# Criterion of unitary similarity for upper triangular matrices in general position

Douglas Farenick* Vyacheslav Futorny† 
Tatiana G. Gerasimova‡ Vladimir V. Sergeichuk§ 
Nadya Shvai‡

Dedicated to the Memory of Aleksandr Shvai, the husband of Nadya Shvai, who died tragically at age 23

**Abstract**

Each square complex matrix is unitarily similar to an upper triangular matrix with diagonal entries in any prescribed order. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices that

- are not similar to direct sums of matrices of smaller sizes, or
- are in general position and have the same main diagonal.

We prove that $A$ and $B$ are unitarily similar if and only if

$$\|h(A_k)\| = \|h(B_k)\| \qquad \text{for all } h \in \mathbb{C}[x] \text{ and } k = 1, \dots, n,$$

where $A_k := [a_{ij}]_{i,j=1}^k$ and $B_k := [b_{ij}]_{i,j=1}^k$ are the principal $k \times k$ submatrices of $A$ and $B$ and $\|\cdot\|$ is the Frobenius norm.

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*Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada. Email: Doug.Farenick@uregina.ca. Supported in part by NSERC.

†Department of Mathematics, University of São Paulo, São Paulo, Brazil. Email: futorny@ime.usp.br. Supported in part by the CNPq grant (301743/2007-0) and by the Fapesp grant (2010/50347-9).

‡Faculty of Mechanics and Mathematics, Kiev National Taras Shevchenko University, Volodymyrska 64, Kiev, Ukraine. Emails: gerasimova@imath.kiev.ua (T.G. Gerasimova) and nadiia.shvai@gmail.com (N. Shvai). The work was started while these authors were visiting the University of Regina supported by NSERC, the University of Regina Visiting Graduate Student Research Program, and the Svjatoslav Vakarchuk Foundation "People of Future".

§Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine. Email: sergeich@imath.kiev.ua. Supported in part by the Fapesp grant (2010/07278-6).

# 1 Introduction

A classical problem in linear algebra is the following one: if $A$ and $B$ are square complex matrices, then how can one determine whether $A$ and $B$ are unitarily similar (i.e., $U^{-1}AU = B$ for a unitary $U$)? More precisely, which invariants completely determine a matrix up to unitary similarity?

Let us recall the most known solutions to this problem:

**Specht's theorem.** *Matrices $A$ and $B$ are unitarily similar if and only if*

$$\text{trace}\,\omega(A, A^*) = \text{trace}\,\omega(B, B^*)$$

*for all words $\omega$ in two noncommuting variables,* see [7].

**Littlewood's canonical matrices.** Littlewood [5] constructed an algorithm that reduces each square complex matrix $A$ by transformations of unitary similarity to some matrix $A_{\text{can}}$ in such a way that $A$ and $B$ are unitarily similar if and only if they are reduced to the same matrix $A_{\text{can}} = B_{\text{can}}$. Thus, the matrices that are not changed by Littlewood's algorithm are *canonical with respect to unitary similarity.* We use Littlewood's canonical matrices in this paper (see Remark 7). Systems of linear mappings on unitary and Euclidean spaces (i.e., unitary and Euclidean representations of quivers) were studied in [6] using Littlewood's algorithm.

**Arveson's criterion.** Let $A$ and $B$ be $n \times n$ complex matrices such that each of them is not unitarily similar to a direct sum of square matrices of smaller sizes. Arveson [1, Theorems 2 and 3] proved that $A$ *and* $B$ *are unitarily similar if and only if*

$$\|H_0 \otimes I_n + H_1 \otimes A\|_{op} = \|H_0 \otimes I_n + H_1 \otimes B\|_{op} \tag{1}$$

*for all $H_0, H_1 \in \mathbb{C}^{n\times n}$, where $\|M\|_{op} := \max_{|v|=1} |Mv|$ is the operator norm and $|\cdot|$ stands for the Euclidean norm of vectors.*

For each matrix polynomial

$$H(x) = H_0 + H_1x + \cdots + H_tx^t \in \mathbb{C}^{k\times k}[x],$$

whose coefficients $H_i$ are $k \times k$ matrices, we define its value at an $n \times n$ matrix $M$ as follows:

$$H(M) := H_0 \otimes I_n + H_1 \otimes M + \cdots + H_t \otimes M^t \in \mathbb{C}^{kn\times kn}.$$

The condition (1) means that

$$\|H(A)\|_{op} = \|H(B)\|_{op} \tag{2}$$

for all matrix polynomials $H \in \mathbb{C}^{n\times n}[x]$ of degree at most 1. For some class of operators on a Hilbert space, Arveson [2, Theorem 2.3.2] proved that two operators $A$ and $B$ are unitarily similar if and only if the condition (2) holds for all (possibly, nonlinear) $H \in \mathbb{C}^{k\times k}[x]$.

The purpose of this paper is to give a criterion of unitary similarity of matrices that is analogous to Arveson's criterion (2), but in which polynomials over $\mathbb{C}$ are used instead of linear polynomials over $\mathbb{C}^{n\times n}$. All matrices that we consider are complex matrices.

We study only the finite dimensional case, and so we can and will use the *Frobenius norm*

$$\|A\| := \sqrt{\sum |a_{ij}|^2}, \qquad \text{where } A = [a_{ij}] \in \mathbb{C}^{n\times n},$$

instead of the operator norm. The *Frobenius norm of a linear operator* on a unitary space is the Frobenius norm of its matrix in any orthonormal basis. This definition is correct since the Frobenius norm of a matrix does not change under multiplication by unitary matrices. Hence, if $A$ and $B$ are unitarily similar matrices, then $\|A\| = \|B\|$; moreover,

$$\|h(A)\| = \|h(B)\| \qquad \text{for all } h \in \mathbb{C}[x]. \tag{3}$$

The converse statement is not true; the condition (3) does not ensure the unitary similarity of matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

are not unitarily similar and satisfy (3); see Lemma 9. But their $2 \times 2$ principal submatrices

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

do not satisfy (3). (By the $k \times k$ *principal submatrix* $M_k$ of a matrix $M$, we mean the submatrix at the intersection of the first $k$ rows and the first $k$

columns.) For this reason, we give a criterion of unitary similarity, in which the condition (3) is imposed not only on $n \times n$ matrices $A$ and $B$, but also on their principal submatrices:

$$\|h(A_k)\| = \|h(B_k)\| \qquad \text{for all } h \in \mathbb{C}[x] \text{ and } k = 1, \dots, n. \tag{5}$$

We prove that the condition (5) ensures the unitary similarity of *upper triangular* $n \times n$ matrices $A$ and $B$ in two cases:

- if $A$ and $B$ are not similar to direct sums of square matrices of smaller sizes (Theorem 1), and
- if $A$ and $B$ are in general position (Theorem 4).

We consider only upper triangular matrices because of the Schur unitary triangularization theorem [4, Theorem 2.3.1]: *every square matrix $A$ is unitarily similar to an upper triangular matrix $B$ whose diagonal entries are complex numbers in any prescribed order*; say, in the lexicographical order:

$$a + bi \preccurlyeq c + di \qquad \text{if either } a < c, \text{ or } a = c \text{ and } b \leqslant d. \tag{6}$$

A unitary matrix $U$ that transforms $A$ to $B = U^{-1}AU$ is easily constructed: we reduce $A$ by similarity transformations to an upper triangular matrix $S^{-1}AS$ with diagonal entries in the prescribed order (this matrix can be obtained from the Jordan form of $A$ by simultaneous permutations of rows and columns), then apply the Gram-Schmidt orthogonalization to the columns of $S$ and obtain a desired unitary matrix $U = ST$, where $T$ is upper triangular.

# 2 Main results

## 2.1 Criterion for indecomposable matrices and unicellular operators

We say that a matrix is *indecomposable for similarity* if it is not similar to a direct sum of square matrices of smaller sizes. This means that the matrix is similar to a Jordan block. Thus, a matrix is indecomposable with respect to similarity if and only if it is unitarily similar to a matrix of the form

$$A = \begin{bmatrix} \lambda & a_{12} & \dots & a_{1n} \\ & \lambda & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda \end{bmatrix}, \qquad \text{all } a_{i,i+1} \neq 0. \tag{7}$$

In Section 3 we prove the following theorem, which is the first main result of the paper.

**Theorem 1.** *Let $A$ and $B$ be $n \times n$ upper triangular matrices that are indecomposable with respect to similarity. Then $A$ and $B$ are unitarily similar if and only if*

$$\|h(A_k)\| = \|h(B_k)\| \qquad \text{for all } h \in \mathbb{C}[x] \text{ and } k = 1, \dots, n, \tag{8}$$

*where $A_k$ and $B_k$ are the principal $k \times k$ submatrices of $A$ and $B$.*

Now we give the operator form of this criterion. Two operators $\mathcal{A}$ and $\mathcal{B}$ on a unitary space are *unitarily similar* if there exists a unitary operator $\mathcal{U}$ such that $\mathcal{U}^{-1}\mathcal{A}\mathcal{U} = \mathcal{B}$. A linear operator $\mathcal{A} : U \to U$ on an $n$-dimensional unitary space $U$ is said to be *unicellular* if it satisfies one of the following 3 equivalent conditions:

- its matrix is indecomposable with respect to similarity;
- there exist no invariant subspaces $U'$ and $U''$ of $\mathcal{A}$ such that

$$\dim U' + \dim U'' = n, \qquad U' \cap U'' = 0;$$

- all invariant subspaces of $\mathcal{A}$ form the chain

$$0 \subset U_1 \subset U_2 \subset \dots \subset U_n = U, \qquad \dim U_i = i \text{ for all } i.$$

**Corollary 2.** (a) *Let $\mathcal{A}$ and $\mathcal{B}$ be unicellular linear operators on an $n$-dimensional unitary space $U$ with the chains of invariant subspaces*

$$0 \subset U_1 \subset U_2 \subset \dots \subset U_n = U, \qquad 0 \subset V_1 \subset V_2 \subset \dots \subset V_n = U,$$

*and let*

$$\mathcal{A}_i := \mathcal{A}|U_k, \qquad \mathcal{B}_k := \mathcal{B}|V_k$$

*be the restrictions of $\mathcal{A}$ and $\mathcal{B}$ to their invariant subspaces. Then $\mathcal{A}$ and $\mathcal{B}$ are unitarily similar if and only if*

$$\|h(\mathcal{A}_k)\| = \|h(\mathcal{B}_k)\| \qquad \text{for all } h \in \mathbb{C}[x] \text{ and } k = 1, \dots, n. \tag{9}$$

(b) *In particular, two nilpotent linear operators $\mathcal{A}$ and $\mathcal{B}$ of rank $n-1$ on an $n$-dimensional unitary space are unitarily similar if and only if* (9) *holds for the restrictions $\mathcal{A}_k$ and $\mathcal{B}_k$ of the operators to the images of $\mathcal{A}^k$ and $\mathcal{B}^k$.*

Let (9) hold. Then $\mathcal{A}$ and $\mathcal{B}$ have the same eigenvalue: if $\lambda$ is the eigenvalue of $\mathcal{A}$ and $h(x) := (x-\lambda)^n$, then $\|h(\mathcal{B})\| = \|h(\mathcal{A})\| = 0$ and so $\lambda$ is the eigenvalue of $\mathcal{B}$. Hence, the canonical isomorphism of one-generated algebras

$$\mathbb{C}[\mathcal{A}_k] \simeq \mathbb{C}[\mathcal{B}_k], \qquad \mathcal{A}_k \mapsto \mathcal{B}_k \tag{10}$$

is defined correctly: the algebras are isomorphic to $\mathbb{C}[x]/(x-\lambda)^k\mathbb{C}[x]$.

**Corollary 3.** *Two unicellular linear operators $\mathcal{A}$ and $\mathcal{B}$ on an $n$-dimensional unitary space are unitarily similar if and only if they have the same eigenvalue and the canonical isomorphism* (10) *is isometric* (*i.e., it preserves the norm*) *for each $k = 1, \dots, n$.*

## 2.2 Criterion for matrices in general position

Theorem 1 is not extended to matrices with several eigenvalues: we prove in Lemma 10 that each two matrices of the form

$$A := \begin{bmatrix} 0 & 1 & -1 & a \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 1 & -1 & b \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad \begin{array}{c} a \neq b, \\ |a| = |b| = 1, \end{array} \tag{11}$$

are not unitarily similar but satisfy (8). Nevertheless, in this section we extend Theorem 1 to "almost all" upper triangular matrices as follows.

Let

$$X_n := \begin{bmatrix} x_{11} & \dots & x_{1n} \\ & \ddots & \vdots \\ 0 & & x_{nn} \end{bmatrix} \tag{12}$$

be a matrix whose upper triangular entries are variables; denote by $\mathbb{C}[x_{ij} | i \leqslant j \leqslant n]$ the set of polynomials in these variables. For simplicity of notation, we write $f\{X_n\}$ instead of $f(x_{11}, x_{12}, x_{22}, \dots)$.

For each $f \in \mathbb{C}[x_{ij}|i \leqslant j \leqslant n]$, write

$$M_n(f) := \{A \in \mathbb{C}^{n\times n} \,|\, A \text{ is upper triangular and } f\{A\} \neq 0\}. \tag{13}$$

For example, if

$$\varphi_n\{X_n\} := x_{12}x_{23}\cdots x_{n-1,n}\prod_{i<j}(x_{ii}-x_{jj}), \tag{14}$$

then $M_n(\varphi_n)$ consists of matrices of the form

$$\begin{bmatrix} \lambda_1 & a_{12} & \dots & a_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}, \qquad \begin{matrix} \lambda_i \neq \lambda_j \text{ if } i \neq j, \\ \text{all } a_{i,i+1} \neq 0. \end{matrix} \tag{15}$$

We say that $n \times n$ upper triangular matrices *in general position* possess some property if there exists a nonzero polynomial $f_n \in \mathbb{C}[x_{ij}|i \leqslant j \leqslant n]$ such that all matrices in $M_n(f_n)$ possess this property. Thus, this property holds for all matrices in $\mathbb{C}^{n\times n}$ except for matrices from an algebraic variety of smaller dimension.[1]

The second main result of the paper is the following theorem.

**Theorem 4.** *Two $n \times n$ upper triangular matrices $A$ and $B$ in general position with lexicographically ordered eigenvalues on the main diagonal* (*see* (6)) *are unitarily similar if and only if*

$$\|h(A_k)\| = \|h(B_k)\| \qquad \text{for all } h \in \mathbb{C}[x] \text{ and } k = 1,\dots,n, \tag{16}$$

*where $A_k$ and $B_k$ are the principal $k \times k$ submatrices of $A$ and $B$.*

Theorem 4 is an existence theorem: "$A$ and $B$ in general position" means "$A, B \in M_n(f_n)$ for some $f_n$". In Theorem 5, we give $f_n$ in an explicit form.

[1] In algebraic geometry, when a family of objects $\{X_p\}_{p\in\Sigma}$ is parametrized by the points of an irreducible algebraic variety $\Sigma$, the statement that "the general object $X$ has a property $\mathcal{P}$" is taken to mean that "the subset of points $p \in \Sigma$ such that the corresponding object $X_p$ has the property $\mathcal{P}$ contains a Zariski open dense subset of $\Sigma$", see [3, p. 54].

For each $n \geqslant 2$ and $r = 1, 2, \dots, n$, define the $n \times n$ matrix

$$G^{(n,r)}\{X_n\} = [g_{ij}^{(n,r)}\{X_n\}]$$
$$:= \begin{cases} (X_n - x_{22}I_n)(X_n - x_{33}I_n)\cdots(X_n - x_{nn}I_n) & \text{if } r = 1, \\ (X_n - x_{11}I_n)(X_n - x_{22}I_n)\cdots(X_n - x_{r-1,r-1}I_n) & \text{if } r > 1. \end{cases} \tag{17}$$

Its entries $g_{ij}^{(n,r)}\{X_n\}$ are polynomials in entries of (12). Write

$$f_n := \begin{cases} \varphi_n & \text{if } n = 1, 2, 3, \\ \varphi_n \cdot g_{14}^{(4,1)} g_{15}^{(5,1)} \cdots g_{1n}^{(n,1)} \cdot g_{13}^{(3,3)} g_{14}^{(4,4)} \cdots g_{1,n-1}^{(n-1,n-1)} & \text{if } n \geqslant 4. \end{cases} \tag{18}$$

in which $\varphi_n$ is defined in (14). Theorem 4 results from the following theorem, which is proved in Section 5.

**Theorem 5.** *Matrices $A, B \in M_n(f_n)$ are unitarily similar and have the same main diagonal if and only if* (16) *holds.*

By this theorem and the top equality in (18), *two matrices $A$ and $B$ of the form* (15) *of size at most $3 \times 3$ are unitarily similar if and only if* (16) *holds.*

# 3 Proof of Theorem 1

**Lemma 6.** (a) *For each*

$$A = \begin{bmatrix} \lambda_1 & a_{12} & \dots & a_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}, \qquad \textit{all } a_{i,i+1} \neq 0, \tag{19}$$

*there exists a diagonal unitary matrix $U$ such that all the entries of the first superdiagonal of $U^{-1}AU$ are positive real numbers.*

(b) *Let*

$$A = \begin{bmatrix} \lambda_1 & a_{12} & \dots & a_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}, \qquad \textit{all } a_{i,i+1} \textit{ are positive real,} \tag{20}$$

*and*

$$B = \begin{bmatrix} \lambda_1 & b_{12} & \dots & b_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & b_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}, \qquad \text{all } b_{i,i+1} \text{ are positive real.}$$

*If $A$ and $B$ are unitarily similar, then $A = B$. Moreover, if $U^{-1}AU = B$ and $U$ is a unitary matrix, then $U = uI_n$ for some $u \in \mathbb{C}$ with $|u| = 1$.*

*Proof.* (a) Write $a_{i,i+1}$ in the form $r_i u_i$, in which $r_i$ is a positive real number and $|u_i| = 1$. Then

$$U^{-1} := \operatorname{diag}(1, u_1, u_1u_2, u_1u_2u_3, \dots)$$

is the desired matrix.

(b) Let $U^{-1}AU = B$, in which $U$ is a unitary matrix. Equating the entries of $AU = UB$ along diagonals starting at the lower left diagonal (i.e., from the entry $(n,1)$) and finishing at the main diagonal, we find that $U$ is upper triangular. Since $U$ is unitary, it is a diagonal matrix: $U = \operatorname{diag}(u_1, \dots, u_n)$. Equating the entries of $AU = UB$ along the first superdiagonal, we find that $u_1 = \cdots = u_n$. Hence, $U = uI_n$ and $A = B$. □

**Remark 7.** By Lemma 6(b), any two matrices of the form (20) in which the diagonal entries are lexicographically ordered (i.e., $\lambda_1 \preccurlyeq \cdots \preccurlyeq \lambda_n$, see (6)) are either equal or unitarily dissimilar. These matrices are Littlewood's canonical forms of matrices (19); see the beginning of Section 1.

**Lemma 8.** *Each matrix of the form*

$$A = \begin{bmatrix} \lambda & a_{12} & \dots & a_{1n} \\ & \lambda & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda \end{bmatrix}, \qquad \text{all } a_{i,i+1} \text{ are positive real,} \tag{21}$$

*is fully determined by the indexed family of real numbers*

$$\{\|h(A_k)\|\}_{(h,k)}, \qquad \text{in which } h \in \mathbb{C}[x] \text{ and } k = 1, \dots, n. \tag{22}$$

*Proof.* Let $h$ be a nonzero polynomial of minimal degree such that $\|h(A)\| = 0$. Then $h(A) = 0$ and so $h(A) = (x - \lambda)^n$. Thus, $\lambda$ is determined by (22).

Write $B := A - \lambda I_n$. Then (22) determines the family

$$\{\|h(B_k)\|\}_{(h,k)}, \qquad \text{in which } h \in \mathbb{C}[x] \text{ and } k = 1, \dots, n. \tag{23}$$

The positive real number $a_{12}$ is determined by (23) since $\|B_2\| = a_{12}$. This proves the lemma for $n = 1$ and 2.

Reasoning by induction on $n$, we assume that $n \geqslant 3$ and $B_{n-1}$ is determined by (23). Since all entries of $B^{n-1}$ are zero except for the $(1, n)$ entry, which is the positive real number

$$c := a_{12}a_{23} \cdots a_{n-1,n},$$

we have $\|B^{n-1}\| = c$. Thus, $a_{n-1,n}$ is determined by (23).

Reasoning by induction, we assume that $a_{n-1,n}, a_{n-2,n}, \dots, a_{r+1,n}$ are determined by (23) and find $a_{rn}$. Let $\alpha$ be a complex number for which $\|B^r - \alpha B^{n-1}\|$ is minimal. Then the $(1, n)$ entry of $B^r - \alpha B^{n-1}$ is

$$a_{12}a_{23} \cdots a_{r-1,r}a_{rn} + \cdots + \alpha c = 0.$$

Since the unspecified summands do not contain $a_{rn}$ and only $a_{rn}$ is unknown in this equality, it determines $a_{rn}$. □

*Proof of Theorem 1.* Let $M$ be an $n \times n$ upper triangular matrix that is indecomposable with respect to similarity. By Lemma 6(a), $M$ is unitarily similar to a matrix $A$ of the form (21) via a diagonal unitary matrix. Then $\|h(M_k)\| = \|h(A_k)\|$ for all $h \in \mathbb{C}[x]$ and $k = 1, \dots, n$. Thus, it suffices to prove Theorem 1 for matrices of the form (21).

"$\Rightarrow$" Let $A$ and $B$ of the form (21) be unitarily similar. By Lemma 6(b), $A = B$, and so (8) holds.

"$\Leftarrow$" Let $A$ and $B$ of the form (21) satisfy (8). By Lemma 8, $A = B$ since their indexed families $\{\|h(A_k)\|\}_{(h,k)}$ and $\{\|h(B_k)\|\}_{(h,k)}$ coincide. □

# 4 Counterexamples

## 4.1 Condition (3) does not ensure the unitary similarity

In this section, we give examples of matrices of the form (20) (and even of the form (21)), for which the condition (3) does not ensure their unitary similarity.

For each square matrix $A$, denote by $A^S$ its transpose with respect to the secondary diagonal:

$$A^S = ZA^T Z, \qquad Z := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

For instance, $B = A^S$ in (4).

**Lemma 9.** *Let $A$ be a matrix of the form* (20) *such that $A \neq A^S$ and the main diagonals of $A$ and $A^S$ coincide* (*i.e., the main diagonal of $A$ is symmetric*)*. Then $A$ and $B := A^S$ satisfy* (3)*, but they are not unitarily similar.*

*Proof.* The condition (3) holds for $A$ and $B = A^S$, because $\|h(A^S)\| = \|h(ZA^T Z)\| = \|Zh(A^T)Z\| = \|h(A^T)\| = \|h(A)^T\| = \|h(A)\|$.

Since $A \neq A^S$, $A$ and $A^S$ are not unitarily similar by Lemma 6(b). ☐

## 4.2 Theorem 1 is not extended to matrices with several eigenvalues

Theorem 1 was proved for matrices of the form (7); let us show that it is not extended to matrices of the form (15).

**Lemma 10.** *Matrices $A$ and $B$ of the form* (11) *are not unitarily similar but satisfy* (8)*.*

*Proof.* By Lemma 6(b), $A$ and $B$ are not unitarily similar since $a \neq b$.

Let us prove, that $A$ and $B$ satisfy (8). Write

$$M_c := \begin{bmatrix} 0 & 1 & -1 & c \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad \text{in which } c \in \mathbb{C} \text{ and } |c| = 1,$$

and take any $h \in \mathbb{C}[x]$.

It suffices to prove that $\|h(M_c)\|$ does not depend on $c$. Let $r(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$ be the residue of division of $h(x)$ by the characteristic polynomial of $M_c$. Then

$$h(M_c) = r(M_c) = \alpha I_4 + \beta M_c + \gamma \begin{bmatrix} 0 & 1 & -1 & 3c \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 9 \end{bmatrix} + \delta \begin{bmatrix} 0 & 1 & -1 & 9c \\ 0 & 1 & 7 & 19 \\ 0 & 0 & 8 & 19 \\ 0 & 0 & 0 & 27 \end{bmatrix},$$

and so

$$\begin{aligned} \|h(M_c)\|^2 = \|r(M_c)\|^2 =& \|r(M_0)\|^2 + |\beta c + \gamma 3c + \delta 9c|^2 \\ =& \|r(M_0)\|^2 + |\beta + 3\gamma + 9\delta|^2 |c|^2 \\ =& \|r(M_0)\|^2 + |\beta + 3\gamma + 9\delta|^2 \end{aligned}$$

does not depend on $c$. ☐

Note that $M_c \notin M_4(f_4)$ (in which $M_4(f_4)$ from Theorem 5) since $g_{13}^{(3,3)}\{M_c\} = 0$.

# 5 Proof of Theorem 5

In this section, $M_n(f)$ is the set (13) and $f_n$ is the polynomial (18).

**Lemma 11.** *Let $G^{(n,r)}$ be the matrix defined in* (17).

(a) *Only the first row of $G^{(n,1)}$ is nonzero.*

(b) *The matrix $G^{(n,r)}$ with $2 \leqslant r \leqslant n$ has the form*

$$\begin{bmatrix} 0_{r-1} & * \\ 0 & T \end{bmatrix}, \tag{24}$$

*in which $0_{r-1}$ is the $(r-1) \times (r-1)$ zero matrix and $T$ is upper triangular.*

(c) *The matrix $G^{(r,r)}$ with $2 \leqslant r < n$ is the $r \times r$ principal submatrix of $G^{(n,r)}$.*

*Proof.* For every $i = 1, \dots, n$, let $P_i$ be any $n \times n$ upper triangular matrix, in which the $(i,i)$ entry is zero. Then

$$P_1 \cdots P_n = \begin{bmatrix} 0 & & & * \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix} \begin{bmatrix} * & & & * \\ & 0 & & \\ & & * & \\ 0 & & & \ddots \end{bmatrix} \cdots \begin{bmatrix} * & & & * \\ & \ddots & & \\ & & * & \\ 0 & & & 0 \end{bmatrix} = 0. \tag{25}$$

This equality is proved by induction on $n$: if it holds for $n-1$, then the product of the $(n-1)\times(n-1)$ principal submatrices of $P_1,\dots,P_{n-1}$ is zero, and so

$$(P_1\cdots P_{n-1})P_n = \begin{bmatrix} 0 & \dots & 0 & * \\ & \ddots & \vdots & \vdots \\ & & 0 & * \\ 0 & & & * \end{bmatrix} \begin{bmatrix} * & & & * \\ & \ddots & & \\ & & * & \\ 0 & & & 0 \end{bmatrix} = 0.$$

(a) In the product of matrices (17) that defines $G^{(n,1)}$, we remove the first row and the first column in each of its factors. Then apply (25) to the obtained product.

(b) In the product of matrices (17) that defines $G^{(n,r)}$ with $r \geqslant 2$, we replace each factor by its $(r-1)\times(r-1)$ principal submatrix. Then apply (25) to the obtained product.

(c) This statement follows from (17). □

**Lemma 12.** *If $A \in M_n(f_n)$ and $S$ is a nonsingular diagonal matrix, then $S^{-1}AS \in M_n(f_n)$.*

*Proof.* Let $A \in M_n(f_n)$ and let $S$ be a nonsingular diagonal matrix. For each $i$, the $(i,i)$ entries of $A$ and $S^{-1}AS$ coincide and $S^{-1}AS - a_{ii}I_n = S^{-1}(A - a_{ii}I_n)S$. Thus, $G^{(n,r)}\{S^{-1}AS\} = S^{-1}G^{(n,r)}\{A\}S$ for each $r$, and so the corresponding entries of $G^{(n,r)}\{A\}$ and $G^{(n,r)}\{S^{-1}AS\}$ are simultaneously zero or nonzero. Taking into account the definition (18) of $f_n$, we get $S^{-1}AS \in M_n(f_n)$. □

The following lemma is analogous to Lemma 8.

**Lemma 13.** *Each matrix $A \in M_n(f_n)$, in which all entries of the first superdiagonal are positive real numbers, is fully determined by the indexed family of real numbers*

$$\{\|h(A_k)\|\}_{(h,k)}, \qquad \text{in which } h \in \mathbb{C}[x] \text{ and } k = 1,\dots,n. \tag{26}$$

*Proof.* The matrix $A$ has the form

$$\begin{bmatrix} \lambda_1 & a_{12} & \dots & a_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}, \qquad \begin{matrix} \lambda_i \neq \lambda_j \text{ if } i \neq j, \\ \text{all } a_{i,i+1} \text{ are positive real.} \end{matrix} \tag{27}$$

For each $k$, the minimal polynomial $\mu_k(x)$ of $A_k$ is determined by the family (26). Since $\lambda_i \neq \lambda_j$ if $i \neq j$, $\mu_k(x)$ is the characteristic polynomial of $A_k$, and so $\mu_k(x)/\mu_{k-1}(x) = x - \lambda_k$. Thus, the main diagonal of $A$ is determined by (26).

The entry $a_{12}$ of $A$ is also determined by (26) since $a_{12}$ is a positive real number, $\|A_2\|$ is determined by (26), and

$$\|A_2\|^2 = |\lambda_1|^2 + |\lambda_2|^2 + a_{12}^2.$$

This proves Lemma 13 for $n = 2$.

Let $n \geqslant 3$. Since $f_{n-1}$ divides $f_n$, we can use induction on $n$, and so we assume that (26) determines $A_{n-1}$.

Let us find $a_{n-1,n}$. For each $r = 2, 3, \ldots, n-1$, define the $n \times n$ matrix

$$B^{(r)} = [b_{ij}^{(r)}] := G^{(n,r)}\{A\}(A - \lambda_n I_n) = (A - \lambda_1 I_n) \cdots (A - \lambda_{r-1} I_n)(A - \lambda_n I_n).$$

Since $G^{(n,n-1)}\{A\}$ has the form (24) and the $(n, n)$ entry of $A - \lambda_n I_n$ is zero, the last column of $B^{(n-1)}$ is $va_{n-1,n}$, in which

$$v := [g_{1,n-1}^{(n,n-1)}\{A\}, \ldots, g_{n-1,n-1}^{(n,n-1)}\{A\}, 0]^T \tag{28}$$

is the $(n-1)$th column of $G^{(n,n-1)}\{A\}$. The column $v$ is known since it is determined by $A_{n-1}$. The first coordinate of $v$ is nonzero since by Lemma 11(c)

$$g_{1r}^{(n,r)}\{A\} = g_{1r}^{(r,r)}\{A\} \neq 0, \qquad r = 2, \ldots, n-1; \tag{29}$$

these are nonzero since each $g_{1r}^{(r,r)}$ divides $f_n$ (note that $g_{12}^{(2,2)} = x_{12}$).

Thus, $\|v\| \neq 0$. The positive real number $a_{n-1,n}$ is fully determined by the equality

$$\|B^{(n-1)}\|^2 = \|B_{n-1}^{(n-1)}\|^2 + \|v\|^2 a_{n-1,n}^2,$$

in which $B_{n-1}^{(n-1)}$ is the $(n-1)$-by-$(n-1)$ principal submatrix of $B^{(n-1)}$.

We have determined the matrix $B^{(n-1)}$ too since its last column is $va_{n-1,n}$.

Let us consider the space $\mathbb{C}^{n \times n}$ of $n$-by-$n$ matrices as the unitary space with scalar product

$$(X, Y) := \sum_{i,j} x_{ij} \bar{y}_{ij}, \qquad X = [x_{ij}],\ Y = [y_{ij}] \in \mathbb{C}^{n \times n}.$$

This scalar product is expressed via the Frobenius norm due to the polarization identity

$$(X,Y)=\frac{1}{4}(\|X+Y\|^2-\|X-Y\|^2)+\frac{i}{4}(\|X+iY\|^2-\|X-iY\|^2).$$

By Lemma 11(a),

$$C:=G^{(n,1)}\{A\}=(A-\lambda_2 I_n)\cdots(A-\lambda_n I_n)=\begin{bmatrix}c_1&c_2&\dots&c_n\\ &0&\dots&0\\ &&\ddots&\vdots\\ 0&&&0\end{bmatrix},$$

in which $c_1,\dots,c_{n-1}$ are known. Since the main diagonal of $A$ is determined by (26), $\|C\|$ is determined by (26) too. Using the polarization identity, we find $(B^{(n-1)},C)$. Then we find $c_n$ from the equality

$$(B^{(n-1)},C)=b_{11}^{(n-1)}\bar c_1+\dots+b_{1,n-1}^{(n-1)}\bar c_{n-1}+b_{1n}^{(n-1)}\bar c_n,$$

in which $b_{1n}^{(n-1)}=g_{1,n-1}^{(n,n-1)}\{A\}a_{n-1,n}\neq 0$ (see (28) and (29)).

Reasoning by induction, we assume that $a_{n-1,n},a_{n-2,n},\dots,a_{r+1,n}$ are known and find $a_{rn}$ for each $r\leqslant n-2$.

Suppose first that $r\geqslant 2$. Then $n\geqslant 4$, and $c_n\neq 0$ because $g_{1n}^{(n,1)}$ divides $f_n$. Since $\|B^{(r)}\|$ is determined by (26) and $C$ is known, we determine $(B^{(r)},C)$ using the polarization identity. We determine $b_{1n}^{(r)}$ from

$$(B^{(r)},C)=b_{11}^{(r)}\bar c_1+\dots+b_{1,n-1}^{(r)}\bar c_{n-1}+b_{1n}^{(r)}\bar c_n.$$

By (24), the first $r-1$ columns of $G^{(n,r)}$ are zero, and so

$$b_{1n}^{(r)}=g_{1r}^{(n,r)}\{A\}a_{rn}+g_{1,r+1}^{(n,r)}\{A\}a_{r+1,n}+\dots+g_{1,n-1}^{(n,r)}\{A\}a_{n-1,n}.$$

This equality determines $a_{rn}$ because only $a_{rn}$ is unknown and $g_{1r}^{(n,r)}\{A\}\neq 0$ by (29).

Suppose now that $r=1$. Write $C$ in the form $D(A-\lambda_n I_n)$, in which

$$D=[d_{ij}]:=(A-\lambda_2 I_n)(A-\lambda_3 I_n)\cdots(A-\lambda_{n-1}I_n).$$

Then

$$c_n=d_{11}a_{1n}+d_{12}a_{2n}+\dots+d_{1,n-1}a_{n-1,n}.$$

This equality determines $a_{1n}$ since only $a_{1n}$ is unknown and

$$d_{11} = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_{n-1}) \neq 0.$$

Therefore, we have determined all entries of $A$. $\square$

*Proof of Theorem 5.* By Lemmas 6(a) and 12, each matrix $A \in M_n(f_n)$ is unitarily similar to a matrix $A' \in M_n(f_n)$ of the form (27) via a diagonal unitary matrix. Then $\|h(A_k)\| = \|h(A'_k)\|$ for all $h \in \mathbb{C}[x]$ and $k = 1, \dots, n$. Thus, it suffices to prove Theorem 5 for matrices $A, B \in M_n(f_n)$ of the form (27).

"$\Rightarrow$" Let $A, B \in M_n(f_n)$ of the form (27) be unitarily similar and have the same main diagonal. By Lemma 6(b), $A = B$ and so (16) holds.

"$\Leftarrow$" Let $A, B \in M_n(f_n)$ of the form (27) satisfy (16). By Lemma 13, $A = B$ since their indexed families $\{\|h(A_k)\|\}_{(h,k)}$ and $\{\|h(B_k)\|\}_{(h,k)}$ coincide. $\square$

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