Reduced Gröbner Bases of Certain Toric Varieties; A New Short Proof

Ibrahim Al-Ayyoub

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Abstract

Let K be a field and let $m_0, ..., m_n$ be an almost arithmetic sequence of positive integers. Let C be a toric variety in the affine $(n+1)$-space, defined parametrically by $x_0 = t^{m_0}, \ldots, x_n = t^{m_n}$. In this paper we produce a minimal Gröbner basis for the toric ideal which is the defining ideal of C and give sufficient and necessary conditions for this basis to be the reduced Gröbner basis of C , correcting a previous work of [Sen] and giving a much simpler proof than that of [Ayy].

Introduction

Let $n \geq 2$, K a field and let $x_0, \ldots, x_n, t$ be indeterminates. Let $m_0, \ldots, m_n$ be an almost arithmetic sequence of positive integers, that is, some $n-1$ of these form an arithmetic sequence, and assume $gcd(m_0, \ldots, m_n) = 1$. Let P be the kernel of the K-algebra homomorphism $\eta : K[x_0, \ldots, x_n] \to K[t]$, defined by $\eta(x_i) = t^{m_i}$. Such an ideal is called a *toric ideal* and the variety $V(P)$, the zero set of P, is called an *affiine toric variety.* The definition of toric variety that we us is the same as the definition given in [Stu1]. This differs from the definition found in the algebraic geometry literature (as in [Ful]) which requires the variety to be normal. Toric ideals are an interesting kind of ideals that have been studied by many authors, for example, see [Stu2] and Chapter 4 of [Stu1]. The theory of toric varieties plays an important role at the crossroads of geometry, algebra and combinatorics.

A set of generators for the ideal P was explicitly constructed in [PaSi]. We call these generators the *Patil-Singh generators.* Out of this generating set, Patil [Pat] constructed a minimal generating set Ω for the ideal P. We call the elements of Ω the *Patil generators.* Sengupta [Sen] proved that Ω forms a Gröbner basis for the relation ideal P with respect to the grevlex monomial order, however, Al-Ayyoub [Ayy] showed that Sengupta's proof is not complete, as in fact Ω is not a Gröbner basis in all cases, see Remark 1.6 and Remark 1.7. The proof introduced by Al-Ayyoub [Ayy] is computational as it uses the Buchberger criterion and the division algorithm and it did not characterize whether the given Gröbner basis is reduced. The goal of this paper is to produce a minimal Gröbner basis for P, give sufficient and necessary conditions for this basis to be reduced, and to give a new proof that is based on a lemma of Aramova et al. [AHH]. The proof given in this paper is much shorter and simpler than the computational work given in [Ayy] or [Sen]. The author thanks the referee for suggesting to use a result of [AHH] that shortened the proof.

1 Generators for Toric Varieties

In this part we recall the construction, given in [PaSi] and [Pat], of the generating set of the defining ideal P of certain monomial curves (toric varieties), and we also recall the result of [Ayy] proving that the set given in [Pat] is not a Gröbner basis for P. We shall use the notation and the terminology from [PaSi] and [Pat] with a slight difference in naming some variables and constants. Let $n \geq 2$ be an integer and let $p = n-1$. Let $m_0, \ldots, m_p$ be an arithmetic sequence of positive integers with $0 < m_0 < \cdots < m_p$, let m_n be arbitrary, and $gcd(m_0, \ldots, m_n) = 1$. Let Γ denote the numerical semigroup that is generated by $m_0, \ldots, m_n$ i.e. $\Gamma = \sum_{i=0}^{n} \mathbb{N}_0 m_i$ with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We assume throughout that Γ is minimally generated by $m_0, \ldots, m_n$. Put $\Gamma' = \sum_{i=0}^{p} \mathbb{N}_0 m_i$. Thus $\Gamma = \Gamma' + \mathbb{N}_{\mathbf{0}} m_n$. Let $S = \{\gamma \in \Gamma \mid \gamma - m_0 \notin \Gamma\}$.

Notation 1.1 *For $a, b \in \mathbb{Z}$ let $[a, b] = \{t \in \mathbb{Z} \mid a \leq t \leq b\}$. For $t \geq 0$, let $q_t \in \mathbb{Z}$, $r_t \in [1, p]$ and $g_t \in \Gamma'$ be defined by $t = q_t p + r_t$ and $g_t = q_t m_p + m_{r_t}$.*

The following lemma provides us with the parameters and the equalities that are crucial for the new proof.

Lemma 1.2 *(Lemmas 3.1 and 3.2, [PaSi]) Let $u = min\{t \geq 0 \mid g_t \notin S\}$ and $\upsilon = min\{b \geq 1 \mid bm_n \in \Gamma'\}$.*
(a) There exist unique integers $w \in [0, \upsilon - 1]$, $z \in [0, u-1]$, $\lambda \geq 1$, $\mu \geq 0$, and $\nu \geq 2$ such that
(i) $g_u = \lambda m_0 + w m_n$;
(ii) $\upsilon m_n = \mu m_0 + g_z$;
(iii) $g_{u-z} + (\upsilon - w) m_n = \nu m_0$, where $\nu = \begin{cases} \lambda + \mu + 1, & \text{if } \ r_{u-z} < r_u; \\ \lambda + \mu, & \text{if } \ r_{u-z} \geq r_u. \end{cases}$

(b) Let $V = [0, u-1] \times [0, \upsilon-1]$ *and* $W = [u-z, u-1] \times [\upsilon-w, \upsilon-1]$. *Then every element of* Γ *can be expressed uniquely in the form* $am_0 + g_s + bm_n$ *with* $a \in \mathbb{N}_0$ *and* $(s, b) \in V - W$.

Notation 1.3 *Let* $q = q_u$, $r = r_u$, $q' = q_{u-z}$, $r' = r_{u-z}$. *From now on, the symbols* q, q', r, r', u, υ, w, z, λ, μ, ν, V *and* W *will have the meaning assigned to them by this notation and the lemma above.*

Remark 1.4 *Note that for* $1 \leq i \leq p$ *we have* $g_i - m_0 = m_i - m_0$. *Then by the minimality assumption on the generators of* Γ *it follows that* $u > p$, *hence* $q > 0$.

We recall the construction and the result given in [PaSi]: let $p = n-1$ and let

$$\xi_{i,j} = \begin{cases} x_i x_j - x_0 x_{i+j}, & \text{if} \quad i+j \leq p; \\ x_i x_j - x_{i+j-p} x_p, & \text{if} \quad i+j > p, \end{cases}$$
$$\varphi_i = x_{r+i} x_p^q - x_0^{\lambda-1} x_i x_n^w,$$
$$\psi_j = x_{r'+j} x_p^{q'} x_n^{\upsilon-w} - x_0^{\nu-1} x_j,$$
$$\theta = \begin{cases} x_n^\upsilon - x_0^\mu x_{r-r'} x_p^{q-q'}, & \text{if} \ \ r' < r; \\ x_n^\upsilon - x_0^\mu x_{p+r-r'} x_p^{q-q'-1}, & \text{if} \ \ r' \geq r. \end{cases}$$

The following intervals are introduced by [Pat] in the process of producing minimal generating sets.

$$I = \begin{cases} [0, p-r], & \text{if } \mu \neq 0 \text{ or } W = \phi; \\ [\max(r_z - r + 1, 0), p-r], & \text{if } \mu = 0 \text{ and } W \neq \phi, \end{cases}$$
$$J = \begin{cases} \phi, & \text{if} \ \ W = \phi; \\ [0, \min(z-1, p-r')], & \text{if} \ \ W \neq \phi. \end{cases}$$

Theorem 1.5 *(Theorem 4.5, [PaSi]) The set*

$$\{\xi_{i,j} \mid 1 \leq i \leq j \leq p-1\} \cup \{\theta\} \cup \{\varphi_i \mid 0 \leq i \leq p-r\} \cup \{\psi_j \mid 0 \leq j \leq p-r'\}$$

forms a generating set for the ideal P. *The elements in this set are called the Patil-Singh generators. Also, (Theorem 4.5, [Pat]) the set*

$$\Omega = \{\xi_{i,j} \mid 1 \leq i \leq j \leq p-1\} \cup \{\theta\} \cup \{\varphi_i \mid i \in I\} \cup \{\psi_j \mid j \in J\}$$

forms a minimal generating set for the ideal P. *The elements in this set are called the Patil generators.*

Considering the indices we note that handling the Patil-Singh generators is simpler than the Patil generators.

Sengupta [Sen] tried to prove that the set Ω forms a Gröbner basis for P with respect to the grevlex monomial order using the grading $wt(x_i) = m_i$ with $x_0 < x_1 < \cdots < x_n$. In this ordering $\prod_{i=0}^{n} x_i^{a_i} >_{grevlex} \prod_{i=0}^{n} x_i^{b_i}$ if in the ordered tuple $(a_1 - b_1, \ldots, a_n - b_n)$ the left-most nonzero entry is negative. Al-Ayyoub [Ayy] proved that Sengupta's proof works for arithmetic sequences, but it is incomplete for the almost arithmetic sequences. Below we recall the work of [Ayy] for the convenience of the reader;

Remark 1.6 *Assume $r' \geq r$, $\mu = 0$, and $W \neq \phi$. Then Patil generators are not a Gröbner basis with respect to the grevlex monomial ordering with $x_0 < x_1 < \cdots < x_n$ and with the grading $wt(x_i) = m_i$.*

Proof. As $u - z = (q - q_z)p + (r - r_z)$ then $r' \geq r$ if and only if $r_z \geq r$. Assume $r' \geq r$, then $r_z - r + 1 > 0$ and also $\theta = x_n^{\upsilon} - x_0^{\mu} x_{p+r-r'} x_p^{q-q'-1}$. Assume also that $\mu = 0$ and $W \neq \phi$, then $I = [\max(r_z - r + 1, 0), p - r] = [r_z - r + 1, p - r]$. Under these assumptions the S-polynomial $S(\psi_k, \theta)$ can not be reduced to zero modulo Ω: for $0 \leq k < r_z - r + 1$ consider $S(\psi_k, \theta) = x_0^{\mu} S_1$ where $S_1 = x_0^{\lambda-1} x_k x_n^w - \underline{x_{r'+k} x_{p+r-r'} x_p^{q-1}}$, with the leading monomial underlined. We note that $LM(S_1)$, the leading monomial of S_1, is a multiple of $LM(\xi_{r'+j,p+r-r'})$ only. Hence, the only possible way to reduce S_1 with respect to Ω is by using $\xi_{r'+j,p+r-r'}$. However, none of the terms of the binomial $S_1 + x_p^{q-1} \xi_{r'+j,p+r-r'} = x_{r+k} x_p^q - x_0^{\lambda-1} x_k x_n^w$ is a multiple of any of the leading terms of Patil generators. Therefore, it can not be reduced to 0 modulo Ω. ■

The following shows that the hypothesis of the remark above are satisfied by an infinite family of toric varieties:

Remark 1.7 *Let $m_0 \geq 5$ be an odd integer. Let P be the defining ideal of the toric variety that corresponds to the almost arithmetic sequence $m_0, m_0 + 1, m_0 - 1$. Then the Patil generators for the ideal P are not a Gröbner basis with respect to the grevlex monomial ordering with $x_0 < x_1 < x_2$ and with the grading $wt(x_i) = m_i$.*

Proof. Observe: $p = 1, n = 2$, and $g_i = i(m_0 + 1)$ for all i.

Let υ, μ, and z be as defined in Lemma 1.2. Then $\upsilon(m_0 - 1) = \mu m_0 + z(m_0 + 1)$ for some integers $\mu, z \geq 0$. This implies $\mu + z < \upsilon$. Note that $\upsilon(m_0 - 1) = \mu m_0 + z(m_0 + 1) = (\mu + z)(m_0 - 1) + \mu + 2z$. Thus $\mu + 2z = s(m_0 - 1)$ for some

$s \geq 1$. Hence, $\upsilon > \mu + z \geq \dfrac{\mu}{2} + z = \dfrac{s}{2}(m_0 - 1) \geq \frac{m_0-1}{2}$. Thus,

$$\upsilon \geq \frac{m_0 + 1}{2}. \tag{1}$$

On the other hand, note that

$$\frac{m_0 + 1}{2}(m_0 - 1) = \frac{m_0 - 1}{2}(m_0 + 1) \in \Gamma'. \tag{2}$$

Therefore, by the minimality of υ we must have

$$\upsilon \leq \frac{m_0 + 1}{2}. \tag{3}$$

By (1) and (3) it follows that $\upsilon = \dfrac{m_0 + 1}{2}$.

Let u, λ, w, and g_u be as defined in Lemma 1.2. Note

$$\frac{m_0 + 1}{2}(m_0 + 1) - m_0 = \frac{m_0 - 1}{2}(m_0 - 1) + m_0 \in \Gamma. \tag{4}$$

Therefore,

$$u \leq \frac{m_0 + 1}{2}. \tag{5}$$

Claim $w > 0$: if $w = 0$ then $g_u = \lambda m_0$, thus $u(m_0 + 1) = \lambda m_0$. But m_0 and $m_0 + 1$ are relatively prime, therefore, we must have $u = bm_0$ for some $b \geq 1$, a contradiction to (5). Thus $w > 0$.

Claim $\lambda < u$: by Lemma 1.2 we have $u(m_0 + 1) = \lambda m_0 + w(m_0 - 1)$. If $\lambda \geq u$ then $w(m_0 - 1) = u(m_0 + 1) - \lambda m_0 = u + (u - \lambda)m_0$, which implies $u \geq m_0 - 1$ as $w > 0$, a contradiction to (5). Thus $\lambda < u$.

Now consider $w(m_0 - 1) = u(m_0 + 1) - \lambda m_0 = (u - \lambda)(m_0 - 1) + 2u - \lambda$. As $w(m_0 - 1) > 0$ and $u > \lambda$ we must have $2u - \lambda = c(m_0 - 1)$ for some $c \geq 1$. But if $u \leq \dfrac{m_0 - 1}{2}$ then $2u - \lambda \leq m_0 - 1 - \lambda$, a contradiction as $\lambda \geq 1$. Therefore,

$$u > \frac{m_0 - 1}{2}. \tag{6}$$

By (5) and (6) it follows that $u = \dfrac{m_0 + 1}{2}$.

Now by the uniqueness in Lemma 1.2 and as of (2) and (4) it follows that $\mu = 0$, $z = \dfrac{m_0 - 1}{2}$, $\lambda = 2$ and $w = \dfrac{m_0 - 1}{2}$. Finally, note that $r = p = r' = 1$. Therefore, the parameters z, w, μ, p, r, and r' all satisfy the assumptions of the previous remark, hence done. ■

2 Reduced Gröbner Bases

In the following we combine the results of [PaSi] and [Pat] to obtain the set of generators that we prove to be a minimal (the reduced) Gröbner Basis. In particular, we pick an appropriate set of indices (different from Sengupta [Sen]), as well as, we modify the form of the binomial θ as follows; let u, z, q, r, $q' = q_{u-z}$, and $r' = r_{u-z}$ be as in Lemma 1.2 and Notation 1.3. Let $z = q_z p + r_z$ with $q_z \in \mathbb{Z}$ and $r_z \in [1, p]$. By Notation 1.1 it is clear that $q_z \leq q$ since $0 \leq z \leq u-1$. As $u - z = (q - q_z)p + (r - r_z)$, it follows that $q' = q - q_z - \varepsilon$ and $r' = \varepsilon p + r - r_z$ where $\varepsilon = 0$ or 1 according as $r > r_z$ or $r \leq r_z$. Therefore, $r' < r$ if and only if $r_z < r$. Thus we rewrite $\theta = x_n^{\upsilon} - x_0^{\mu} x_{r_z} x_p^{q_z}$. Then the generators that we prove to be a minimal (the reduced) Gröbner basis are as follows (with the leading monomial underlined);

$$
\begin{array}{lll}
\varphi_i & = \underline{x_{r+i} x_p^{q}} - x_0^{\lambda-1} x_i x_n^{w}, & \text{for} \quad 0 \leq i \leq p-r; \\
\psi_j & = \underline{x_{r'+j} x_p^{q'} x_n^{\upsilon-w}} - x_0^{\nu-1} x_j, & \text{for} \quad j \in J; \\
\theta & = \underline{x_n^{\upsilon}} - x_0^{\mu} x_{r_z} x_p^{q_z}, & \\
\xi_{i,j} & = \begin{cases} \underline{x_i x_j} - x_0 x_{i+j}, & \text{if} \quad i+j \leq p; \\ \underline{x_i x_j} - x_{i+j-p} x_p, & \text{if} \quad i+j > p, \end{cases} & \text{for} \quad 1 \leq i \leq j \leq p-1.
\end{array}
$$

Note that this set of generators contains the set of Patil generators and it is contained in the set of Patil-Singh generators.

Definition 2.1 *Let I be a polynomial ideal and G a Gröbner basis for I such that:*
(i) $LC(f) = 1$ for all $f \in G$, where $LC(f)$ is the leading coefficient of f.
(ii) For all $f \in G$, $LM(f) \notin \langle LM\{G - \{f\}\}\rangle$.
(ii') For all $f \in G$, no monomial appearing in f lies in $\langle LM\{G - \{f\}\}\rangle$.
*Then G is called **minimal** if it satisfies (i) and (ii), and it is called **reduced** if it satisfies (i) and (ii').*

Condition 2.2 *Let C1 and C2 refer to the conditions as follows*
***C1**: $J \neq \phi, q' = 0, \upsilon - w \leq w, \lambda = 1,$ and $r' \leq p - r$.*
***C2**: $q = 1$ and $r \leq p - 2$.*

The following is the main result of this paper.

Theorem 2.3 *The set*

$$G = \{\varphi_i \mid 0 \leq i \leq p-r\} \cup \{\psi_j \mid j \in J\} \cup \{\theta\} \cup \{\xi_{i,j} \mid 1 \leq i \leq j \leq p-1\}$$

is a minimal Gröbner basis for the ideal P with respect to the grevlex monomial order with $x_0 < x_1 < \cdots < x_n$ and with the grading $wt(x_i) = m_i$. Moreover, G is reduced if and only if none of the conditions C1 and C2 holds.

Proof. The proof that G is a Gröbner basis is after Lemma 2.5 below. Here we prove that G is minimal (or reduced).

It is clear that $LM(\theta) \notin \langle LM(G - \{\theta\})\rangle$. Since $w < \upsilon$ (by Lemma 1.2) and since $q > 0$ (by Remark 1.4) it is clear that $LM(\varphi_i) \notin \langle LM(G - \{\varphi_i\})\rangle$ and $LM(\xi_{i,j}) \notin \langle LM(G - \{\xi_{i,j}\})\rangle$. To show $LM(\psi_j) \notin \langle LM(G - \{\psi_j\})\rangle$ it is clear that it suffices to show that $LM(\psi_j)$ is not a multiple of any of $LM(\varphi_i)$. If $q_z > 0$ or $\varepsilon > 0$, then this is clear since $q' < q$ (as $q' = q - q_z - \varepsilon$) and since $\upsilon - w < \upsilon$ whenever $J \neq \phi$. If $q_z = 0$ and $\varepsilon = 0$, then $r' = r - r_z$ and $z - 1 = r_z - 1 < p - r + r_z = p - r'$. Thus there is no overlap between the indices of the leading monomials of φ_i and those of ψ_j. This shows G is minimal.

Define $SM(f) = f - LM(f)$ with f a binomial. Recalling that $\nu \geq 2$ and x_0 divides no $LM(f)$ for any $f \in G$, it follows that $SM(\psi_j) \notin \langle LM(G - \{\psi_j\})\rangle$. Also, recalling that $w < \upsilon$ and $z < u$, it follows that $SM(\theta) \notin \langle LM(G - \{\theta\})\rangle$ and $SM(\xi_{i,j}) \notin \langle LM(G - \{\xi_{i,j}\})\rangle$. If any of the parts of condition *C1* does not hold, then it follows that $SM(\varphi_i) \notin \langle LM\{\psi_i\}; j \in J\rangle$ which suffices to show $SM(\varphi_i) \notin \langle LM(G - \{\varphi_i\})\rangle$. To show $SM(\xi_{i,j}) \notin \langle LM(G - \{\xi_{i,j}\})\rangle$, it is enough to show $SM(\xi_{i,j}) \notin \langle LM(\varphi_k); 0 \leq k \leq p - r\rangle$ whenever $i + j > p$ because $w < \upsilon$ and $r > 0$. But this clear if any of the parts of condition *C2* does not hold (recall $i+j-p \leq p-2$). This proves that if none of *C1* and *C2* holds, then G is reduced.

Conversely, assume *C1* holds. Then as $q' = 0$ and $\lambda = 1$ then $LM(\psi_0) = x_{r'}x_n^{\upsilon - w}$. On the other hand, since $r' \leq p - r$ then $SM(\varphi_{r'}) = x_{r'}x_n^{w}$. Thus $SM(\varphi_{r'})$ is a multiple of $LM(\psi_0)$ whenever $\upsilon - w \leq w$. Thus G is not reduced. Assume *C2* holds. Choose $i = p-1$ and $j = r+1$ (note that $j \leq p-1$ since $r \leq p-2$ by assumption). Then $SM(\xi_{i,j}) = x_r x_p = LM(\varphi_0)$. Hence G is not reduced. ■

Note the toric varieties in Remark 1.7 do not satisfy any of the conditions *C1* or *C2* as $r = p = r' = 1$. This provides a family of toric varieties with reduced Gröbner bases, while the following example provides a mimimal Gröbner basis which is not reduced.

Example 2.4 *Let $m_0 = 5, m_1 = 6, m_2 = 7, m_3 = 8,$ and $m_4 = 9$ so that $n = 4$ and $p = 3$. Note $g_4 - m_0 = m_3 + m_1 - m_0 = 9 = m_4 \in \Gamma$. Hence, $u = 4$. Thus $q = 1$ and $r = 1$. Thus C2 holds. Also, $\upsilon = 2$ as $2(m_4) = 2m_0 + m_3$. Note $g_4 = m_0 + m_4$, hence $\lambda = 1$ and $w = 1$. Also, $\upsilon m_4 = 3m_0 + m_3$, thus $z = 3$. Now, $q' = q_{u-z} = 0$ and $r' = r_{u-z} = 1$. Thus C1 holds.*

To prove the main theorem we use the following lemma of Aramova et al.

Lemma 2.5 *(Lemma 1.1, [AHH]) Let $I \subset R = K[x_0, \ldots, x_n]$ be a graded ideal and G a finite subset of homogenous elements of I. Given a term order $<$, there exist a unique monomial K-basis B of $R/(in_<(G))$. If B is a K-basis of R/I, then G is a Gröbner basis of I with respect to $<$.*

Remark 2.6 *Let $P \subset R = K[x_0, \ldots, x_n]$ be the kernel of the K-algebra homomorphism $\eta : R \to K[t]$ defined by $\eta(x_i) = t^{m_i}$ with $m_0, \ldots, m_n$ an almost arithmetic sequence of positive integers with $gcd(m_0, \ldots, m_n) = 1$. Then a set B is a K-basis of R/P if and only if $l_1 - l_2 \notin P$ for any two monomials $l_1, l_2 \in B$ with $l_1 \neq l_2$.*

Proof. Assume there exist $l_1, \ldots, l_s \in B$ and $c_1, \ldots, c_s \in K$ not all zero such that $\sum c_i l_i \in P$. This implies that $\sum c_i \eta(l_i) = 0$. Hence by the definition of η, there exist $i \neq j$ such that $\eta(l_i) = \eta(l_j)$. This implies that $l_i - l_j \in P$. ■

Proof. (of Theorem 2.3) Let G be as in the theorem (it consists of homogenous binomials according to the grading $wt(x_i) = m_i$). By Lemma 2.5 let B be the unique monomial K-basis of $R/(in_<(G))$. Assume $0 \neq l_1 - l_2 \in P$ for some monomials $l_1, l_2 \in B$. Then we show there is a contradiction to Lemma 1.2, and hence the proof is done by the above lemma and remark.

Throughout the proof let i, j, and δ_k be positive integers such that $1 \leq i, j \leq p - 1$ and $\delta_k = 0$ or 1. Also, we will use the sentence "without loss of generality" repeatedly. The usage of this sentence will be in instances as follows. If a monomial β divides l_1 and l_2, then write $l_1 - l_2 = \beta\,(l'_1 - l'_2)$ with β does not divide l'_1 or β does not divide l'_2. Note $l_1 - l_2 \in P$ if and only if $\eta(l_1) - \eta(l_2) = 0$ if and only if $\eta(l'_1) - \eta(l'_2) = 0$ if and only if $l'_1 - l'_2 \in P$.

First, we work the proof under the assumption that x_n divides either l_1 or l_2. Without loss of generality assume $x_n^{a_1}$ divides l_1 for some $a_1 < \upsilon$ but x_n does not divide l_2. Consider two cases:

<u>Case x_0 divides neither l_1 nor l_2</u>: then $x_p^{a_2}$ must divide l_2 for some a_2, otherwise $l_2 = x_j$ for some $1 \leq j \leq p - 1$ (as $x_i x_j \notin B$ for $1 \leq i \leq j \leq p - 1$). But this is a contradiction to the minimality of the generating set of Γ. We may assume that x_p does not divide l_1, therefore, we have $l_1 = x_j^{\delta_1} x_n^{a_1}$ and $l_2 = x_i^{\delta_2} x_p^{a_2}$ with $a_2 < q + \sigma$ and $\sigma = 1$ or 0 according as $i < r$ or $i \geq r$. Since $\eta(l_1) = \eta(l_2)$ we get the following equality

$$\delta_1 m_j + a_1 m_n = \delta_2 m_i + a_2 m_p \tag{1}$$

If $\delta_1 = 0$, then $a_1 m_n \in \Gamma'$, but $a_1 < \upsilon$, thus this gives a contradiction to the minimality of υ in Lemma 1.2, hence done. Therefore, assume $\delta_1 = 1$. If $\delta_2 = 0$, then the above equality becomes $m_0 + a_1 m_n = (a_2 - 1)m_p + m_{p-j}$. Note the right-hand side is $g_{(a_2-1)+(p-j)}$. Thus $g_{(a_2-1)+(p-j)} - m_0 = a_1 m_n \in \Gamma$. This gives a contradiction to the minimality of u in Lemma 1.2 as $a_2 - 1 < q$ and hence $(a_2 - 1) + (p - j) < u$. If $\delta_2 = 1$ then (1) becomes $(1-\gamma)m_0 + a_1 m_n = (a_2 - \gamma)m_p + m_{\gamma p+i-j}$ with $\gamma = 0$ or 1 according as $i > j$ or $i < j$. If $\gamma = 1$, then this gives a contradiction to the minimality of υ, on the other hand, if $\gamma = 0$, then we get a contradiction to the minimality of u (noting $a_2 \le q$ if $i < r$ and $a_2 < q$ if $i \ge r$).

<u>Case x_0 divides either l_1 or l_2</u>:

Consider four subcases:

<u>Subcase 1</u>: x_0^b divides l_1 for some $b \ge 1$ (and without loss of generality x_0 does not divide l_2). Then $x_p^{a_2}$ must divide l_2 for some a_2 (we may assume that x_p does not divide l_1), otherwise $l_2 = x_j$ for some $1 \le j \le p-1$ which is a contradiction to the minimality of the generating set of Γ. Therefore, we have $l_1 = x_0^b x_j^{\delta_1} x_n^{a_1}$ and $l_2 = x_i^{\delta_2} x_p^{a_2}$ with $a_2 < q + \sigma$ and $\sigma = 1$ or 0 according as $i < r$ or $i \ge r$. Since $\eta(l_1) = \eta(l_2)$ we get $bm_0 + \delta_1 m_j + a_1 m_n = \delta_2 m_i + a_2 m_p$. This is a contradiction to the minimality of u as $a_2 p + i < qp + r = u$ and $b \ge 1$.

<u>Subcase 2</u>: x_0^b divides l_2 for some $b \ge 1$ (and without loss of generality x_0 does not divide l_1). There are three subcases;

<u>Subsubcase 2-1</u>: x_p does not divide any of l_1 or l_2. Then $l_1 = x_j^{\delta_1} x_n^{a_1}$ and $l_2 = x_0^b x_i^{\delta_2}$. Note that if $\delta_1 = 1$, $q' = 0$, and $a_1 \ge \upsilon - w$, then we must have $j < r'$, otherwise l_1 is a multiple of $LM(\psi_{j-r'})$ and hence is not in B. Since $\eta(l_1) = \eta(l_2)$ we get

$$\delta_1 m_j + a_1 m_n = bm_0 + \delta_2 m_i \tag{2}$$

If $\delta_1 = 0$, then (2) becomes $a_1 m_n = bm_0 + \delta_2 m_i$. This is a contradiction to the minimality of υ. If $\delta_1 = 1$ and $\delta_2 = 0$, then (2) becomes $m_j + a_1 m_n = bm_0$. By Part (iii) and the uniqueness of the parameters in Lemma 1.2, this equality suggests that $a_1 = \upsilon - w$, $\nu = b + 1$, and $q' = 0$. This implies $u - z = j$. But $j < r'$ by the note above, hence $u - z < r'$ which is impossible (see Notations 1.3 and 1.1). If $\delta_1 = 1$, $\delta_2 = 1$, and $j > i$, then (2) becomes $m_{j-i} + a_1 m_n = (b+1)m_0$. By Part (iii) and the uniqueness of the parameters in Lemma 1.2, this equality suggests that $a_1 = \upsilon - w$, $\nu = b + 2$, and $q' = 0$. This implies $u - z = j - i < r'$ which is impossible. If $\delta_1 = 1$, $\delta_2 = 1$, and $j < i$, then (2) becomes $a_1 m_n = (b-1)m_0 + m_{i-j}$. This is a contradiction to the minimality of υ in Lemma 1.2.

<u>Subsubcase 2-2</u>: $x_p^{a_2}$ divides l_2 for some a_2. Then we have $l_1 = x_j^{\delta_1} x_n^{a_1}$ and $l_2 = x_0^b x_i^{\delta_2} x_p^{a_2}$ with $a_2 < q + \sigma$ and $\sigma = 1$ or 0 according as $i < r$ or $i \ge r$. Since $\eta(l_1) = \eta(l_2)$ we get $\delta_1 m_j + a_1 m_n = bm_0 + \delta_2 m_i + a_2 m_p$. Thus $a_1 m_n = (b - \delta_1)m_0 + \delta_2 m_i + (a_2 - \delta_1)m_p + \delta_1 m_{p-j} \in \Gamma'$. This is a contradiction to the

minimality of υ.

Subsubcase 2-3: $x_p^{a_2}$ divides l_1 for some a_2. Then we have $l_1 = x_j^{\delta_1} x_p^{a_2} x_n^{a_1}$ with $l_2 = x_0^b x_i^{\delta_2}$ with $\delta_k = 0$ or 1 and with appropriate values of a_1, a_2, i, and j so that $l_1, l_2 \in B$. Assume $\delta_1 = 1$. Since $\eta(l_1) = \eta(l_2)$ we get $\delta_1 m_j + a_2 m_p + a_1 m_n = bm_0 + \delta_2 m_i$. Thus we have

$$m_{\gamma p + j - i\delta_2} + (a_2 - \gamma) m_p + a_1 m_n = (b + \delta_2 - \gamma) m_0$$

where $\gamma = 1$ or 0 according as $i > j$ or $i < j$. By part (iii) and the uniqueness in Lemma 1.2, this equality suggests that $u - z = (a_2 - \gamma)p + \gamma p + j - i\delta_2$, $a_1 = \upsilon - w$, and $\nu = b + \delta_2 - \gamma$. This is a contradiction to the uniqueness of z and ν since δ_2 and γ may vary non-simultaneously. Similarly, we get a contradiction for the case $\delta_1 = 0$.

Finally, we finish the proof by taking care of the remaining case where x_n divides neither l_1 nor l_2. Consider two cases:

Case x_0 divides neither l_1 nor l_2: in such a case $l_1 = x_{i_1}^{\delta_1} x_p^{a_1}$ and $l_2 = x_{i_2}^{\delta_2} x_p^{a_2}$ with $1 \leq i_j \leq p-1$ and $a_k < q + \sigma$ and $\sigma = 1$ or 0 according as $i_k < r$ or $i_k \geq r$. Following similar process as above one can easily show that there is a contradiction.

Case x_0 divides either l_1 or l_2: then, and without loss of generality, we have $l_1 = x_{i_1}^{\delta_1} x_p^{a_1}$ and $l_2 = x_0^b x_{i_2}^{\delta_2} x_p^{a_2}$. Following similar process as above one can easily show that there is a contradiction. ■

Patil and Singh [PaSi] constructed a generating set (but not minimal) for the defining ideal P. We call the elements of this set the *Patil-Singh generators*. The generators in this set are the same as before but with different indices as follows (with q, r, q_z, r_z, q', r', and ε as before);

$$\{\xi_{i,j} \mid 1 \leq i \leq j \leq p-1\} \cup \{\theta\} \cup \{\varphi_i \mid 0 \leq i \leq p-r\} \cup \{\psi_j \mid 0 \leq j \leq p-r'\}.$$

Note that the sets of indices of φ_i and of ψ_i in the Patil-Singh generators are $[0, p-r]$ and $[0, p-r']$, respectively. On the other hand, the set of indices of φ_i and of ψ_i in the Patil generators are I and J, respectively. It turned out that the Patil set in contained in G (where G as in Theorem 2.3) which in turn is contained in the Patil-Singh set. Also, note that the set of Patil-Singh generators has the advantage of a simpler set of indices than the set G. Therefore, whenever the minimality is not an issue, it is much easier to deal with the set of Patil-Singh generators than with G. The theorem below proves that the set of Patil-Singh generators is indeed a Gröbner basis . To prove the theorem below we need the following proposition which helps to visualize the interval J given by Patil [Pat].

Proposition 2.7 *Let $z > 0$ and let $z = q_z p + r_z$ with $q_z \in \mathbb{Z}$ and $r_z \in [1, p]$. Then*

$$min\{z-1, p-r'\} = \begin{cases} p-r', & \text{if } r \le r_z; \\ p-r', & \text{if } r > r_z \text{ and } z > p; \\ z-1, & \text{if } r > r_z \text{ and } z \le p. \end{cases}$$

Moreover, $z \le p$ if and only if $q_z = 0$.

Proof. First note that $p - r' = (1-\varepsilon)p + r_z - r$ where $\varepsilon = 0$ or 1 according as $r > r_z$ or $r \le r_z$. It is obvious that if $z > 0$ then $q_z \ge 0$. Consider three cases:
Case $r \le r_z$: since $r \in [1, p]$ then $z - 1 = q_z p + r_z - 1 \ge r_z - 1 \ge r_z - r = p - r'$.
Case $r > r_z$ and $z > p$: this implies $q_z \ge 1$. Therefore, $z - 1 = q_z p + r_z - 1 \ge p + r_z - 1 \ge p + r_z - r = p - r'$.
Case $r > r_z$ and $z \le p$: this implies $q_z = 0$. Therefore, $z-1 = r_z-1 \le r_z-1+p-r < p + r_z - r = p - r'$. ■

Therefore, whenever $W \neq \phi$ we write J as follows

$$J = \begin{cases} [0, p-r'], & \text{if } q_z > 0 \text{ or } \varepsilon > 0; \\ [0, r_z - 1], & \text{if } q_z = 0 \text{ and } \varepsilon = 0. \end{cases}$$

Theorem 2.8 *The set $S = \{\varphi_i \mid 0 \le i \le p - r\} \cup \{\psi_j \mid 0 \le j \le p - r'\} \cup \{\theta\} \cup \{\xi_{i,j} \mid 1 \le i \le j \le p - 1\}$, that is, the set of Patil-Singh generators, is a Gröbner basis (not minimal) for the ideal P with respect to the grevlex monomial order with $x_0 < x_1 < \cdots < x_n$ and with the grading $wt(x_i) = m_i$.*

Proof. Recall $q' = q - q_z - \varepsilon$ and $r' = \varepsilon p + r - r_z$ where $\varepsilon = 0$ or 1 according as $r > r_z$ or $r \le r_z$. If $q_z > 0$ or $\varepsilon > 0$, then $J = [0, p - r']$ and the set of Patil-Singh generators coincides with the set G of Theorem 2.3, hence done. If $q_z = 0$ and $\varepsilon = 0$, then $q' = q$, $r' = r - r_z$, and $J = [0, r_z - 1]$. Also note $r_z \le r_z + p - r = p - r'$. Now consider $LM(\psi_j)$ where j runs over $[r_z, p - r']$ (this indicates the binomials that exist in Patil-Singh but not in G) we get $\{LM(\psi_j) = x_{j+r'} x_p^{q'} x_n^{v-w} \mid r_z \le j \le p - r'\} = \{x_j x_p^q x_n^{v-w} \mid r \le j \le p\} = x_n^{v-w}\{LM(\varphi_i) = x_{j+r} x_p^q \mid 0 \le j \le p-r\}$. Therefore, the monomial K-basis of $R/(in_<(S))$ is essentially the same as the monomial K-basis of $R/(in_<(G))$ where S is the set of the Patil-Singh generators. Hence done by Lemma 2.5. ■

Finally, we finish this paper by noting that Patil-Singh generators do not form a Gröbner basis in all cases if we consider the grevlex monomial order with the same grading as before but with $x_0 > x_1 > \cdots > x_n$ (in this case $\prod_{i=0}^{n} x_i^{a_i} >_{grevlex} \prod_{i=0}^{n} x_i^{b_i}$ if in the ordered tuple $(a_1 - b_1, \ldots, a_n - b_n)$ the right-most nonzero entry is negative). In the following we prove this and give an example.

Remark 2.9 *Assume $r < r_z < p$ (hence $\varepsilon = 0$), $\lambda > 1$, and $w > 0$. Then Patil-Singh generators are not a Gröbner basis with respect to the grevlex monomial ordering with $x_0 > x_1 > \cdots > x_n$ and with the grading $wt(x_i) = m_i$.*

Proof. First note $LT(\varphi_i) = x_{i+r}x_p^q$ if $w > 0$ and $LT(\varphi_i) = x_0^{\lambda-1}x_i$ if $w = 0$. Also, $LT(\psi_j) = x_0^{\lambda+\mu-\varepsilon}x_j$, $LT(\theta) = x_0^{\mu}x_{r_z}x_p^{q_z}$, and $LT\left(\xi_{i,j}\right) = x_ix_j$. If $r < r_z < p$ (hence $\varepsilon = 0$), $\lambda > 1$, and $w > 0$, then none of the terms of $S(\xi_{1,r_z}, \theta) = x_1x_n^{\upsilon} - x_0^{\mu+1}x_{r_z+1}x_p^{q_z}$ is a multiple of any of the leading terms of the Patil-Singh generators. ■

Example 2.10 *Let $m_0 = 20, m_1 = 21, m_2 = 22, m_3 = 23, m_4 = 24$, and $m_5 = 29$. Note $n = 5$ and $p = 4$. Let P be the kernel of the K-algebra homomorphism $\eta : K[x_0, \ldots, x_5] \to K[t]$ defined by $\eta(x_i) = t^{m_i}$. Recall the parameters in Lemma 1.2. It is easy to check that $\upsilon = 3$, hence by the uniqueness condition we must have $\mu = 2$, $q_z = 1$, and $r_z = 3$, thus $z = 7$. For $1 \leq i \leq 3$ note that in order for $am_4+m_i-m_0$ to be in Γ we must have $a \geq 2$. Note $g_{2p+1} = 2(24)+21 = 2(20)+29$. Therefore, we conclude that $u = 2p + 1 = 9$, thus $q = 2$ and $r = 1$. Hence, $\lambda = 2$, $w = 1$, $r' = 2$, and $q' = 1$. Therefore, Patil-Singh generators are as follows: $G = \{\varphi_i \mid 0 \leq i \leq 3\} \cup \{\psi_j \mid 0 \leq j \leq 2\} \cup \{\theta\} \cup \{\xi_{i,j} \mid 1 \leq i \leq j \leq 3\}$ where $\varphi_i = \underline{x_{i+1}x_4^2} - x_0x_ix_5$, and $\psi_j = x_{j+2}x_5^2 - \underline{x_0^3x_j}$, and $\theta = x_5^3 - \underline{x_0^2x_3x_4}$ and $\xi_{i,j} = \underline{x_ix_j} - x_0^{(1-\gamma)}x_{i+j-\gamma p}x_p^{\gamma}$ with $\gamma = 0$ or 1 according as $i + j \leq p$ or $i + j > p$. The set G is not Gröbner basis with respect to the grevlex monomial ordering with $x_0 > x_1 > \cdots > x_5$ and with the grading $wt(x_i) = m_i$: consider $S(\theta, \xi_{1,3}) = x_1x_5^3 - x_0^3x_4^2$. Note that neither term of $S(\theta, \xi_{1,3})$ is a multiple of any of the leading terms above.*

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References

[Ayy] I. Al-Ayyoub. The Ratliff-Rush closure and a minimal Groebner basis for the defining ideals of certain affine monomial curves. Ph.D. thesis, New Mexico State University. Las Cruces, NM (2004).

[AHH] A. Aramova, J. Herzog and T. Hibi, Finite lattices and lexicographic Gröbner bases, *Europ. J. Combin.* **21** (2000), 431 – 439.

[Ful] W. Fulton. *Introduction to Toric Varieties*, Princeton University Press, Princeton, NJ (1993).

[Pat] D. P. Patil, Minimal sets of generators for the relation ideals of certain monomial curves, *Manuscripta Math.* **80** (1993), 239 – 248.

[PaSi] D. P. Patil and B. Singh, Generators for the derivation modules and the relation ideals of certain curves, *Manuscripta Math.* **68** (1990), 327 – 335.

[Sen] I. Sengupta, A Gröbner basis for certain affine monomial curves, *Comm. Algebra.* **31** (3) (2003), 1113 – 1129.

[Stu1] B. Sturmfels, Gröbner Bases and Convex Polytopes, Amer. Math. Soc., Providence, RI (1995).

[Stu2] B. Sturmfels, Gröbner bases of toric varieties, Tohoku Mathematical Journal **43** (1991), 249-261.

Ibrahim Al-Ayyoub, assistant professor
Department of Mathematics and Statistics
Jordan University of Science and Technology
P O Box 3030, Irbid 22110, Jordan.
Email address: iayyoub@just.edu.jo