Fock-state view of weak-value measurements and implementation with photons and atomic ensembles

Christoph Simon[1] and Eugene S. Polzik[2]
[1] *Institute for Quantum Information Science and Department of Physics and Astronomy, University of Calgary, Calgary T2N 1N4, Alberta, Canada*
[2] *Niels Bohr Institute, Danish Quantum Optics Center - QUANTOP, Copenhagen University, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
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Weak measurements in combination with post-selection can give rise to a striking amplification effect (related to a large "weak value"). We show that this effect can be understood by viewing the initial state of the pointer as the ground state of a fictional harmonic oscillator, helping us to clarify the transition from the weak-value regime to conventional dark-port interferometry. We then describe how to implement fully quantum weak-value measurements combining photons and atomic ensembles.

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In 1988 Aharonov, Albert and Vaidman discovered that weak measurements in combination with pre- and post-selection give rise to a striking amplification effect, e.g. giving a "weak value" greater than 100 for the spin of a spin-1/2 particle [1]. Such weak-value measurements have been studied in various contexts, including the foundations of quantum mechanics [2], superluminal light propagation in dispersive materials [3], polarization effects in optical networks [4], and cavity QED [5]. Most recently they were used for precision measurements [6], spurring further research in that direction [7, 8].

The standard scenario of weak-value measurements involves a system and a pointer that interact via a Hamiltonian $H = \chi\sigma_z P$, where σ_z is a spin-like observable for the system and P is a momentum-like observable for the pointer. If the interaction is weak, and if the initial and post-selected states of the system are almost, but not quite, orthogonal, the final state of the pointer is displaced in X (the canonical variable conjugate to P) by an amount that is much greater than one would naively expect based on the interaction strength. In the first part of the present paper we describe an instructive way of understanding this effect. The initial state of the pointer can be seen as the ground state of a fictional harmonic oscillator, while the state of the pointer that is prepared by the post-selection is a superposition of the ground and first excited states, with coefficients that depend on the interaction strength and on the post-selection measurement. This gives a simple view of the relationship between the weak-value regime and other measurement techniques.

This new perspective on weak-value measurements is very helpful when thinking about possible implementations of such measurements combining photons and atomic ensembles, which is the second topic of the present paper. So far weak values were primarily observed with light, where typically polarization plays the role of the system and the external degrees of freedom of the photons play the role of the pointer [2–4, 6, 9]. These optical experiments can be understood using the weak value formalism, but also in purely classical terms. Recently there have been proposals to implement weak values with quantum dots [10]. Here we propose to implement weak values in a fully quantum way via the coupling of light to atomic ensembles. The interaction of light with atomic ensembles has been widely used in quantum information processing [11], implementing important paradigms such as quantum memories [12] and quantum teleportation [13], making atomic ensembles very attractive for the realization of quantum repeaters [14–17]. We show that weak-value experiments can be implemented by a modification of the spontaneous Raman scattering experiment that lies at the heart of the well-known Duan-Lukin-Cirac-Zoller (DLCZ) quantum repeater protocol [15, 16].

Fock-state view of weak-value measurements. As is customary in the weak-value literature, let us consider the case where the initial wave function of the pointer is a Gaussian, $\psi_0(x) = e^{-\frac{x^2}{2w^2}} w^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$, where x is the variable corresponding to the operator X and w is the width of the Gaussian. Any Gaussian can be seen as the ground state of a fictional harmonic oscillator Hamiltonian. That is, one can define an annihilation operator $a = \frac{1}{\sqrt{2}}(\frac{X}{w} + iwP)$ (satisfying $[a, a^\dagger] = \mathbb{1}$) which annihilates the initial state, $a|\psi_0(x)\rangle = 0$. One can then identify the initial pointer state with the ground state of the thought harmonic oscillator, $|\psi_0(x)\rangle \equiv |0\rangle$. The Hamiltonian $H = \chi\sigma_z P$ can be rewritten in terms of a and $a^\dagger$ as $H = -i\frac{\chi}{\sqrt{2}w}\sigma_z(a - a^\dagger)$.

Suppose that the system is initially prepared in the state $|x+\rangle$, which satisfies $\sigma_x|x+\rangle = |x+\rangle$ and $\sigma_z|x+\rangle = |x-\rangle$, where $|x-\rangle$ is the other eigenstate of σ_x, satisfying $\sigma_x|x-\rangle = -|x-\rangle$. Then the time evolution of the system and the pointer for a weak interaction is given by

$$e^{-iHt}|x+\rangle|0\rangle \approx (\mathbb{1} - iHt)|x+\rangle|0\rangle = |x+\rangle|0\rangle + \kappa|x-\rangle|1\rangle, \quad (1)$$

where $\kappa = \frac{\chi t}{\sqrt{2}w}$ is supposed to be much smaller than 1, and $|1\rangle \equiv a^\dagger|0\rangle$ is the first excited state of the fictional harmonic oscillator. The interaction has created an entangled state between the system and the pointer. If $\kappa \ll 1$ the entanglement is very weak, and the effect of the interaction may be hard to detect. In particular, the probability to detect the pointer in the state $|1\rangle$ is κ^2.

The weak-value protocol proceeds by measuring the state of the system in a basis that is close, but not identical, to the σ_x basis, and post-selecting those results for which the measured state is almost orthogonal to the initial state $|x+\rangle$. That is, we project the system onto a state that can approximately be written as $\phi|x+\rangle+|x-\rangle$, where $\phi \ll 1$ is the small angle between the preparation and post-selection bases. The resulting state of the pointer is seen from Eq. (1) to be

$$\phi|0\rangle + \kappa|1\rangle, \tag{2}$$

showing that the probability for the projection to occur is $\phi^2+\kappa^2$. The usual weak-value regime corresponds to the conditions $\kappa \ll \phi \ll 1$. In this case the probability for a successful post-selection is approximately equal to ϕ^2, and the pointer state can be written in renormalized form as $|0\rangle + \frac{\kappa}{\phi}|1\rangle$.

By applying the creation operator $a^\dagger$ to the ground state wave function one finds the wave function of the state $|1\rangle$, $\psi_1(x) = \frac{\sqrt{2}x}{w}\psi_0(x) = -\sqrt{2}w\frac{d\psi_0(x)}{dx}$. As a consequence, the wave function of the final pointer state is

$$\psi_0(x) - \frac{\kappa}{\phi}\sqrt{2}w\frac{d\psi_0(x)}{dx} \approx \psi_0(x-\sqrt{2}w\kappa/\phi), \tag{3}$$

which clearly shows that the wave function of the pointer is displaced by an amount that is inversely proportional to the small parameter ϕ. This corresponds to a large "weak value" of the system observable σ_z of $\frac{1}{\phi}$. This terminology is due to the fact that, for the Hamiltonian $H = \chi\sigma_z P$ and in the absence of post-selection, the displacement of the pointer would be determined only by κ, w and the initial value of σ_z. We see now that post-selection allows for much larger displacements. This approach can be used to amplify the effect of the small interaction (κ) at the expense of a small post-selection probability (ϕ^2).

The amplified displacement is obtained in the regime where $\frac{\kappa}{\phi} \ll 1$, so that the final wave function of the pointer is still approximately a Gaussian. However, Eq. (2) also allows us to easily understand other regimes in which the usual weak-value description no longer applies. In particular, as ϕ is decreased and becomes smaller than κ, the final pointer wave function is increasingly dominated by $\psi_1(x)$, which has the typical "derivative" shape. This regime was recently discussed in the weak-value context in Ref. [8]. The limiting case of $\phi = 0$ corresponds to observing the dark port in conventional interferometry.

It is an interesting question under what exact conditions the choice of $\phi \gg \kappa$ vs. $\phi \lesssim \kappa$ is advantageous for measuring the small parameter κ. The weak-value regime ($\phi \gg \kappa$) seems to be advantageous when κ is so small that dark-port detections (which occur with a probability of order κ^2) are dominated by background noise, i.e. $\kappa^2 \ll \beta$, where β is the noise level, whereas ϕ^2 can be made greater than β. Note that the weak-value regime still corresponds to $\phi \ll 1$. Increasing ϕ to of order one corresponds to the conventional "bright-port" regime, where there is no post-selection, but also no amplification effect. Weak-value measurements have the potential to outperform "bright-port" interferometry in certain cases, for example for the measurement of longitudinal phase shifts in the presence of alignment errors, see Ref. [7]. Weak-value techniques thus correspond to an interesting and potentially useful intermediate regime between bright-port and dark-port interferometry.

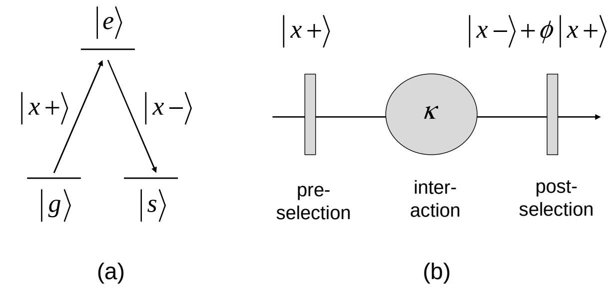

FIG. 1: Implementation of weak-value measurements with photons and atomic ensembles. (a) An incoming photon has a small probability amplitude κ to be scattered from the polarization state $|x+\rangle$ to the state $|x-\rangle$, causing the atomic ensemble to go from the state $|0\rangle$ of Eq. (4) to the state $|1\rangle$ of Eq. (5). This creates an entangled state between the photon and the ensemble, see Eq. (1). (b) Photons are observed in the forward direction and projected onto the almost orthogonal polarization state $|x-\rangle + \phi|x+\rangle$, where $1 \gg \phi \gg \kappa$. This projects the atomic ensemble onto the state $\phi|0\rangle+\kappa|1\rangle$, which is the initial atomic state displaced by an amount that is proportional to $\frac{\kappa}{\phi}$, see Eq. (3). This amplified displacement of the atomic state can be detected by "atomic homodyne detection", see text.

Implementation with photons and atomic ensembles. It is possible to realize an interaction Hamiltonian of the form $H = \chi\sigma_z P$ through the off-resonant interaction of light with atomic ensembles [11–13]. However, we have learned from the above discussion that the key point is in fact the creation of the entangled state Eq. (1). This can be done most directly through spontaneous Raman scattering, similarly to the DLCZ quantum repeater proposal [15], see Figure 1. The "system" is realized by the polarization states of a photon, where $x+$ is the polarization of the "write" photon, whereas the $x-$ polarization corresponds to the Raman-scattered "Stokes" photon. The scattering of the photon from $x+$ to $x-$ polarization is accompanied by the creation of an atomic excitation. The atomic system, which is initially in the state

$$|0\rangle \equiv |g\rangle_1|g\rangle_2...|g\rangle_{N_A}, \tag{4}$$

with all N_A atoms in the ground state g, makes a transition to the state

$$|1\rangle \equiv \frac{1}{\sqrt{N_A}} \left(|s\rangle_1 |g\rangle_2 ... |g\rangle_{N_A} + ... + |g\rangle_1 ... |g\rangle_{N_A-1} |s\rangle_{N_A} \right), \quad (5)$$

which is the symmetric state with one atom in the state s. It is again possible to see the states $|0\rangle$ and $|1\rangle$ as the ground and first excited states of a harmonic oscillator, where the number of excitations in Fock space now corresponds to the number of atoms in s.

This can best be seen in two steps [11]. One begins by introducing a quasi-spin operator $\sigma_x = |g\rangle\langle g| - |s\rangle\langle s|$ for each atom, with corresponding expressions for σ_y and σ_z. The associated collective spin operators J_x, J_y, J_z, where $J_x = \frac{1}{2}\sum_{k=1}^{N_A} \sigma_x^{(k)}$ etc., fulfill commutation relations $[J_y, J_z] = iJ_x$. The state $|0\rangle$ of Eq. (4) is the eigenstate of J_x with maximum eigenvalue $N_A/2$. For the phase space region in the vicinity of this state (i.e. as long as there are just a few atoms in s) one can then employ the Holstein-Primakoff approximation and introduce new canonical variables $X = J_y/\sqrt{N_A/2}$ and $P = J_z/\sqrt{N_A/2}$ that satisfy $[X, P] = i$ [11]. The angular momentum ladder operator $J_- = J_y - iJ_z$, which creates excitations in s, is then proportional to the harmonic-oscillator creation operator $a^\dagger = \frac{1}{\sqrt{2}}(X - iP)$, and the states of Eqs. (4) and (5) are related by $|1\rangle = a^\dagger|0\rangle$ as before.

Starting with one photon in the state $|x+\rangle$ and the atomic ensemble in the state $|0\rangle$, spontaneous Raman scattering with an amplitude κ will create the entangled state of Eq. (1). In typical experiments [16, 17] one detects the Stokes photon, corresponding to a projection onto the $|x-\rangle$ state, which furthermore projects the atomic ensemble onto the state $|1\rangle$. In fact, the idea of the protocol of Ref. [15] is to detect a Stokes photon that could have come from either of two ensembles, thus creating a single, but delocalized, atomic excitation, which forms the basic unit of entanglement for the quantum repeater protocol. It is important in this context to distinguish the Stokes photons from the write photons, i.e. to perform a very accurate projection onto $|x-\rangle$.

Now suppose that our task is not to implement a quantum repeater, but to measure κ, which might be very small in some situations. One possible approach is to directly detect the Stokes photons as before. However, this may be impossible for very small κ because of various sources of background noise. The alternative, weak-value, approach is to detect light in the forward direction, but to project not exactly onto the Stokes mode $|x-\rangle$, but onto a superposition state $\phi|x+\rangle + |x-\rangle$ which has a slight admixture of the write beam mode, see Figure 1(b). This will project the atomic ensemble onto a superposition state of the form $\phi|0\rangle + \kappa|1\rangle$ as before. Generating such superposition states for the atomic ensemble could also be a motivation in itself for this approach. The weak-value regime corresponds to $1 \gg \phi \gg \kappa$, where ϕ^2 is chosen such that it is above the noise level. In the x-representation the atomic state will have the form of Eq. (3) with $w = 1$, i.e. it corresponds to the vacuum state displaced by $\frac{\sqrt{2}\kappa}{\phi}$.

In typical experiments the write beam contains more than a single photon. However, this does not change the principle of the proposed implementation, as long as the probability of scattering a photon into the polarization mode $x-$ and thus exciting the atomic state $|1\rangle$ remains small. In the above discussion one can simply replace the single-photon state $|x+\rangle$ by the state $|N\rangle_{x+}|0\rangle_{x-}$, which describes N photons of polarization $x+$, and the single-photon state $|x-\rangle$ by the state $|N-1\rangle_{x+}|1\rangle_{x-}$, in which a single photon has been scattered into the $x-$ polarization. The final detection of a single photon is described by the annihilation operator $\phi a_{x+} + a_{x-}$, where a_{x+} describes the annihilation of a photon of $x+$ polarization etc. The presence of N photons enhances both the scattering amplitude and the amplitude of detecting a photon from the write beam by a factor of $\sqrt{N}$, but the final atomic state is the same as before.

In order to explicitly show the displacement of the atomic pointer state, one has to measure the distribution of results for the observable $X = J_y/\sqrt{N_A/\sqrt{2}}$. This can be done by first applying a $\pi/2$ pulse to the atomic state, followed by a measurement of the population difference in the levels g and s (i.e. J_x). The latter measurement can be performed using a quantum non-demolition (QND) interaction with off-resonant light followed by homodyning of the transmitted light [11]. Note that the described procedure can be seen as a homodyne detection of the atomic state, where the $\pi/2$ pulse corresponds to the beam splitter in conventional homodyne detection. The population differences between the states $|0\rangle$ and $|1\rangle$ of Eqs. (4) and (5), for example, are too small to be detected directly through the QND interaction, but the two states are well distinguishable with the described "atomic homodyning" technique. The precision of this measurement technique for X is sufficient to detect displacements that are smaller than the width of the vacuum wave packet [11, 18].

We have described an instructive new perspective on weak-value measurements and proposed a feasible, fully quantum experimental implementation with photons and atomic ensembles. In the future we intend to explore the application of the present approach to precision measurements with atomic ensembles.

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