ for all i.

The numerical range, or field of values, $W(R)$ of any $R \in \mathrm{M}_n$ is given analytically by

$$W(R) = \bigcap_{\alpha, \beta \in \mathbb{C}} \{ z \in \mathbb{C} \,|\, |\alpha z + \beta| \leq \|\alpha R + \beta 1\| \} .$$

Hence, $W(A) = W(Q)$. Let $\Re(R) = \frac{1}{2}(R + R^*)$, for any $R \in \mathrm{M}_n$, and observe that $\frac{1}{2} + \Re(Q) = \frac{1}{2}\, \xi \otimes \xi$, where $\xi = \sum_{i=1}^{n} e_i \in \mathbb{C}^n$ and $\xi \otimes \xi$ denotes the outer product $\xi \xi^* \in \mathrm{M}_n$ of ξ (a column vector) with its conjugate transpose ξ^*. Thus, for every unit vector $\gamma \in \mathbb{C}^n$, we have that the real part of $\langle Q\gamma, \gamma \rangle$ satisfies the inequality

$$\Re\left(\langle Q\gamma, \gamma \rangle \right) \geq -\frac{1}{2} .$$

Because A and Q have the same numerical range, it is also true that $\Re(A)$ has the same property. Now, if P_i is the projection of $\mathbb{C}^n$ onto $\operatorname{Span}\{e_i, e_{i+1}\}$, for each $1 \le i \le n-1$, then $P_i A P_i$ as a linear transformation on the range of P_i is given by

$$\begin{bmatrix} 0 & a_{i,i+1} \\ 0 & 0 \end{bmatrix}.$$

Therefore, the numerical range of $P_i A P_i$ is a disc of radius $\frac{1}{2}a_{i,i+1}$ centered at the origin. Because $W(P_i A P_i) \subseteq W(A) \subset \{z \in \mathbb{C} \,|\, \Re(z) \ge -1/2\}$, we conclude that each $a_{i,i+1} \le 1$. However, under these conditions the equation $1 = \|A^{n-1}\| = a_{12}a_{23}\cdots a_{n-1,n}$ holds only if $a_{i,i+1} = 1$ for all $1 \le i \le n-1$. Hence, A has the structure given in the hypothesis of Lemma 2.3. Moreover, by Lemma 2.2,

$$1 \;=\; \|S\| \;=\; \left\| \sum_{k=1}^{\infty} (-1)^{k+1} Q^k \right\| \;=\; \left\| \sum_{k=1}^{\infty} (-1)^{k+1} A^k \right\|.$$

Thus, A satisfies all of the hypotheses of Lemma 2.3, yielding $Q = A$. □

Theorem 2.1 allows us to identify some other Toeplitz operators that are completely determined by the polynomial norm condition.

To do this, let $\mathrm{UpperToepl}_n \subset \mathrm{M}_n$ denote the algebra of $n \times n$ upper triangular Toeplitz matrices, and let $\operatorname{Alg} A$ denote the unital abelian algebra generated by $A \in \mathrm{M}_n$, namely the set of all $f(A)$, where $f \in \mathbb{C}[t]$.

Corollary 2.4. *If $\varrho : \mathrm{UpperToepl}_n \to \mathrm{M}_n$ is any unital isometric homomorphism of $\mathrm{UpperToepl}_n$, then there exists $U \in \mathrm{U}_n$ such that $\varrho(X) = U^*XU$ for all $X \in \mathrm{UpperToepl}_n$.*

Proof. By Lemma 2.2, $S \in \operatorname{Alg} Q$. Because $\operatorname{Alg} S = \mathrm{UpperToepl}_n$ and because Q itself is an upper triangular Toeplitz operator, Q generates $\mathrm{UpperToepl}_n$. Thus, if $A = \varrho(Q)$, then, for every $f \in \mathbb{C}[t]$,

$$\|f(A)\| \;=\; \|f(\varrho(Q))\| \;=\; \|\varrho(f(Q))\| \;=\; \|f(Q)\|.$$

Therefore, by Theorem 2.1, there is a $U \in \mathrm{U}_n$ such that $A = U^*QU$. Because the matrix A generates the range of ϱ, we conclude that $\varrho(X) = U^*XU$ for all $X \in \mathrm{UpperToepl}_n$. □

Corollary 2.5. *If $R \in \mathrm{M}_n$ is a generator of the algebra of upper triangular Toeplitz matrices, then $R \sim B$ for any $B \in \mathrm{M}_n$ that satisfies $\|f(B)\| = \|f(R)\|$ for every $f \in \mathbb{C}[t]$. This is true, in particular, of any Jordan block $J(\lambda) \in \mathrm{M}_n$.*

3. Necessary and Sufficient Conditions for Unitary Similarity

If $A \in \mathrm{M}_n$ is unicellular — say with spectrum $\{\lambda\}$ — and if $B \in \mathrm{M}_n$ is any matrix for which $\|f(A)\| = \|f(B)\|$ for all $f \in \mathbb{C}[t]$, then A and B are similar, as the condition implies that $\sigma(B) = \sigma(A)$ and that $(B - \lambda 1)^{n-1} \ne 0$. But, unlike the case for generators of the upper triangular Toeplitz matrices, A and B need not be unitarily equivalent (Proposition 3.1 below). Therefore, one can have an invertible matrix $Z \in \mathrm{M}_n$ with

$$\|f(A)\| \;=\; \|Zf(A)Z^{-1}\|, \;\; \text{for all } f \in \mathbb{C}[t],$$

and yet Z can fail to be unitary.

Proposition 3.1. *If $0 < \alpha < \beta$, then the unicellular matrices*

$$Q \;=\; \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad Q' \;=\; \begin{bmatrix} 0 & \beta & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy $\|f(Q')\| = |f(Q)\|$ for all $f \in \mathbb{C}[t]$, but $Q' \not\sim Q$.

Proof. Note that $Q' = W^*Q^tW$, where $X \mapsto X^t$ denotes the transpose map and

$$W \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Because the norm is transpose invariant, $\|f(Q')\| = \|f(Q^t)\| = \|f(Q)^t\| = \|f(Q)\|$, for all $f \in \mathbb{C}[t]$. On the other hand, because $0 < \alpha < \beta$, one verifies directly that the equation $UQ' = QU$ is impossible to satisfy with $U \in \mathrm{U}_3$. □

One way to explain the failure of Q and Q' above to be unitarily similar is by way of the observation that the restrictions of Q and Q' to their common invariant subspace Span $\{e_1, e_2\}$ have unequal norms (α and β, respectively). This particular observation about restrictions to invariant subspaces motivates our approach in this section to the problem of unitary similarity for unicellular matrices.

For any matrix $R \in \mathrm{M}_n$ and unitary $U \in \mathrm{U}_n$ for which U^*RU is an upper triangular matrix, the subspaces

$$\mathrm{Span}\,\{e_1, \dots, e_\ell\}\,, 1 \le \ell \le n\,,$$

are invariant under the action of U^*RU. If R is unicellular, then subspaces are the only subspaces, besides $\{0\}$ and $\mathbb{C}^n$, that are invariant under U^*RU [7, §2.5].

Let $P_\ell \in \mathrm{U}_n$ denote the projection onto Span $\{e_1, \dots, e_\ell\}$; that is, $P_\ell = 1_\ell \oplus 0_{n-\ell}$. The following theorem is the main result of this paper.

Theorem 3.2. *Assume that $Q, Q' \in \mathrm{M}_n$ are upper triangular unicellular matrices such that the entries of the superdiagonal above the main diagonal are positive. Then the following statements are equivalent:*

1. *$\|P_i f(Q) P_i\| = \|P_i f(Q') P_i\|$ for every $1 \le i \le n$ and all $f \in \mathbb{C}[t]$;*
2. *$Q = Q'$.*

Remark. As $A \sim B$ implies $\|f(A)\| = \|f(B)\|$ for every polynomial f, there is no loss in generality in assuming that Q and Q' are in upper triangular form. The entries of the superdiagonal above the main diagonal of an upper triangular unicellular matrix must be nonzero — for otherwise the matrix would fail to be unicellular — and so by a diagonal unitary similarity transformation these superdiagonal elements can be assumed to be positive. Thus, the statement of Theorem 3.2 does not impose any special conditions on Q and Q' other than unicellularity.

Proof. We need only prove that (1) implies (2), as the converse is trivial. Because Q and Q' have one point λ of spectrum, by scalar translation we may assume this to be $\lambda = 0$.

The matrix Q has the form

$$Q = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & a_{n-1n} \\ 0 & & & & 0 \end{bmatrix}, \tag{3.1}$$

where $a_{\ell,\ell+1} > 0$, for every $1 \leq \ell \leq n-1$.

We show below that the entries of Q in (3.1) are completely determined from the values of $\|P_i f(Q) P_i\|$ for all $1 \leq i \leq n$ and all $f \in \mathbb{C}[t]$.

We shall proceed by induction on $n \geq 3$.

Let $n = 3$. Thus,

$$Q = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

The value of a_{12} is determined via the fact that $\|P_2 Q P_2\| = a_{12}$, and so the value of a_{23} is determined from the equation $a_{12}a_{23} = \|Q^2\|$. Using $f(t) = t$, we have

$$\|Q\|^2 = \frac{1}{2}\left(a_{12}^2 + a_{23}^2 + |a_{13}|^2 + \sqrt{(a_{12}^2 + a_{23}^2 + |a_{13}|^2)^2 - 4a_{12}^2 a_{23}^2}\right),$$

which determines the value of $|a_{13}|$. Two similar calculations using the polynomials $f(t) = t - \frac{1}{a_{12}a_{23}}t^2$ and $g(t) = t - \frac{i}{a_{12}a_{23}}t^2$ determine the values of $|a_{13} - 1|$ and $|a_{13} - i|$. These last two quantities together with the value of $|a_{13}|$ determine the complex number a_{13}, thereby establishing the base case for the induction.

Assume now that the statement holds for all spaces up to including dimension $n-1$; we will show the statement also holds for spaces of dimension n.

For convenience, we denote the entries of Q^k, where $Q = \{a_{ij}\}_{i>j}$, by $a_{ij}^{(k)}$. By the inductive hypothesis, all elements of leading principal $(n-1)\times(n-1)$ submatrix of Q are uniquely determined by the norms $\|P_j f(Q) P_j\|$, for various $f \in \mathbb{C}[t]$ and $1 \leq j \leq (n-1)$. The only elements left to consider are those in the final column of Q: a_{in}, $1 \leq i \leq (n-1)$. We shall obtain these entries in an argument that requires $n-1$ steps, where each step uses the conclusion of the previous step.

Step 1. Recall $Q^n = 0$ and $Q^{n-1} \neq 0$. All of the elements of Q^{n-1} are zero except in the $(1,n)$ position, where we have

$$\|Q^{n-1}\| = |a_{1n}^{(n-1)}| = a_{12}a_{23}\dots a_{n-2,n-1}a_{n-1,n}.$$

Hence, a_{n-1n} is uniquely determined by the norms $\|P_j f(Q) P_j\|$ for various $f \in \mathbb{C}[t]$ and $1 \leq j \leq n$. This means, in addition, all of the entries of Q^{n-1} are now determined.

Step i. Assume $3 \leq i \leq (n-1)$ and that we have completed Steps 1 to $i-1$, giving us the values of $a_{j,n}$, for $j = n-i-1, \dots, n-1$ and the entries of each Q^{n-j}, for $j = 1, \dots, i-1$. We aim to show that the value of $a_{n-i,n}$ is determined from the norms of various $P_j f(Q) P_j$.

For each complex number $z \in \mathbb{C}$, let $g_z \in \mathbb{C}[t]$ be given by $g_z(t) = t^{n-i} + \frac{z}{a_{12}q} t^{n-1}$, where $q = a_{2,n}^{(n-2)}$ (as in Step 2). Thus,

$$g_z(Q) = \begin{bmatrix} 0 & \dots & 0 & a_{1,n-i+1}^{(n-i)} & a_{1,n-i+2}^{(n-i)} & \dots & a_{1n}^{(n-i)} + z \\ & & & 0 & a_{2,n-i+2}^{(n-i)} & \dots & a_{2n}^{(n-i)} \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 0 & a_{in}^{(n-i)} \\ & & & & & & 0 \\ & & & & & & \vdots \\ 0 & & & & & & 0 \end{bmatrix}.$$

Observe that $g_z(Q)$ is a rank-1 perturbation of Q^{n-i}: namely, $g_z(Q) = Q^{n-i} + zE_{1,n}$. Suppose that there is a complex number $\tilde{a}_{1,n}^{(n-i)}$ such that $\|\tilde{Q}^{n-i} + zE_{1,n}\| = \|Q^{n-i} + zE_{1,n}\|$ for all $z \in \mathbb{C}$, where $\tilde{Q}^{n-i}$ is the matrix obtained from $Q^{n-i} + zE_{1,n}$ by replacing $a_{1,n}^{(n-i)}$ by $\tilde{a}_{1,n}^{(n-i)}$. We shall prove that $\tilde{a}_{1,n}^{(n-i)} = a_{1,n}^{(n-i)}$. Define a function $h : \mathbb{C} \to \mathrm{R}_+$ by $h(z) = \|Q^{n-i} + zE_{1n}\|$ and let $\gamma = \tilde{a}_{1,n}^{(n-i)} - a_{1,n}^{(n-i)}$. Thus, $h(z) = h(z + \gamma)$, for all $z \in \mathbb{C}$. In particular, $h(0) = h(k\gamma)$, for all positive integers k. However, as it is clear that $|h(z)| \to \infty$ as $|z| \to \infty$, the equations $h(0) = h(k\gamma)$, for all positive integers k, can hold only if $\gamma = 0$.

Thus, we have shown that the $(1, n)$-entry of Q^{n-i}, namely $a_{1,n}^{(n-i)}$, is determined uniquely by the norms of various $P_j f(Q) P_j$.

Because the first $n-i-1$ entries in the first row of Q^{n-i-1} are zero and because the first $n-i-1$ entries of the last column of Q are $a_{1,n} \dots, a_{1,n-i-1}$, we obtain from $Q^{n-i} = Q^{n-i-1}Q$ that the $(1, n)$-entry of Q^{n-i} is given by

$$a_{1,n}^{(n-i)} = a_{1,n-i}^{(n-i-1)} a_{1,n-i} + \sum_{k=1}^{i-1} a_{1,n-i+k}^{(n-i-1)} a_{1,n-i+k} \tag{3.3}$$

Because the entries $a_{1,n}^{(n-i)}$, $a_{1,n-i}^{(n-i-1)}$, $a_{1,n-i+k}^{(n-i-1)}$, and $a_{1,n-i+k}$, for $1 \leq k \leq i-1$, have already been determined from the norms of various $P_j f(Q) P_j$ using the induction hypothesis and Steps 1 to $i-1$, (3.3) implies that the value of $a_{1,n-i}$ is determined uniquely from the norms of various $P_j f(Q) P_j$.

This completes the induction and, hence, the proof of the theorem. □

4. Remarks

4.1. **The Volterra Integral Operator.** In the theory of integral equations, the classical *Volterra operator* V of integration has some remarkably special properties [6]. The operator V is defined as follows: for each $f \in \mathrm{L}^2([0,1])$, let $Vf \in \mathrm{L}^2([0,1])$ be given by

$$Vf(t) = 2i \int_t^1 f(s)\, ds, \quad f \in \mathrm{L}^2([0,1]),\ t \in [0,1].$$

In the context of our work in this paper, the operator V is unicellular, which in infinite dimensions is to say that its closed invariant subspaces are totally ordered by inclusion.

A question raised many years ago by W. Arveson [1, page 218] asks whether the norms $\|f(V)\|$, for $f \in \mathbb{C}[t]$, determine the unitary similarity class of V in the set of irreducible compact operators on $L^2([0,1])$. Although this question remains open, we point out below that given any $\varepsilon > 0$ there is a unicellular piece Q of the Volterra operator whose norms $\|f(Q)\|$ determine its unitarily similarity class and such that Q is within ε of V uniformly on $L^2([0,1])$.

Proposition 4.1. *For every $\varepsilon > 0$ there is a finite-dimensional subspace $\mathrm{L} \subset L^2([0,1])$ such that, if P denotes the projection onto L, then*

(1) *$PVP_{|\mathrm{L}}$ is a unicellular operator whose unitary similarity orbit, as an operator on L, is completely determined by the norms $\|f(PVP_{|\mathrm{L}})\|$, for $f \in \mathbb{C}[t]$, and*
(2) *$\|PVP - V\| < \varepsilon$.*

Proof. We shall use an approximation scheme of E.B. Davies and B. Simon [4] which they employed to compute the norm of V. For each positive integer m, let H_m be the Hilbert space spanned by the m orthonormal functions $\sqrt{m}\chi_{E_j}$, $0 \le j \le n-1$, where $E_j = [\frac{j}{m}, \frac{j+1}{m})$. If P_m is the projection with range H_m, then $P_m V P_m$ considered as an operator on H_m has a matrix representation with respect to this orthonormal basis of H_m that is given by

$$P_m V P_{m|\mathrm{H}_m} \;=\; \frac{i}{m}\,(1+2Q)\;,$$

where Q is the unicelluar Toeplitz operator acting on $\mathbb{C}^m$ described in Theorem 2.1. As $1+2Q$ is a generator of the upper triangular Toeplitz matrices, the unitary similarity orbit of $P_m V P_{m|\mathrm{H}_m}$ is completely determined by the norms $\|f(P_m V P_{m|\mathrm{H}_m})\|$, for $f \in \mathbb{C}[t]$.

The sequence $\{P_m\}_m$ of finite-rank projections P_m converges strongly to the identity operator. Hence, because V is a compact operator, there is an m such that $\|P_m V P_m - V\| < \varepsilon$. □

4.2. **Scalar versus higher-dimensional phenomena.** Theorem 3.2 is linked to higher dimensional phenomena encoded by the *matricial spectrum* of $A \in \mathrm{M}_n$ [5].

Because for every $A \in \mathrm{M}_n$ the unital algebra $\mathrm{Alg}\,A$ is abelian, there exist unital homomorphisms $\mathrm{Alg}\,A \to \mathrm{M}_k$, for all $1 \le k \le n-1$. For a given k, let $\mathrm{Hom}\,(A, \mathrm{M}_k)$ denote the set of all unital homomorphisms $\mathrm{Alg}\,A \to \mathrm{M}_k$. If $\rho \in \mathrm{Hom}\,(A, \mathrm{M}_k)$, then there is a k-dimensional subspace $\mathrm{L} \subseteq \mathbb{C}^n$ such that $\rho(A) \sim PAP_{|\mathrm{L}}$, where $P \in \mathrm{M}_n$ is the unique (selfadjoint) projection with range L. This subspace L is necessarily *semi-invariant* under A; conversely, every k-dimensional semi-invariant subspace of A determines an element $\rho \in \mathrm{Hom}\,(A, \mathrm{M}_k)$ [7, Theorem 3.3.1].

It is natural to consider the values of $\rho \in \mathrm{Hom}\,(A, \mathrm{M}_k)$ as higher order spectra. Specifically, consider the k-th matricial spectrum of A:

$$\sigma_k(A) \;=\; \{\Lambda \in \mathrm{M}_k \,|\, \Lambda = \rho(A) \text{ for some } \rho \in \mathrm{Hom}\,(A, \mathrm{M}_k)\}\,.$$

This set is closed under unitary similarity, and is itself a unitary similarity invariant of A. Theorem 3.2 is formulated in the context of invariant subspaces; if one strengthens that to semi-invariant subspaces on the one hand, then a slightly weaker hypothesis on B is afforded on the other hand.

Proposition 4.2. *Assume $A, B \in \mathrm{M}_n$ and that A is unicellular. If for each $1 \le k \le n$ and each $\rho \in \mathrm{Hom}\,(A, \mathrm{M}_k)$ there is a $\varrho \in \mathrm{Hom}\,(B, \mathrm{M}_k)$ such that*

$$\|\varrho(f(B))\| \;=\; \|\rho(f(A))\|\,,\ \forall\, f \in \mathbb{C}[t]\,,$$

then $B \sim A$.

Proof. As in the proof of Theorem 3.2, one can assume A and B are nilpotent and in upper triangular form with nonnegative entries along the superdiagonal above the main diagonal. With A, the entries $a_{i,i+1}$ are positive. The proof of Theorem 3.2 uses the unicellularity of B for only one purpose: to deduce $b_{i,i+1} > 0$, for $1 \leq i < n$. But this can be achieved, for a given i, by considering $\Lambda = PAP_{|\mathrm{L}}$ and $\Omega = PBP_{|\mathrm{L}}$, where $\mathrm{L} = \mathrm{Span}\,\{e_i, e_{i+1}\}$. In this case, $\rho(X) = PXP_{|\mathrm{L}}$, for $X \in \mathrm{Alg}\,A \cup \mathrm{Alg}\,B$, defines an element of $\mathrm{Hom}\,(A, \mathrm{M}_2)$ and $\mathrm{Hom}\,(B, \mathrm{M}_2)$ such that

$$\Lambda \sim \begin{bmatrix} 0 & a_{i,i+1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Omega \sim \begin{bmatrix} 0 & b_{i,i+1} \\ 0 & 0 \end{bmatrix}.$$

Thus, $0 \neq \|\Lambda\| = \|\Omega\| = b_{i,i+1}$. □

The power of working in higher dimensions is amply illustrated by an important theorem of Arveson [2]: If $A, B \in \mathrm{M}_n$ are irreducible, then $A \sim B$ if and only if $\|A \otimes C + 1 \otimes D\| = \|B \otimes C + 1 \otimes D\|$, for all $C, D \in \mathrm{M}_n$. This is to say that the norms of polynomials (of degree at most 1) in A, over the ring M_n, determine A up to unitary similarity. In comparison, Theorem 3.2 and Proposition 4.2 represent a hybrid of the matricial and scalar environments.

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