WARMTH AND MOBILITY OF RANDOM GRAPHS

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Abstract. Brightwell and Winkler introduced the graph parameters warmth and mobility in the context of combinatorial statistical physics. They related both parameters to lower bounds on chromatic number. Here we study these parameters for random graphs.

Although warmth is not a monotone graph property we show it is nevertheless "statistically monotone,", and that for Erdős-Rényi random graphs $G(n,p)$ with $p = O(n^{-\alpha})$, $\alpha > 0$, the warmth is concentrated on at most two values, and that for most p it is concentrated on one value. We also study the uniform case $p = 1/2$, and as a corollary obtain that a conjecture of Lovász holds for almost all graphs.

1. Introduction

Graph coloring is well studied from the points of view of combinatorics [25] and theoretical computer science [12], and finding the chromatic number is thought to be quite difficult in general. It is known, for example, that deciding whether a graph is 3-colorable is NP-complete. Finding general bounds on chromatic number is of considerable interest for both theoretical and practical reasons [16, 21].

Lower bounds on chromatic number are of particular interest, since it is generally hard to rule out all possible k-colorings of a graph. In recent years, lower bounds coming from algebraic topology [5, 4, 22], and statistical physics [9, 8, 24] have been developed. These methods have in common that they obtain restrictions on maps from a graph H (such as colorings of H) by probing with maps into H by "test graphs" T.

In the topological setting, T is usually a finite graph such as an edge or an odd cycle, equipped with a group action. For an introduction to topological obstructions to graph colorings, see [4]. In the statistical physics setting, T is usually an infinite graph with a high degree of symmetry, such as a tree or lattice, or else an infinite family of graphs such as all finite graphs of maximum degree $\leq d$. In each case, the more freedom there is in mapping T into H, the larger the chromatic number of H.

It is natural to ask how good these bounds are for typical graphs, or for which graphs they will tend to be close to the truth. In [17] it is shown that certain topological bounds on chromatic number are far from the truth for Erdős-Rényi random graphs.

In this article we show that the statistical physics bounds warmth and mobility, introduced by Brightwell and Winkler [9], are also far from the truth for almost all graphs. This includes sparse and dense Erdős-Rényi random graphs, as well as random regular graphs. Our efforts focus on computing facts about warmth of

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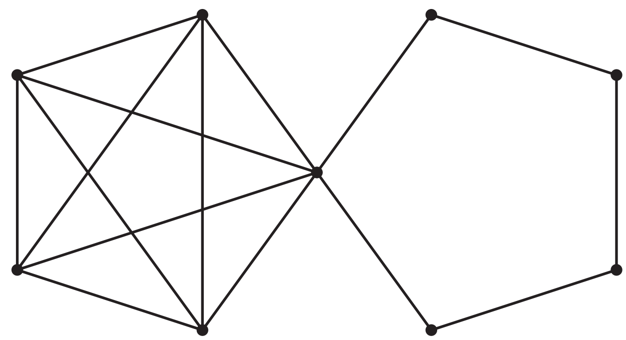

FIGURE 1. This graph shows that warmth is not a monotone graph parameter. One can compute that the warmth is 3, but deletion of any of the five edges on the right results in warmth of 5.

random graphs, and then we compare our results with well known facts about the chromatic number of random graphs ([6], [1], [23], [2], [10]). As a corollary of our results we establish a conjecture of Lovász for almost all graphs.

Although most graph parameters studied in random graph theory are *monotone* in the sense that they only decrease (or increase) with the deletion of edges [7, 11], it turns out that warmth is not such a property. For an example of the non-monotonicity of warmth see Figure 1.

2. Background

We first reproduce a few definitions and results of Brightwell and Winkler [9].

2.1. Definitions.

All the graphs we consider here are simple graphs, meaning undirected and without multiple edges or loops.

Definition 2.1. For graphs G and H, a function $\phi : V(G) \to V(H)$ is a *graph homomorphism* if it induces a map between edges $E(G) \to E(G)$. The set of all homomorphisms of a graph G to a graph H is denoted $\mathrm{Hom}(H, G)$. It is naturally endowed with a graph structure as follows: the elements of $\mathrm{Hom}(G, H)$ are the vertices of a graph, with an edge between two homomorphisms whenever they differ in exactly one vertex.

(There is a natural higher-dimensional structure on $\mathrm{Hom}(G,H)$ as well, important for topological applications. See [5] for a nice introduction.)

Graph homomorphisms generalize graph coloring, since an n-coloring of H is equivalent to a homomorphism $H \to K_n$. (For an overview of graph homomorphisms, see the book [15].)

Let T^d denote the d-branching rooted tree, as illustrated in Figure 2.

Definition 2.2. A map φ in $\mathrm{Hom}(T^d, G)$ is said to be *cold* if there is a node a of G such that for any k no $\psi \in \mathrm{Hom}(T^d, G)$ agrees with ψ on the sites at distance k from the root r and has $\psi(r) = a$. Graph G is said to be *d-warm* if $\mathrm{Hom}(T^{d-2}, G)$ does not contain any cold maps. Further, the *warmth* $w(G)$ of G is defined to be the largest d for which G is d-warm.

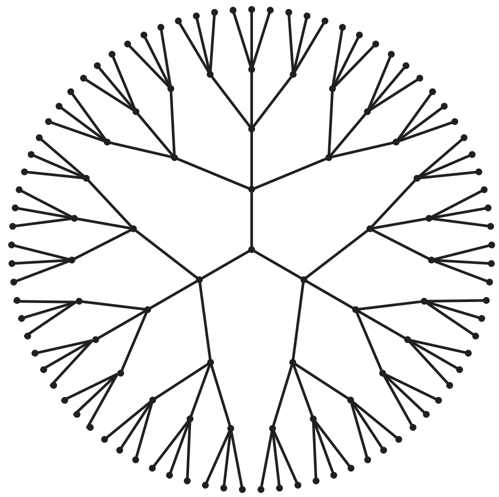

FIGURE 2. The first few levels of the 3-branching rooted tree T^3. Note that T^d is not quite a Cayley tree, since it is regular of degree d except at the root.

Definition 2.3. Let $\mathcal{H}_d$ denote the set of all finite simple graphs of maximum degree d. Graph G is said to be *d-mobile* if $\mathrm{Hom}(H, G)$ is connected for all $H \in \mathcal{H}_{d-2}$. (Note: the right interpretation of Brightwell and Winkler's definition of d-mobile is probably to say that either $\mathrm{Hom}(H, G)$ is connected or it is empty, for all $H \in \mathcal{H}_{d-2}$. See comments in the last section.) Further, the largest d for which G is d-mobile is called the *mobility* $m(G)$ of G.

Definition 2.4. For a graph G and a vertex $u \in G$ let $N(u)$ denote the set of vertices in G adjacent to u. $N(u)$ is called the *neighborhood* of u.

For a set of vertices A, define the neighborhood $N(A) = \cup_{a\in A} N(a)$. A family of subsets $\{A_i\}_1^w$ of G is called a *d-stable* family if for all $1 \le i \le w$ there exist $A_{i_1} \cdots A_{i_d}$ such that

$$\bigcap_{i_1}^{i_d} N(A_{i_j}) = A_i.$$

2.2. **Known results and a conjecture.**

We state two theorems of Brightwell and Winkler. First, an equivalent characterization of warmth.

Theorem 2.5 (Brightwell and Winkler [9])**.** *Given a constraint graph H and $d \ge 1$, the following are equivalent:*

- *H is not $(d + 2)$-warm;*
- *There exists a d-stable family of subsets of H.*

We also have the following inequalities relating warmth, mobility, and chromatic number.

Theorem 2.6 (Brightwell and Winkler [9])**.** *For every finite graph H,*

- *$w(H) \leq \chi(H)$ for unlooped H,*
- *$m(H) \leq 2w(H) - 2$, and*
- *$m(H) \leq 2\chi(H) - 2$.*

A conjecture attributed to Lovász is that the last inequality can be improved to the following.

Conjecture 2.7. *For all finite loopless graphs H,*

$$m(H) \leq \chi(H).$$

Brightwell and Winkler have shown this for $\chi(H) \leq 3$, and Lovász has an unpublished proof for $\chi(H) \leq 4$ as noted in [9], but in general this conjecture is open. One of the main contributions of this article is to establish the conjecture for many graphs with large chromatic number.

3. Statement of Results

In the following sections we study the warmth of various types of random graphs (Erdős-Rényi as well as random regular) and show that in all these cases warmth is much smaller than the chromatic number. By the Brightwell-Winkler inequalities, the mobility is also relatively small, and comparing with known results about chromatic numbers of random graphs gives Conjecture 2.7 for "almost all" graphs. This conjecture seems to only be known for relatively sparse graphs so far; in particular it is apparently known to be true for all graphs H with $\chi(H) \leq 4$ in unpublished notes of Lovász [9]. Since our results hold for dense random graphs as well as sparse, this verifies the conjecture for many graphs for which it was previously unknown.

The following is a summary of results. We use Bachmann–Landau and related notations: $O, o, \Omega, \omega, \Theta$. In every case, the asymptotic notation is to be understood as the number of vertices $n \to \infty$. The statement of the theorems and proofs are similar for the sparse and dense cases – however it eases notation and simplifies the proofs to treat the sparse and dense cases separately.

In the following $\alpha, \delta > 0$ are constant.

Theorem 3.1. *(Upper bound – sparse regime) If $p = O(n^{-\alpha})$, then a.a.s.*

$$w(G(n,p)) \leq \lfloor 1/\alpha + 2 \rfloor.$$

Theorem 3.2. *(Upper bound – dense / uniform case) If $p = 1/2$, then a.a.s.*

$$w(G(n,p)) \leq (1+\delta)\log_2 n + 1.$$

Theorem 3.3. *(Lower bound – sparse regime) If $p = \Omega(n^{-\alpha})$, then a.a.s.*

$$w(G(n,p)) \geq \lceil 1/\alpha + 1 \rceil.$$

Theorem 3.4. *(Lower bound – dense / uniform case) If $p = 1/2$, then a.a.s.*

$$w(G(n,p)) \geq (1-\delta)\log_2 n.$$

As a corollary we have that in the sparse regime $p = O(n^{-\alpha})$ the warmth is concentrated on at most two values and that for "most" sequences $p = p(n)$ is concentrated on one value. For example if $p = \theta(n^{-\alpha})$ with $-1/k < \alpha < -1/k + 1$ then a.a.s. we have $w = k + 2$. See Figure 3.

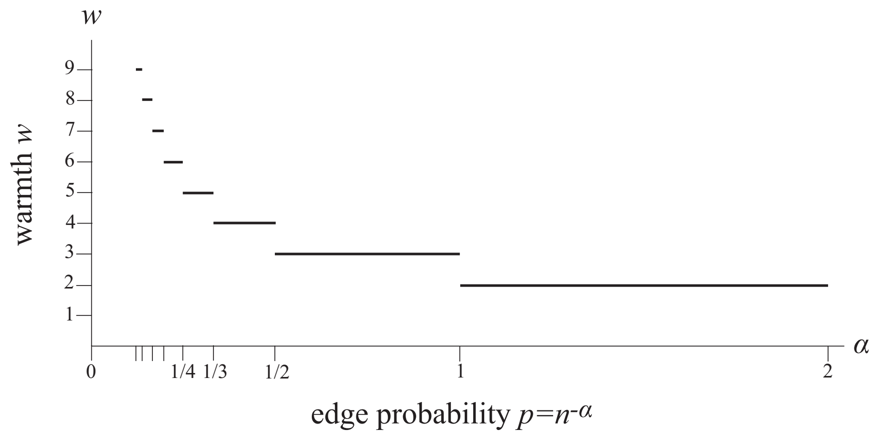

FIGURE 3. A summary of our results for $G(n,p)$ in the sparse regime.

We also consider the dense case of $p = 1/2$ (i.e. the uniform distribution on all labeled graphs on n vertices).

By establishing upper bounds on warmth, we also obtain upper bounds on mobility of random graphs, by Theorem 2.6. In particular for p in the sparse regime, both warmth and mobility are $O(1)$ as $n \to \infty$. In contrast, as long as $p = \omega(1/n)$, it is known that $\chi(G(n,p))$ tends to infinity as $n \to \infty$ (e.g. see Theorem 11.29 in [7]).

Our results should also establish the Loász conjecture for smaller p: if $p = O(1/n)$ then we have $w(G(n,p)) \leq 3$ a.a.s. by Theorem 3.1. So by Theorem 2.6, $m(G(n,p)) \leq 4$, and this is covered by the case of small chromatic number already established by Lovász.

We also use results on chromatic number of random regular graphs and others to check that the conjecture of Lovász holds for almost all random regular graphs $G_{n,d}$ for a wide range of degree $d = d(n)$ in Section 6.

4. Upper bounds

In this section we prove Theorems 3.1 and 3.2.

For upper bounds in the sparse case it is convenient to assume that

$$p \geq \frac{\log n + \omega(\log\log n)}{n},$$

which insures that $G(n,p)$ is a.a.s. connected and that for any fixed k, $d(G(n,p)) \geq k$ [7]. (Here $d(H)$ denotes the minimum degree of H.) One can consider smaller p as a separate case. For example, an adaptation of the argument below to handle small degree vertices and possibly disconnected graphs shows that if $p = O(n^{-0.99})$ then a.a.s. $w(G(n,p) \leq 3$.

Similarly for the dense case, it is useful to observe that if $p = 1/2$ then

$$P(d(G(n,p)) \leq (1-\delta)\log_2 n \to 0$$

as $n \to \infty$.

Proof of Theorems 3.1 and 3.2. Let c be a constant in the sparse case and $c = O(\log n)$ a function in the dense case; (in both cases c will be specified more precisely later.) Since $P(d(G(n,p)) \leq c) \to 0$ we shall assume that $d(G) > c$.

In the sparse case we assume that $p \leq \mu n^{-\alpha}$ for some constants $\alpha, \mu > 0$; in this case set $s = \lfloor 1/\alpha + 1 \rfloor$. In the dense case we assume that $p = 1/2$ and $\delta > 0$, and set $s = \lfloor (1+\delta)\log_2 n \rfloor$. In both cases we show that

$$w(G(n,p)) \leq s + 1.$$

Consider the family $\mathcal{F} = \{v \mid v \in V\}$ of singleton vertices of G. We show that a.a.s. $\mathcal{F}$ is an s-stable family, i.e. for every vertex v there exist $\{u_1, \cdots, u_s\} \subseteq N(v)$ such that

$$\bigcap_{i=1}^{s} N(u_i) = v.$$

This is sufficient to bound warmth by Brightwell and Winkler's results. We call such a set $\{u_1, \ldots, u_s\}$ an *s-representative* of v in $\mathcal{F}$.

For $u_1, \cdots, u_s$ neighbors of v, the probability that they are also adjacent to vertex w is p^s by edge independence. Let $A(w)(w_1, \cdots, w_s)$ denote the event that $w_1, \cdots, w_s$ are in the neighborhood $N(w)$ of w. Then,

$$P(A(u)(u_1, \cdots, u_s) \text{ for some } u \in V) \leq \sum_{u \in V} P(A(u)(u_1, \cdots, u_s)) \leq np^s,$$

where, $P(A(u)(u_1, \cdots, u_s)$ for some $u \in V)$ is the probability that v shares $u_1, \cdots, u_s$ as neighbors with some other vertex in V. Let $\gamma = \lfloor c/s \rfloor$. We consider $N_1, \cdots, N_\gamma$ some disjoint subsets of the neighbors of v, such that $|N_i| = s$. We can do so by the assumption that $|N(v)| \geq c$. Let

$$N(v,s) = \{U \subset N(v), |U| = s\}.$$

For $M \in N(v,s)$ let $A(M)$ denote the event that $M \subseteq N(u)$ for some $u \in V$. We have

$$P\left(\bigcap_{M \in N(v,s)} A(M)\right) \leq P\left(\bigcap_{i=1}^{\gamma} A(N_i)\right). \tag{1}$$

That is, the probability that v shares every s-subset of $N(v)$ with some other vertex of V is less than the probability that v shares each of the N_i as neighbors with some other vertex in V.

Since the N_i are disjoint, we have,

$$P\left(\bigcap_{i=1}^{\gamma} A(N_i)\right) = O((np^s)^\gamma).$$

Thus, the probability that some vertex does not have a s-representative in $\mathcal{F}$ is

$$O\left(\sum_{v \in V} (np^s)^\gamma\right) = O(n(np^s)^\gamma) \leq O(n(np^s)^\gamma) = O(n^{1+\gamma(1-s\alpha)}),$$

where $\alpha = \frac{1}{\log_2 n}$ in the dense case.

Note that in the sparse case,

$$s\alpha = \alpha\lfloor 1/\alpha + 1 \rfloor > 1 \text{ and } s\alpha \text{ is constant.} \tag{2}$$

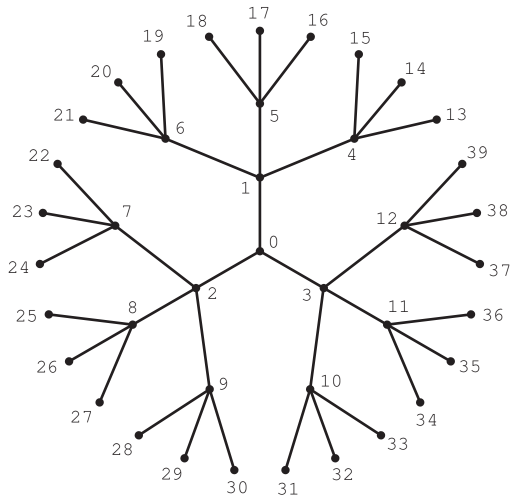

FIGURE 4. The labeling of vertices for the three branching tree.

And in the dense case,

$$sα = \frac{\lfloor (1+\delta)\log_2 n \rfloor}{\log_2 n} \geq 1 + \delta/2. \tag{3}$$

Now since $s\alpha - 1$ is greater than zero in both cases we can choose $c(n) = \frac{2s(n)}{s\alpha - 1} = O(\log n)$, so that $\gamma(1 - s\alpha) \leq -2$. Thus, the probability that G does not have a s-stable family is

$$\leq n \times n^{\gamma(1-s\alpha)} \leq n^{-1} \to 0, \text{ by our choice of } c.$$

This and the lemma above prove that a.a.s. $G(n,p)$ has a s-stable family of subsets and hence by Theorem 2.5, a.a.s. $w(G(n,p)) \leq s+1$, as desired. This completes the proof.

□

5. Lower bounds

In this section we prove Theorems 3.3 and 3.4. We will discuss maps from the s-branching tree T^s and it is convenient to label its vertices with the root labeled 0 and its children labeled $1, 2, 3 \ldots$, as shown in Figure 4.

Proving that $w(H) \geq s+2$ means showing that $\mathrm{Hom}(T^s, H)$ has no cold maps. So in the sparse case let $s = \lceil 1/\alpha - 1 \rceil$ and in the dense case set $s = (1-\delta)\log_2 n$.

Let T^s_v denote the truncated s-branching tree T^s with v vertices, labeled 0 to $v-1$. Note that when $v \equiv 1 \pmod{s}$, the set L of leaves of the tree 'generates' the remaining nodes in the sense that every other node in T^s has an ancestor in L. Hence we restrict to this case.

Let $D = D(T^s_v)$ denote the set of leaves of T^s_v, together with the root; i.e.

$$D = \{0\} \cup L.$$

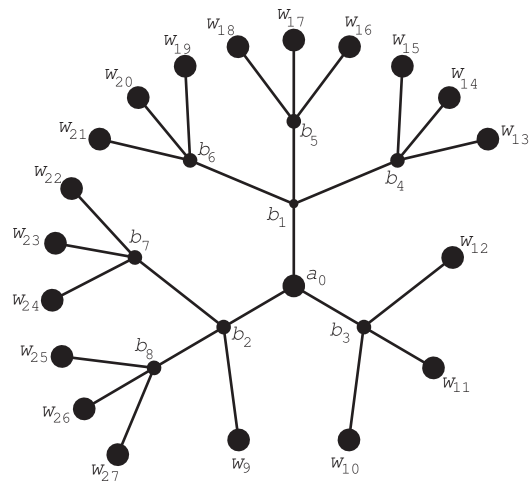

FIGURE 5. An extendable function $f : D \to V(H)$ and accompanying graph map $\phi : T^3_{28} \to H$. Here $D = \{0, 9, 10, \dots, 27\}$, and $a_0, b_1, \dots, b_8, w_9, \dots, w_{27} \in V(H)$, and f is illustrated with larger vertices and ϕ with smaller ones.

Let H be any graph, and $f : D \to V(H)$ any function. We say that f is H-*extendable* if there exists a graph homomorphism $\phi : T^s_v \to H$ such that $\phi|_D = f$, and that H has *property* $\mathcal{P}^s_v$ if every function $f : D(T^s_v) \to V(H)$ is H-extendable.

Property $\mathcal{P}^s_v$ precludes any cold maps $T^s \to H$, so by definition such graphs have warmth at least $s+2$. A useful observation is that property $\mathcal{P}^s_v$ is monotone, even though warmth is not. Thus if property $\mathcal{P}^s_v$ holds for $G(n,p)$ a.a.s. then it also holds for $G(n,p')$ a.a.s. whenever $p \le p'$.

For the sparse case we assume that $p = \Omega(n^{-\alpha})$ and prove that a.a.s. $w(G(n,p)) \ge \lceil 1/\alpha + 1 \rceil$. In this case we set $\epsilon = 1 - \alpha \lceil 1/\alpha - 1 \rceil$. (The main point is that ϵ is bounded away from 0 as $n \to \infty$.)

In the dense case we assume that $p = 1/2$ and prove that $w(G(n,p)) \ge (1-\delta)\log_2 n$. In this case set

$$\epsilon = 1 - \frac{\lceil (1-\delta)\log_2 n \rceil - 2}{\log_2 n}.$$

Here ϵ is still bounded away from 0 as $n \to \infty$, since δ is constant.

Thus $\mu_1 n^{-(1-\epsilon)/s} \le p \le \mu_2 n^{-(1-\epsilon)/s}$ for some constants $\mu_1, \mu_2 > 0$ in the sparse case and for $\mu = 1$ in the dense case. Also $\inf_n \epsilon > 0$.

Now we will make precise the choice of v. We choose v such that,

$$s^2\left(1 + \frac{1}{\epsilon}\right) + 1 < v < s^2\left(1 + \frac{1}{\epsilon}\right) + 3s + 1. \tag{4}$$

Note that since $\inf_n \epsilon > 0$, we have $v = O(s^2)$ and hence, $O(\log^2 n)$.

Now we show that maps into $G(n,p)$ are all extendable for this choice of v.

Lemma 5.1. *Let $a, w_{m+1}, \dots, w_{v-1}$ be some fixed vertices in $G(n,p)$. The probability P that a function $f : D \to V(G(n,p))$, defined by $f(0) = a$ and $f(i) = w_i$, is* *not extendable is bounded above by*

$$P \leq e^{-Cn^{\epsilon}/\log^2 n} \tag{5}$$

for some absolute constant $C > 0$, and ϵ depending on p as defined above.

Proof. We apply the extended Janson's inequality to prove this claim. Let $m = v - |L| - 1$ and $W = \{w_{m+1}, \ldots, w_{v-1}\}$. Note, $m = (v-1)/s$. Also let,

$$\mathcal{A} = \{A \subset G, \text{ labeled subgraphs} : |V(A)| = m, V(A) \cap (\{a\} \cup W) = \phi\}.$$

Note that $|\mathcal{A}| \geq \binom{n-|L|-1}{m}$ since there are at least $n - |L| - 1$ vertices to choose the labeled vertices of A from. For $A_i \in \mathcal{A}$ let $\{b_1^i, \ldots b_m^i\}$ denote the set of vertices of A_i ordered according to it's labeling. Let B_i denote the event that $\beta_{A_i} : T^s \to G$ given by,

$$\beta_{A_i}(j) = \begin{cases} a & \text{if } i = 0 \\ b_j^i & \text{if } 1 \leq jm \\ w_j & \text{if } m+1 \leq j \leq (v-1), \end{cases} \tag{6}$$

is a homomorphism. Note that since a, w_j are fixed and b_i are all distinct and $\{b_1, \ldots, b_m\} \cap (\{a\} \cup W) = \phi$, the number of distinct edges in the image of T_v^s i.e. $|\beta_{A_i}(E(T_v^s))|$, depends only on a and W and not on A_i. We denote this number by e. Note that, $m \leq e \leq sm + s$. Also, $Pr[B_i] = p^e$. We say $x \sim y$ if the edges of A_x and A_y have a non-trivial intersection. Then, assuming

$$\Delta \geq \mu \tag{7}$$

the extended Janson's inequality [3] gives us,

$$Pr[\wedge_j \overline{B_j}] \leq \exp\left(-\frac{\mu^2}{2\Delta}\right) \tag{8}$$

where,

$$\begin{aligned} \mu &= \sum_j Pr[B_j] \\ \text{and } \Delta &= \sum_{x \sim y} Pr[B_x \wedge B_y]. \end{aligned} \tag{9}$$

Note that,

$$\mu = \sum_j Pr[B_j] = \Theta(n^m \times p^e). \tag{10}$$

Now we bound the sum Δ. Let $A' = A \setminus (\{a\} \cup W)$. We split the sum in Δ according to the size of the set $A'_x \cap A'_y$. Let

$$f_z = \max_{x,y}\{|E(A'_x \cap A'_y)| : |V(A'_x \cap A'_y)| = z\} \tag{11}$$

denote the maximum over all pairs x, y of the number of edges that A_x and A_y intersect in for A'_x and A'_y intersecting in exactly z vertices. The edge sets of A_x and A_y have a non-trivial intersection. Hence $z \geq 1$. Then, $Pr[A_x \wedge A_y] \leq p^{2e-f_z}$. Note that $f_z \leq sz$ and hence $Pr[A_x \wedge A_y] \leq p^{2e-sz}$. There are $O(n^{2m-z})$ such pairs A_x and A_y since $2m - z$ vertices determine these subgraphs up to order. Thus,

$$\Delta = \Theta\left(\sum_{z=1}^{v-l-1} n^{2m-z} \times p^{2e-sz}\right) \tag{12}$$
$$\geq \Omega(n^{2m-1} \times p^{2e-s})$$

Also,

$$\Delta = \Theta\left(\sum_{z=1}^{v-l-1} n^{2m-z} \times p^{2e-sz}\right) \tag{13}$$
$$= O((v-l-1)n^{2m-1}p^{2e-s}) = O(\log^2 n \times n^{2m-1}p^{2e-s})$$

It follows that $\Delta \geq \mu$ from the choice of v in equation 4 and the observation that $e > m = (v-1)/s$.

Also,

$$\frac{\mu^2}{2\Delta} \geq R\left(\frac{n \times p^{-s}}{\log^2 n}\right) \geq C\left(\frac{n^\epsilon}{\log^2 n}\right) \text{ for some constants } R, C. \tag{14}$$

Hence we have

$$Pr[\wedge_j \overline{B_j}] \leq \exp\left(-\frac{\mu^2}{\Delta}\right) \leq e^{-Cn^\epsilon}. \tag{15}$$

Also, as noted above, $\inf_n \epsilon > 0$. This proves the claim. □

Now we can complete the proof as follows. There are at most $n^{|L|+1}$ choices for a, w_i. Thus the probability that an appropriate β does not exist for at least one such choice of vertices is at most

$$n^{|L|+1}e^{-Cn^\epsilon/\log^2 n} \to 0 \text{ as } n \to \infty.$$

This completes the proof.

6. Random regular graphs

Consider the uniform distribution on the set of d-regular graphs on n vertices [7]. We denote a randomly chosen graph from this distribution by $G_{n,d}$. The following lemma of Krivelevich, Sudakov, Vu, and Wormald gives that for a wide range of $d \to \infty$, with high probability every pair of vertices has the expected number of common neighbors.

Lemma 6.1. *(Lemma 4.1((i) from [19].) Suppose that $\sqrt{n}\log n \leq d \leq n - n/\log n$. Then a.a.s.*

$$|N(u) \cap N(v)| = (1 + o(1))d^2/n$$

for every pair of vertices $\{u, v\}$ in $G_{n,d}$.

The following theorem bounds the chromatic number from below for a certain range of d.

Theorem 6.2. *(Theorem 2.7 from [20].) Let $0 < \alpha < 1/2$ be a positive constant; then for any*

$$n^\alpha < d < n^{1-\alpha},$$

a.a.s. we have,

$$\chi(G_{n,d}) = \Omega(d/\log d).$$

We combine the above theorems and Theorem 2.5 to show that Conjecture 2.7 holds a.a.s. for $G_{n,d}$.

Lemma 6.3. *Let G be a r-regular graph. If $|N(u) \cap N(v)| < k \leq r$ for every pair $u \neq v$ in G, then G is not $(k+2)$-warm, i.e. $w(G) \leq (k+1)$ and hence $m(G) \leq 2k+1$.*

Proof. We show that G has a k-stable family of subsets and hence the lemma will follow from Theorem 2.5. Consider the family $\mathcal{F}$ of all singleton vertices of G. This is a k-stable family of subsets of G. For $v \in G$ we can choose any k of its neighbors $v_1 \cdots v_k$. Then $N(v_1) \cap \cdots \cap N(v_k)$ is the set of all vertices of G that have $v_1 \cdots v_k$ as neighbors. But $|N(u) \cap N(v)| < k$ for every pair $u \neq v$ in G implies that

$$N(v_1) \cap \cdots \cap N(v_k) = \{v\}.$$

Thus we have our k-stable family $\mathcal{F}$ which completes the proof. □

Theorem 6.4. *Let $0 < \alpha < 1/4$ be a positive constant; then for any*

$$n^{1/2+\alpha} < d < n^{1-\alpha},$$

$w(G) \leq \chi(G)$ a.s.

Proof. It follows from Lemmas 6.1 and 6.3 that $m(G_{n,d}) < 4d^2/n$ a.a.s. Also $4d^2/n = o(d/\log d)$ since $d < n^{1-\alpha}$. Thus, using Theorem 6.2 we have $w(G) \leq \chi(G)$ a.a.s. This completes the proof. □

7. Open problems

We have shown that warmth is much smaller than chromatic number for certain types of random graphs, but we could also ask: for what types of graphs is warmth close to the chromatic number? In particular, can one describe large families of graphs, either constructively or probabilistically, for which warmth and chromatic number are equal? Brightwell and Winkler give several examples of such graphs in [9], including bipartite graphs, complete graphs, and "collapsible" graphs.

It would be especially interesting to know how the statistical physics and topological lower bounds on chromatic number compare. On this note, the second author explored topological bounds on chromatic number for Erdős-Rényi graphs in [17]. The connectivity (in the sense of homotopy theory [14]) of the neighborhood complex of a graph H, denoted by $\text{conn}[\mathcal{N}(H)]$, is related to chromatic number by

$$\text{conn}[\mathcal{N}(H)] + 3 \leq \chi(H),$$

and this theorem was Lovász's main tool in proving the Kneser conjecture [22].

In [17] it was shown that for $G(n, 1/2)$, the connectivity of the neighborhood complex is a.a.s. between approximately $\log_2(n)$ and $(4/3)\log_2(n)$. This is interesting to compare with what we have shown in this article, that warmth of $G(n, 1/2)$ is a.a.s. approximately $\log_2(n)$. Are there any inequalities relating the statistical physics and topological lower bounds on chromatic number? We can limit the scope of any possible inequalities by considering a few examples.

Kneser graphs play an important role in the study of graph homomorphisms [13, 22].

Definition 7.1. For $n > 2k$ the *Kneser graph* $K(n,k)$ has $\binom{n}{k}$ vertices, one for each k-subset of an n-set, with edges corresponding to disjoint subsets.

The connectivity of the neighborhood complex of Kneser graphs $K(3n-1, n)$ is $n-2$, as a special case of Lovász's result [22]. However the warmth is bounded at 3 since the set of vertices form a 2-stable family. So in general connectivity of the neighborhood complex can be larger than warmth of the graph, and the strongest inequality relating warmth and connectivity that we might hope for is that

$$w(H) \leq \text{conn}[\mathcal{N}(H)] + 3.$$

We do not at the moment know any counterexamples to this inequality. A much stronger inequality, that we also do not know any counterexamples to, would be

$$m(H) \leq \text{conn}[\mathcal{N}(H)] + 3.$$

If true, this stronger inequality would prove the Lovász conjecture, since

$$\text{conn}[\mathcal{N}(H)] + 3 \leq \chi(H).$$

Another direction for future research is into the threshold behavior for non-monotone graph properties, which have also been found to correspond to topological properties of random simplicial complexes [17, 18]. For example, a celebrated result in random graph theory is Friedgut and Kalai's result that every monotone graph property has a sharp threshold [11]. One wonders if this theorem could be extended to a larger family of graph properties, and in any case we would not be surprised if the threshold from $w(G) = k$ to $w(G) = k + 1$ is sharper than what is proved here.

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