THE CONJECTURAL CONNECTIONS BETWEEN AUTOMORPHIC REPRESENTATIONS AND GALOIS REPRESENTATIONS

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Abstract. We state conjectures on the relationships between automorphic representations and Galois representations, and give evidence for them.

Contents

1. Introduction.

1.1. Given an algebraic Hecke character for a number field F, a classical construction of Weil produces a compatible system of 1-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{F}/F)$. In the 1960s it was realised by Serre and others that this construction might well be the tip of a very large iceberg. Serre conjectured the existence of 2-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to classical modular eigenforms for the group GL_2 over $\mathbb{Q}$, and their existence was established by Deligne not long afterwards. Moreover, Langlands observed that one way to attack Artin's conjecture on the analytic continuation of Artin L-functions might be via first proving that any n-dimensional irreducible complex representation of the absolute Galois group of a number field F came (in some precise sense) from an automorphic representation for GL_n/F, and then analytically continuing the L-function of this automorphic representation instead.

One might ask whether one can associate "Galois representations" to automorphic representations for an arbitrary connected reductive group over a number field. There are several approaches to formalising this problem. Firstly one could insist on working with all automorphic representations and attempt to associate to them complex representations of a "Langlands group", a group whose existence is only conjectural but which, if it exists, should be much bigger than the absolute Galois

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group of the number field (and even much bigger than the Weil group of the number field)—a nice reference for a rigorous formulation of a conjecture here is [Art02]. Alternatively one could restrict to automorphic representations that are "algebraic" in some reasonable sense, and in this case one might attempt to associate certain complex representations of the fundamental group of some Tannakian category of motives, a group which might either be a pro-algebraic group scheme or a topological group. Finally, following the original examples of Weil and Deligne, one might again restrict to algebraic automorphic representations, and then attempt to associate compatible systems of ℓ-adic Galois representations to such objects (that is, representations of the absolute Galois group of the number field over which the group is defined). The advantage of the latter approach is that it is surely the most concrete.

For the group GL_n over a number field, Clozel gave a definition of what it meant for an automorphic representation to be "algebraic". The definition was, perhaps surprisingly, a non-trivial twist of a notion which presumably had been in the air for many years. Clozel made some conjectures predicting that algebraic automorphic representations should give rise to n-dimensional ℓ-adic Galois representations (so his conjecture encapsulates Weil's result on Hecke characters and Deligne's theorem too). Clozel proved some cases of his conjecture, when he could switch to a unitary group and use algebraic geometry to construct the representations.

The goal of this paper is to generalise (most of) Clozel's conjecture to the case where GL_n is replaced by an arbitrary connected reductive group G. Let us explain the first stumbling block in this programme. The naive conjecture would be of the following form: if an automorphic representation π for G is algebraic (in some reasonable sense) then there should be a Galois representation into the $\overline{\mathbb{Q}}_\ell$-points of the L-group of G, associated to π. But if one looks, for example, at Proposition 3.4.4 of [CHT08], one sees that they can associate ℓ-adic Galois representations to certain automorphic representations on certain compact unitary groups, but that the Galois representations are taking values in a group $\mathcal{G}_n$ which one can check is *not* the L-group of the unitary group in question (for dimension reasons, for example). In fact there are even easier examples of this phenomenon: if π is the automorphic representation for $\mathrm{GL}_2/\mathbb{Q}$ attached to an elliptic curve over the rationals, then (if one uses the standard normalisation for π) one sees that π has trivial central character and hence descends to an automorphic representation for $\mathrm{PGL}_2/\mathbb{Q}$ which one would surely hope to be algebraic (because it is cohomological). However, the L-group of $\mathrm{PGL}_2/\mathbb{Q}$ is SL_2 and there is no way of twisting the Galois representation afforded by the ℓ-adic Tate module of the curve so that it lands into $\mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$, because the cyclotomic character has no square root (consider complex conjugation). On the other hand, there do exist automorphic representation for $\mathrm{PGL}_2/\mathbb{Q}$ which have associated Galois representations into $\mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$; for example one can easily build them from automorphic representations on $\mathrm{GL}_2/\mathbb{Q}$ constructed via the Langlands-Tunnell theorem applied to a continuous even irreducible 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\mathrm{SL}_2(\mathbb{C})$ with solvable image. What is going on?

Our proposed solution is the following. For a general connected reductive group G, we believe that there are *two* reasonable notions of "algebraic". For GL_n these notions differ by a twist (and this explains why this twist appears in Clozel's work). For some groups the notions coincide. But for some others—for example PGL_2—the notions are disjoint. The two definitions "differ by half the sum of the positive roots". We call the two notions C-algebraic and L-algebraic. It turns out that cohomological automorphic representations are C-algebraic (hence the C), and that one might expect Galois representations into the L-group attached to L-algebraic automorphic representations (hence the L). Clozel twists C-algebraic representations into L-algebraic ones in his paper, and hence conjectures that there should be Galois representations attached to C-algebraic representations for GL_n, but this trick is not possible in general. In this paper we explicitly conjecture the existence of ℓ-adic Galois representations associated to L-algebraic automorphic representations for a general connected reductive group over a number field.

On the other hand, one must not leave C-algebraic representations behind. For example, for certain rank 2 unitary groups over the rationals, all automorphic representations are C-algebraic and none are L-algebraic at all! It would be a shame to have no conjecture at all in these cases. We show in section 5 that given a C-algebraic automorphic representation for a group G, it can be lifted to an L-algebraic representation for a certain covering group $\tilde{G}$ (a z-extension of G), and hence one might conjecturally expect an associated Galois representation into the L-group not of G but of $\tilde{G}$. For example, if π is the automorphic representation for the group $\mathrm{PGL}_2/\mathbb{Q}$ attached to an elliptic curve over $\mathbb{Q}$, we can verify that π is C-algebraic, and that $\tilde{G} = \mathrm{GL}_2/\mathbb{Q}$ in this case, and hence one would expect a Galois representation into $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ associated to π, which is of course given by the Tate module of the curve. We also verify that in the Clozel-Harris-Taylor unitary group case, the Galois representations that they associate to their C-algebraic automorphic representations are indeed what our conjecture would predict.

In this paper, we explain the phenomenon above in more detail, and formulate a conjecture associating ℓ-adic Galois representations to L-algebraic automorphic representations for an arbitrary connected reductive group over a number field, which appears to essentially include all known theorems and conjectures of this form currently in the literature. We initially imagined that such a conjecture was already "known to the experts". However, our experience has been that this is not the case; in fact, it seems that the issues that arise when comparing the definitions of L-algebraic and C-algebraic representations were a known problem, with no clear solution (earlier attempts to deal with this issue have been by means of redefining the local Langlands correspondence and the Satake isomorphism via a twist, as in [Gro99]; however this trick only works for certain groups). The one example that we found in the literature of a covering group $\tilde{G}$—the group $\mathcal{G}_n$ of [CHT08]—seemed to us to be a construction whose main motivation was that it was the group that worked, rather than the group that came from a more conceptual argument. Consequently we hope that this article will clarify once and for all a variety of issues that occur when leaving the relative safety of GL_n, giving a firm framework for further research on Galois representations on groups other than GL_n.

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2. L-groups and local definitions.

In this section we give an overview of various standard facts concerning L-groups, the Satake isomorphism, the archimedean local Langlands correspondence, and basic Hodge-Tate theory, often with a specific emphasis on certain arithmetic aspects that are not considered relevant in many of the standard references. In summary: our L-groups will be over $\overline{\mathbb{Q}}$, we will keep track of the two different $\mathbb{Q}$-structures in the Satake isomorphism, and our local Langlands correspondence will concern representations of $G(\mathbb{R})$ or $G(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}}$ is an algebraic closure of the reals which we do not canonically identify with $\mathbb{C}$ (note that on the other hand, all our representations will be on $\mathbb{C}$-vector spaces). This section is relatively elementary but contains all of the crucial local definitions.

2.1. **The L-group.** We briefly review the notion of an L-group. We want to view the L-group of a connected reductive group as a group over $\overline{\mathbb{Q}}$, rather than the more traditional $\mathbb{C}$, as we shall later on be considering representations into the $\overline{\mathbb{Q}}_\ell$-points of the L-group. We review the standard definitions from the point of view that we shall be taking.

We take the approach to dual groups explained in section 1 of [Kot84], but work over $\overline{\mathbb{Q}}$. Let k be a field and let G be a connected reductive algebraic group over k. Fix once and for all a separable closure k^{sep} of k, and let Γ_k denote $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. The group G splits over k^{sep}, and for any maximal torus T contained in a Borel subgroup B of $G_{k^{\mathrm{sep}}}$, one can associate the based root datum $\Psi_0(G,B,T) := (X^*(T),\Delta^*(B),X_*(T),\Delta_*(B))$ consisting of the character and cocharacter groups of T, and the roots and coroots which are positive with respect to the ordering defined by B. If B' and T' are another choice of Borel and maximal torus then there is an inner automorphism of $G_{k^{\mathrm{sep}}}$ sending B' to B and T' to T, and all such inner automorphisms induce the same isomorphisms of based root data $\Psi_0(G,B,T) \to \Psi_0(G_{k^{\mathrm{sep}}},B',T')$. Following Kottwitz, we define $\Psi_0(G) := (X^*,\Delta^*,X_*,\Delta_*)$ to be the projective limit of the $\Psi_0(G,B,T)$ via these isomorphisms. This means in practice that given a maximal torus T of $G_{k^{\mathrm{sep}}}$, the group X^* is isomorphic to the character group of T but not canonically; however given also a Borel B containing the torus, there is now a canonical map $X^* = X^*(T)$ (and different Borels give different canonical isomorphisms). There is a natural group homomorphism $\mu_G : \Gamma_k \to \mathrm{Aut}(\Psi_0(G))$ (defined for example in §1.3 of [Bor79]) and if $K \subseteq k^{\mathrm{sep}}$ is a Galois extension of k that splits G then μ_G factors through $\mathrm{Gal}(K/k)$.

We let $\hat{G}$ denote a connected reductive group over $\overline{\mathbb{Q}}$ equipped with a given isomorphism $\Psi_0(\hat{G}) = \Psi_0(G)^\vee$, the dual root datum to $\Psi_0(G)$. There is a canonical group isomorphism $\mathrm{Aut}(\Psi_0(G)) = \mathrm{Aut}(\Psi_0(G)^\vee)$, sending an automorphism of X^* to its inverse (one needs to insert this inverse to ensure the group structures coincide in the bijection), and hence a canonical action of Γ_k on $\Psi_0(G)^\vee$. If we choose a Borel, a torus, and a splitting (also called a pinning) of $\hat{G}$ then, as on p10 of [Spr79], this data induces a lifting $\mathrm{Aut}(\Psi_0(G)^\vee) \to \mathrm{Aut}(\hat{G})$ and hence (via μ_G) a left action of Γ_k on $\hat{G}$. We define the L-group ${}^LG$ of G to be the resulting semidirect product, regarded as a group scheme over $\overline{\mathbb{Q}}$ with connected component $\hat{G}$ and component group Γ_k. For K a field containing $\overline{\mathbb{Q}}$ we have ${}^LG(K) = \hat{G}(K) \rtimes \Gamma_k$. Often in the literature people use ${}^LG$ to be the group that we call ${}^LG(\mathbb{C})$.

Note that there is a fair amount of "ambiguity" in this definition. The group $\hat{G}$ is "only defined up to inner automorphisms", as is the lifting of μ_G. So, even if we fix our choice of k^{sep}, points in ${}^LG(K)$ are "only defined up to conjugation by $\hat{G}(K)$".

If K is an extension of $\overline{\mathbb{Q}}$ and ρ is a group homomorphism $\mathrm{Gal}(\overline{k}/k) \to {}^LG(K)$, then we say that ρ is *admissible* if the map $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(\overline{k}/k)$ induced by ρ and the surjection ${}^LG(K) \to \mathrm{Gal}(\overline{k}/k)$ is the identity.

We fix once and for all an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$. Later on, when talking about Galois representations, we shall fix a prime number p and an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. This will enable us to talk about the groups ${}^LG(\mathbb{C})$ and ${}^LG(\overline{\mathbb{Q}}_p)$.

2.2. **Satake parameters.** In this section, k is a non-archimedean local field and we fix a separable closure $\overline{k}$ of k and set $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. We normalise the reciprocity map $k^\times \to \Gamma_k^{\mathrm{ab}}$ of local class field theory so that it takes a uniformiser to a geometric Frobenius. We follow Tate's definitions and conventions for Weil groups—in brief, a Weil group $W_k = W_{\overline{k}/k}$ for k comes equipped with maps $W_k \to \Gamma_k$ and $k^\times \to W_k^{\mathrm{ab}}$ such that the induced map $k^\times \to \Gamma_k^{\mathrm{ab}}$ is the reciprocity homomorphism of class field theory, normalised as above.

Let G/k be connected reductive group which is furthermore unramified (that is, quasi-split, and split over an unramified extension of k). Then $G(k)$ has hyperspecial maximal compact subgroups; fix one, and call it K. Nothing we do will depend on this choice, but we will occasionally need to justify this. Let B be a Borel in G defined over k, let T be a maximal torus of B, and let T_d be the maximal split sub-torus of T. Let W_d be the subgroup of the Weyl group of G consisting of elements which map T_d to itself. Let ${}^oT$ denote the maximal compact subgroup of $T(k)$. It follows from an easy cohomological calculation (done for example in §9.5 of [Bor79]) that the inclusion $T_d \to T$ induces an isomorphism of groups $T_d(k)/T_d(\mathcal{O}) \to T(k)/{}^oT$. We normalise Haar measure on $G(k)$ so that K has measure 1 (and remark that by 3.8.2 of [Tit79] this normalisation is independent of the choice of hyperspecial maximal compact K). If R is a field of characteristic zero then let $H_R(G(k),K)$ denote the Hecke algebra of bi-K-invariant R-valued functions on $G(k)$ with compact support, and with multiplication given by convolution. Similarly $H_R(T(k),{}^oT)$ is the analogous Hecke algebra for $T(k)$.

The Satake isomorphism (see for example §4.2 of [Car79]) is a canonical isomorphism $H_{\mathbb{C}}(G(k),K) = \mathbb{C}[X_*(T_d)]^{W_d} = H_{\mathbb{C}}(T(k),{}^oT)^{W_d}$, where $X_*(T_d)$ is the cocharacter group of T_d. We normalise the Satake isomorphism in the usual way, so that it does not depend on the choice of the Borel subgroup containing T; this is the only canonical way to do things. This standard normalisation is however not in general "defined over $\mathbb{Q}$"—for example if $k = \mathbb{Q}_p$ and $G = \mathrm{GL}_2$ and $K = \mathrm{GL}_2(\mathbb{Z}_p)$ then the Satake isomorphism sends the characteristic function of $K\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)K$ to a function on $T(\mathbb{Q}_p)$ taking the value $\sqrt{p}$ on the matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This square root of p appears because the definition of the Satake isomorphism involves a twist by half the sum of the positive roots of G (see formula (19) of section 4.2 of [Car79]) and because of this twist, the isomorphism does *not* in general induce a canonical isomorphism $H_{\mathbb{Q}}(G(k),K) = \mathbb{Q}[X_*(T_d)]^{W_d}$.

In [Gro99] and [Clo90] this issue of square roots is avoided by renormalising the Satake isomorphism. Let us stress that we shall *not* do this here, and we shall think of $H_{\mathbb{Q}}(G(k),K)$ and $\mathbb{Q}[X_*(T_d)]^{W_d}$ as giving two possibly distinct $\mathbb{Q}$-structures on the complex algebraic variety $\mathrm{Spec}(H_{\mathbb{C}}(G(k),K))$ which shall perform two different functions—they will give us two (typically distinct) notions of being defined over a subfield of $\mathbb{C}$. We note however that if half the sum of the positive roots of G is in the weight lattice $X^*(T)$ (this occurs for example if G is semi-simple and simply connected, or a torus) then the map $\delta^{1/2}: T(k) \to \mathbb{R}_{>0}$ mentioned in formula (19) of [Car79] is $\mathbb{Q}$-valued (see formula (4) of [Car79] for the definition of δ) and the proof of Theorem 4.1 of [Car79] makes it clear that in this case the Satake isomorphism does induce an isomorphism $H_{\mathbb{Q}}(G(k),K) = \mathbb{Q}[X_*(T_d)]^{W_d}$.

Next we recall how the Satake isomorphism above leads us to an unramified local Langlands correspondence. Say π_v is an unramified complex representation of $G(k)$ (that is, it has non-zero fixed vectors for some hyperspecial maximal compact subgroup of $G(k)$). Assume furthermore that $\pi_v^K \neq 0$ for our given choice of K. Then π_v^K is a 1-dimensional representation of $H_{\mathbb{C}}(G,K)$ and hence gives rise to a $\mathbb{C}$-valued character of $H_{\mathbb{C}}(G,K)$. Now results of Gantmacher and Langlands in section 6 of [Bor79] enable one to canonically associate to π_v an unramified continuous admissible representation $r_{\pi_v}: W_k \to {}^LG(\mathbb{C})$, where "admissible" in this context means a group homomorphism such that if one composes it with the canonical map ${}^LG(\mathbb{C}) \to \Gamma_k$ then one obtains the canonical map $W_k \to \Gamma_k$ (part of the definition of the Weil group) and "unramified" means that the resulting 1-cocycle $W_k \to \hat{G}(\mathbb{C})$ is trivial on the inertia subgroup of W_k. In fact there are two ways of normalising the construction: if we follow section 6 of [Bor79] then in the crucial Proposition 6.7 Borel has chosen the σ that appears there to be an *arbitrary* generator of the Galois group of a finite unramified extension of k which splits G, and we are free to choose σ to be either an arithmetic or a geometric Frobenius. We shall let σ denote a geometric Frobenius: now an easy check (unravelling the definitions in [Bor79] and [Car79]) shows that if $G = \mathrm{GL}_1$ and π is an unramified representation of $\mathrm{GL}_1(k)$ then the corresponding Galois representation $W_k \to \mathrm{GL}_1(\mathbb{C})$ is the one induced by our given isomorphism $k^\times = W_k^{\mathrm{ab}}$. See Remark 3.2.5 for some comments about what would have happened had we chosen an arithmetic Frobenius here.

As we have already mentioned, we have a natural $\mathbb{Q}$-structure on $H_{\mathbb{C}}(G(k),K)$ coming from the $\mathbb{Q}$-valued functions $H_{\mathbb{Q}}(G(k),K)$, and we have another one coming from $\mathbb{Q}[X_*(T_d)]^{W_d}$ via the Satake isomorphism. This means that, for a smooth irreducible admissible representation π of $G(k)$ with a K-fixed vector, there are two (typically distinct) notions of what it means to be "defined over E", for E a subfield of $\mathbb{C}$. Indeed, if π is a smooth admissible irreducible representation of $G(k)$ with a K-fixed vector, then π^K is a 1-dimensional complex vector space on which $H_{\mathbb{C}}(G(k),K)$ acts, and this action induces maps

$$H_{\mathbb{Q}}(G(k),K) \to \mathbb{C}$$

and

$$\mathbb{Q}[X_*(T_d)]^{W_d} \to \mathbb{C}.$$

Definition 2.2.1. Let π be smooth, irreducible and admissible, with a K-fixed vector. Let E be a subfield of $\mathbb{C}$.

(i) We say that π *is defined over* E if the induced map $H_{\mathbb{Q}}(G(k),K) \to \mathbb{C}$ has image lying in E.

(ii) We say that *the Satake parameter of* π *is defined over* E if the induced map $\mathbb{Q}[X_*(T_d)]^{W_d} \to \mathbb{C}$ has image lying in E.

If half the sum of the positive roots of G is in the lattice $X^*(T)$, then these notions coincide. However there is no reason for them to coincide in general and we shall shortly see examples for $\mathrm{GL}_2(\mathbb{Q}_p)$ where they do not.

Note also that it is not immediately clear that these notions are independent of the choice of K: perhaps there is some π with a K-fixed vector and a K'-fixed vector for two non-conjugate hyperspecial maximal compacts (for example, the trivial 1-dimensional representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ has this property), and which is defined over E (or has Satake parameter defined over E) for one choice but not for the other. The reader should bear in mind that for the time being these notions depend on the choice of K, although we will soon see (in Corollary 2.2.3 and Lemma 2.2.4) that they are in fact independent of this choice.

We now discuss some other natural notions of being "defined over E" for E a subfield of $\mathbb{C}$, and relate them to the notions above. So let E be a subfield of $\mathbb{C}$ and let π be a smooth irreducible admissible complex representation of $G(k)$ with a K-fixed vector, for our fixed choice of K. Let V be the underlying vector space for π.

Lemma 2.2.2. *The following are equivalent:*

(i) The representation π is defined over E.

(ii) There is an E-subspace V_0 of V which is $G(k)$-stable and such that $V_0 \otimes_E \mathbb{C} = V$.

(iii) For any (possibly discontinuous) field automorphism σ of $\mathbb{C}$ which fixes E pointwise, we have $\pi \cong \pi^\sigma = \pi \otimes_{\mathbb{C},\sigma} \mathbb{C}$ as $\mathbb{C}$-representations.

Proof. That (ii) implies (iii) is clear—it's an abstract representation-theoretic fact. Conversely if (iii) holds, then (ii) follows from Lemma I.1 of [Wal85] (note: his E is not our E), because V^K is 1-dimensional. This latter lemma of Waldspurger also shows that if (ii) holds then V_0^K is 1-dimensional over E, and hence (ii) implies (i). To show that (i) implies (ii) we look at the explicit construction giving π from the algebra homomorphism $H_{\mathbb{Q}}(G(k), K) \to \mathbb{C}$ given in [Car79]. Given a homomorphism $H_{\mathbb{Q}}(G(k), K) \to \mathbb{C}$ with image landing in E, the resulting spherical function $\Gamma : G(k) \to \mathbb{C}$ defined in equation (30) of [Car79] is also E-valued. Now if we define V_0 to be the E-valued functions on $G(k)$ of the form $f(g) = \sum_{i=1}^n c_i \Gamma(g g_i)$ for $c_i \in E$ and $g_i \in G(k)$, then $G(k)$ acts on V_0 by right translations, $V_0 \otimes_E \mathbb{C}$ is the V_Γ of §4.4 of [Car79], and the arguments in §4.4 of [Car79] show that $\pi \cong V_0 \otimes_E \mathbb{C}$. □

Corollary 2.2.3. *If π is a smooth irreducible admissible unramified representation of $G(k)$, then the notion of being "defined over E" is independent of the choice of hyperspecial maximal compact K for which π^K is non-zero.*

Proof. This is because condition (iii) of Lemma 2.2.2 is independent of this choice. □

We now prove the analogous result for Satake parameters.

Lemma 2.2.4. *If π is a smooth irreducible admissible unramified representation of $G(k)$, then the notion of π having Satake parameter being defined over E is independent of the choice of hyperspecial maximal compact K for which $\pi^K \neq 0$.*

Proof. Say π is an unramified smooth irreducible admissible representation of $G(k)$ with a K-fixed vector. The Satake isomorphism associated to K gives us a character of the algebra $H_{\mathbb{C}}(T(k), {}^oT)^{W_d}$ and hence a W_d-orbit of complex characters of $T(k)$. Now by p45 of [Bor79] and sections 3 and 4 of [Car79], π is a subquotient of the principal series representation attached to any one of these characters, and Theorem 2.9 of [BZ77] then implies that this W_d-orbit of complex characters are the only characters for which π occurs as a subquotient of the corresponding induced representations. Hence the W_d-orbit of characters, and hence the map $\mathbb{Q}[X_*(T_d)]^{W_d} \to \mathbb{C}$ attached to π, does not depend on the choice of K in the case when π has fixed vectors for more than one conjugacy class of hyperspecial maximal compact. In particular the image of $\mathbb{Q}[X_*(T_d)]^{W_d}$ in $\mathbb{C}$ is well-defined independent of the choice of K, and hence the notion of having Satake parameter defined over E is also independent of the choice of K. □

To clarify the meaning of having a Satake parameter defined over E, we now explain that in the case of $G = \mathrm{GL}_n$ the notion becomes a more familiar one. If π is an unramified representation of $\mathrm{GL}_n(k)$ then the formalism above associates to π an algebra homomorphism $\mathbb{C}[X_*(T_d)]^{W_d} \to \mathbb{C}$. But here $T = T_d$ as G is split, and W_d is the usual Weyl group W of G. The ring $\mathbb{C}[X_*(T_d)] = \mathbb{C}[X_*(T)]$ is then just the ring of functions on the dual torus $\hat{T}$, and hence an unramified π gives rise to a W_d-orbit on $\hat{T}$, which can be interpreted as a semisimple conjugacy class S_π in $\mathrm{GL}_n(\mathbb{C})$.

Lemma 2.2.5. *Let G be the group GL_n/k and let π be an unramified representation of $G(k)$. Let E be a subfield of $\mathbb{C}$. Then the Satake parameter of π is defined over E if and only if the conjugacy class S_π contains an element of $\mathrm{GL}_n(E)$.*

Proof. The statement that the Satake parameter is defined over E is precisely the statement that the induced map $\mathbb{Q}[X_*(T)]^W \to \mathbb{C}$ takes values in E, which is the statement that the characteristic polynomial of an element of S_π has coefficients in E. But this is the case if and only S_π contains an element of $\mathrm{GL}_n(E)$, because given a monic polynomial with coefficients in E it is easy to construct a semisimple matrix with this polynomial as characteristic polynomial. □

We leave to the reader the following elementary checks. Let G_1 and G_2 be unramified connected reductive groups over k, and let π_1, π_2 be unramified representations of $G_1(k)$, $G_2(k)$. Then $\pi := \pi_1 \otimes \pi_2$ is an unramified representation of $(G_1 \times G_2)(k)$. One can check that π is defined over E iff π_1 and π_2 are defined over E, and that π has Satake parameter defined over E iff π_1 and π_2 do. Now say k_1/k is a finite unramified extension of non-archimedean local fields, and G/k_1 is unramified connected reductive, and set $H = \mathrm{Res}_{k_1/k}(G)$. Then H is unramified over k_1, and if π is a representation of $G(k_1) = H(k)$ then π is unramified as a representation of $G(k_1)$ if and only if it is unramified as a representation of $H(k)$. Furthermore, the two notions of being defined over E (one for G and one for H) coincide. Moreover, the two notions of having Satake parameter defined over E—one for G and one for H—also coincide; we give the argument for this as it is a little trickier. Let T_d denote a maximal split torus in G and let T denote its centralizer. The Satake homomorphism for G is an injective ring homomorphism from an unramified Hecke algebra for G into $\mathbb{C}[T(k_1)/U]$, with U a maximal compact subgroup of $T(k_1)$. The Satake homomorphism for H is a map between the same two rings, and it can be easily checked from the construction in Theorem 4.1 of [Car79] that it is in fact the same map. The map for G is an isomorphism onto the subring $\mathbb{C}[T(k_1)/U]^{W(G)}$ of $\mathbb{C}[T(k_1)/U]$, with $W(G)$ the relative Weyl group for the pair (G, T_d). The map for H is an isomorphism onto $\mathbb{C}[T(k_1)/U]^{W(H)}$, and hence $\mathbb{C}[T(k_1)/U]^{W(G)} = \mathbb{C}[T(k_1)/U]^{W(H)}$. Now intersecting with $\mathbb{Q}[T(k_1)/U]$ we deduce that $\mathbb{Q}[T(k_1)/U]^{W(G)} = \mathbb{Q}[T(k_1)/U]^{W(H)}$ and hence the two $\mathbb{Q}$-structures—one coming from G and one from H—coincide.

We finish this section by noting that the notion of being defined over E does not coincide with the notion of having Satake parameter defined over E, if $k = \mathbb{Q}_p$ and $G = \mathrm{GL}_2$. For example, if π is the trivial 1-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ then π is defined over $\mathbb{Q}$ but the Satake parameter attached to π has eigenvalues $\sqrt{p}$ and $1/\sqrt{p}$, so the Satake parameter is not defined over $\mathbb{Q}$ (consider traces) but only over $\mathbb{Q}(\sqrt{p})$. Similarly if π is the character $|\det|^{1/2}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ then π_p is not defined over $\mathbb{Q}$ but the Satake parameter of π has characteristic polynomial $(X-1)(X-p)$ and hence is defined over $\mathbb{Q}$. This issue of the canonical normalisation of the Satake isomorphism "introducing a square root of p" is essentially the reason that one sees two normalisations of local Langlands for GL_n in the literature—one used for local questions and one used for local-global compatibility. We are not attempting to unify these two notions—indeed, one of the motivations of this paper is to draw the distinction between the two notions and explain what each is good for.

2.3. **Local Langlands at infinity.** We recall the statements and basic properties of the local Langlands correspondence for connected reductive groups over the real or complex field. In practice, the groups that we will apply these statements to are groups defined over completions of number fields at infinite places, so in fact we work with groups defined over either $\mathbb{R}$ or a degree two extension of $\mathbb{R}$ which will be isomorphic to $\mathbb{C}$ but may not be canonically isomorphic to $\mathbb{C}$. Note however that all our representations will be on $\mathbb{C}$-vector spaces.

Let k be either the real numbers or an algebraic closure of the real numbers. Let G be a connected reductive group over k. Fix an algebraic closure $\overline{k}$ of k and let $T \subseteq B$ be a maximal torus and a Borel subgroup of $G_{\overline{k}}$. If π_∞ is an irreducible admissible complex representation of $G(k)$ then Langlands associates to π_∞, in a completely canonical way, a $\hat{G}(\mathbb{C})$-conjugacy class of admissible homomorphisms $r = r_{\pi_\infty}$ from the Weil group $W_k = W_{\overline{k}/k}$ of k to ${}^LG(\mathbb{C})$. For simplicity let us choose a maximal torus $\hat{T}$ in $\hat{G}_{\mathbb{C}}$; this is just for notational convenience. The group W_k contains a finite index subgroup canonically isomorphic to $\overline{k}^\times$; let us assume that $r(\overline{k}^\times) \subseteq \hat{T}(\mathbb{C})$ (which can always be arranged, possibly after conjugating r by an element of $\hat{G}(\mathbb{C})$). If σ and τ denote the two $\mathbb{R}$-isomorphisms $\overline{k} \to \mathbb{C}$ then one checks easily that for $z \in \overline{k}^\times$ we have $r(z) = \sigma(z)^{\lambda_\sigma}\tau(z)^{\lambda_\tau}$ for $\lambda_\sigma, \lambda_\tau \in X_*(\hat{T}) \otimes \mathbb{C}$ such that $\lambda_\sigma - \lambda_\tau \in X_*(\hat{T})$. Note that because we may not want to fix a preferred choice of isomorphism $\overline{k} = \mathbb{C}$, we might sometimes "have no preference between λ_σ and λ_τ"; this makes our presentation diverge slightly from other standard references.

Because $\hat{T}(\mathbb{C})$ is usually not its own normaliser in $\hat{G}(\mathbb{C})$, there is usually more than one way of conjugating $r(\overline{k}^\times)$ into $\hat{T}(\mathbb{C})$, with the consequence that the pair $(\lambda_\sigma, \lambda_\tau) \in (X_*(\hat{T}) \otimes \mathbb{C})^2$ is not a well-defined invariant of r_{π_∞}; it is only well-defined up to the (diagonal) action of the Weyl group $W = W(G, T)$ on $(X_*(\hat{T}) \otimes \mathbb{C})^2$. For notational convenience however we will continue to refer to the elements λ_σ and λ_τ of $X_*(\hat{T}) \otimes \mathbb{C}$ and will check that none of our important later definitions depend on the choice we have made. If $k = \mathbb{R}$ then recall from the construction of the L-group that there is an action of Γ_k on $X^*(T) \otimes \mathbb{C}$, and the non-trivial element of this group sends the W-orbit of λ_σ to the W-orbit of λ_τ. If k is isomorphic to $\mathbb{C}$ then λ_σ and λ_τ are in general unrelated, subject to their difference being in $X^*(T)$.

The Weyl group orbit of $(\lambda_\sigma, \lambda_\tau)$ in $(X^*(T) \otimes \mathbb{C})^2$ is naturally an invariant attached to the Weil group representation r_{π_∞} rather than to π_∞ itself, but we can access a large part of it (however, not quite all of it) more intrinsically from π_∞ using the Harish-Chandra isomorphism. We explain the story when $k = \mathbb{R}$; the analogous questions in the case $k \cong \mathbb{C}$ can be answered by restriction of scalars.

So, for this paragraph only, we assume $k = \mathbb{R}$. If we regard $G(k)$ as a real Lie group with Lie algebra $\mathfrak{g}$, then our maximal torus T of $G_{\overline{k}}$ gives rise to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} \otimes_{\mathbb{R}} \overline{k}$. If we now break the symmetry and use σ to identify $\overline{k}$ with $\mathbb{C}$, we can interpret the Lie algebra of $T \times_{\overline{k},\sigma} \mathbb{C}$ as a complex Cartan subalgebra $\mathfrak{h}_\sigma^{\mathbb{C}}$ of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We have a canonical isomorphism $\mathfrak{h}_\sigma^{\mathbb{C}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ (this isomorphism implicitly also uses σ, because $X_*(T) = \mathrm{Hom}(\mathrm{GL}_1/\overline{k}, T)$ was computed over $\overline{k}$). Now via the Harish-Chandra isomorphism (normalised in the usual way, so it is independent of the choice of Borel) one can interpret the infinitesimal character of π_∞ as a W-orbit in $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_\sigma^{\mathbb{C}}, \mathbb{C}) = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} = X_*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{C}$. Furthermore, this W-orbit contains λ_σ (this seems to be well-known; see Proposition 7.4 of [Vog93] for a sketch proof). On the other hand, we note that applying this to both σ and τ gives us a pair of W-orbits in $X^*(T) \otimes \mathbb{C}$, whereas our original construction of $(\lambda_\sigma, \lambda_\tau)$ gives us the W-orbit of a pair, which is a slightly finer piece of information (which should not be surprising: there are reducible principal series representations of $\mathrm{GL}_2(\mathbb{R})$ whose irreducible subquotients (one discrete series, one finite-dimensional) have the same infinitesimal character but rather different associated Weil representations).

We go back now to the general case $k = \mathbb{R}$ or $k \cong \mathbb{C}$. We have a W-orbit $(\lambda_\sigma, \lambda_\tau)$ in $(X^*(T) \otimes_{\mathbb{Z}} \mathbb{C})^2$ attached to π_∞. One obvious "algebraicity" criterion that one could impose on π_∞ is that $\lambda_\sigma \in X_*(\hat{T}) = X^*(T)$. Note that λ_σ is only well-defined up to an element of the Weyl group, but the Weyl group of course preserves $X_*(\hat{T}) = X^*(T)$, so the notion is well-defined. Also λ_σ depends on the isomorphism $\sigma : \overline{k} \to \mathbb{C}$, but if we use τ instead then the notion remains unchanged, because $\lambda_\sigma - \lambda_\tau \in X_*(\hat{T})$ and hence $\lambda_\sigma \in X_*(\hat{T})$ if and only if $\lambda_\tau \in X_*(\hat{T})$. This notion of algebraicity is frequently used in the literature—one can give the connected component of the Weil group of k the structure of the real points of an algebraic group $\mathcal{S}$ over $\mathbb{R}$ and one is asking here that the Weil representation associated to π_∞ restricts to a map $\mathcal{S}(\mathbb{R}) \to {}^LG(\mathbb{C})$ induced by a morphism of algebraic groups $\mathcal{S}_{\mathbb{C}} \to {}^LG_{\mathbb{C}}$ via the inclusion $\mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{C})$.

Definition 2.3.1. We say that an admissible Weil group representation $r : W_k \to {}^LG(\mathbb{C})$ is *L-algebraic* if $\lambda_\sigma \in X^*(T)$. We say that an irreducible representation π_∞ of $G(k)$ is *L-algebraic* if the Weil group representation associated to it by Langlands is L-algebraic.

Note that the notion of L-algebraicity for a Weil group representation r depends only on the restriction of r to $\overline{k}^\times$, and the notion of L-algebraicity for a representation of $G(k)$ depends only on the infinitesimal character of this representation when $k = \mathbb{R}$.

Later on we will need the following easy lemma. Say $k = \mathbb{R}$ and $(\lambda_\sigma, \lambda_\tau)$ is a representative of the W-orbit on $X_*(\hat{T})^2$ associated to an L-algebraic π_∞.

Lemma 2.3.2. *If i is a square root of -1 in $\overline{k}$ and j is the usual element of order 4 in W_k then the element $\alpha_\infty := \lambda_\sigma(i)\lambda_\tau(i)r_{\pi_\infty}(j) \in {}^LG(\mathbb{C})$ lies in ${}^LG(\overline{\mathbb{Q}})$, has order dividing 2, and its $\hat{G}(\mathbb{C})$-conjugacy class is well-defined independent of (a) the choice of order of σ and τ, (b) the choice of representative $(\lambda_\sigma, \lambda_\tau)$ of the W-orbit and (c) the choice of square root of -1 in $\overline{k}$.*

Proof. Set $r := r_{\pi_\infty}$. We have $\lambda_\sigma(z)r(j) = r(j)\lambda_\tau(z)$, and $\lambda_\sigma(z)$ commutes with $\lambda_\tau(z')$, and from this it is easy to check that $(\alpha_\infty)^2 = 1$ and that α_∞ is unchanged if we switch σ and τ. Changing representative of the W-orbit just amounts to conjugating r by an element of $\hat{G}(\mathbb{C})$ and hence conjugating α_∞ by this same element. Finally one checks easily that conjugating α_∞ by $\lambda_\sigma(-1)$ gives us the analogous element with i replaced by $-i$. □

The notion of L-algebraicity will be very important to us later, however it is not hard to find automorphic representations that "appear algebraic in nature" but whose infinite components are not L-algebraic. For example one can check that if E is an elliptic curve over $\mathbb{Q}$ and π is the associated automorphic representation of $\mathrm{PGL}_2/\mathbb{Q}$, then π_∞, when considered as a representation of $\mathrm{PGL}_2(\mathbb{R})$, is not L-algebraic: the element λ_σ above is in $X_*(\hat{T}) \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ but not in $X_*(\hat{T})$. What has happened is that the canonical normalisation of the Harish-Chandra homomorphism involves (at some point in the definition) a twist by half the sum of the positive roots, and it is this twist that has taken us out of the lattice in the elliptic curve example.

This observation motivates a *second* notion of algebraicity—which it turns out is the one used in Clozel's paper for the group GL_n. Let us go back to the case of a general connected reductive G over k, either the reals or a field isomorphic to the complexes. Recall that we have fixed $T \subseteq B \subseteq G_{\overline{k}}$ and hence we have the notion of a positive root in $X^*(T)$. Let $\delta \in X^*(T) \otimes \mathbb{C}$ denote half the sum of the positive roots. We observed above that the assertion "$\lambda_\sigma \in X^*(T)$" was independent of the choice of B and of the isomorphism $\overline{k} \cong \mathbb{C}$. But the assertion "$\lambda_\sigma - \delta \in X^*(T)$" is also independent of such choices, for if $\lambda_\sigma - \delta \in X^*(T)$ and w is in the Weyl group, then $w.\lambda_\sigma - \delta = w(\lambda_\sigma - \delta) - (\delta - w.\delta) \in X^*(T)$, and also $\lambda_\tau - \delta = (\lambda_\sigma - \delta) + (\lambda_\tau - \lambda_\sigma) \in X^*(T)$.

Definition 2.3.3. We say that the admissible Weil group representation $r : W_k \to {}^LG(\mathbb{C})$ is *C-algebraic* if $\lambda_\sigma - \delta \in X^*(T)$. We say that the irreducible admissible representation π_∞ of $G(k)$ is *C-algebraic* if the Weil group representation associated to π_∞ by Langlands is C-algebraic.

Again, C-algebraicity for r only depends on the restriction of r to $\overline{k}^\times$, and C-algebraicity for π_∞ only depends on its infinitesimal character when $k = \mathbb{R}$.

Here are some elementary remarks about these definitions. If $\delta \in X^*(T)$ then the notions of L-algebraic and C-algebraic coincide. If G_1 and G_2 are connected reductive over k, if r_i $(i = 1, 2)$ are admissible representations $r_i : W_k \to {}^LG_i(\mathbb{C})$, then there is an obvious notion of a product $r_1 \times r_2 : W_k \to {}^L(G_1 \times G_2)(\mathbb{C})$ and $r_1 \times r_2$ is L-algebraic (resp. C-algebraic) iff r_1 and r_2 are. One can furthermore check that if k denotes an algebraic closure of the reals and G/k is connected reductive, and if $H = \mathrm{Res}_{k/\mathbb{R}}(G)$, and if π is an irreducible admissible representation of $G(k) = H(\mathbb{R})$, then π is L-algebraic (resp. C-algebraic) when considered as a representation of $G(k)$ if and only if it is L-algebraic (resp. C-algebraic) when considered as a representation of $H(\mathbb{R})$. This assertion comes from a careful reading of sections 4 and 5 of [Bor79]. Indeed, if T is a maximal torus of G/k then $\mathrm{Res}_{k/\mathbb{R}}(T)$ is a maximal torus of $H/\mathbb{R}$, and if $\lambda_\sigma, \lambda_\tau \in X^*(T)\otimes\mathbb{C}$ are the parameters attached to a representation of $G(k)$, then $\lambda_\sigma \oplus \lambda_\tau$ and $\lambda_\tau \oplus \lambda_\sigma \in X^*(T)\oplus X^*(T)$ are the parameters attached to the corresponding representation of $H(\mathbb{R})$ (identifying $\hat{H}(\mathbb{C})$ with $\hat{G}(\mathbb{C})^2$), and if δ is half the sum of the positive roots for G then $\delta\oplus\delta$ is half the sum of the positive roots for H. As a consequence, we see that both L-algebraicity and C-algebraicity of a representation π_∞ of $G(k)$ are conditions that only depend on the infinitesimal character of the representation of the underlying real reductive group.

Let us again attempt to illustrate the difference between the two notions of algebraicity by considering the trivial 1-dimensional representation of $\mathrm{GL}_2(\mathbb{R})$. The Local Langlands correspondence associates to this the 2-dimensional representation $|.|^{1/2}\oplus|.|^{-1/2}$ of the Weil group of the reals. If we choose the diagonal torus in GL_2 and identify its character group with $\mathbb{Z}^2$ in the obvious way, then we see that $\lambda_\sigma = \lambda_\tau = \delta = (\frac{1}{2}, -\frac{1}{2})$. In particular, λ_σ is not in $X^*(T)$, but $\lambda_\sigma - \delta$ is. Another example would be the character $|\det|^{1/2}$ of $\mathrm{GL}_2(\mathbb{R})$; this is associated to the representation $|.|\oplus 1$ of the Weil group, and so $\lambda_\sigma = \lambda_\tau = (1, 0)$ (or $(0, 1)$, allowing for the Weyl group action) and on this occasion λ_σ is in $X^*(T)$ but $\lambda_\sigma - \delta$ is not. Finally let us consider the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with trivial central character associated to a weight 2 modular form. The associated representation of the Weil group sends an element z of $\overline{\mathbb{R}}^\times$ to a matrix with eigenvalues $\sqrt{z.\overline{z}}/z$ and $\sqrt{z.\overline{z}}/\overline{z}$, the square root being the positive square root. We see that the set $\{\lambda_\sigma, \lambda_\tau\}$ equals the set $\{(\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})\}$ (with ambiguities due to both the Weyl group action and the two choices of identification of $\overline{\mathbb{R}}$ with $\mathbb{C}$) and neither λ_σ nor λ_τ are in $X^*(T)$, but both of $\lambda_\sigma - \delta$ and $\lambda_\tau - \delta$ are.

2.4. **The Hodge-Tate cocharacter.** In this subsection, let k be a finite extension of the p-adic numbers $\mathbb{Q}_p$. Let H be a (not necessarily connected) reductive algebraic group over our fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Note that we do not fix an embedding $k \to \overline{\mathbb{Q}}_p$. Let $\overline{k}$ denote an algebraic closure of k and let $\rho : \mathrm{Gal}(\overline{k}/k) \to H(\overline{\mathbb{Q}}_p)$ denote a continuous group homomorphism. We say that ρ is crystalline/de Rham/Hodge-Tate if for some (and hence any) faithful representation $H \to \mathrm{GL}_N$ over $\overline{\mathbb{Q}}_p$, the resulting N-dimensional Galois representation is crystalline/de Rham/Hodge-Tate. Let C denote the completion of $\overline{k}$. Then for any injection of fields $i : \overline{\mathbb{Q}}_p \to C$ there is an associated Hodge-Tate cocharacter $\mu_i : (\mathrm{GL}_1)_C \to H_C$ (where the base extension from H to H_C is via i). We know of no precise reference for the construction of μ_i in this generality; if H were defined over $\mathbb{Q}_p$ and ρ took values in $H(\mathbb{Q}_p)$ then μ_i is constructed in [Ser79]. The general case can be reduced to this case in the following way: H descends to group H_0 defined over a finite extension E of $\mathbb{Q}_p$, and a standard Baire category theorem argument shows that ρ takes values in $H_0(E')$ for some finite extension E' of E. Now let $H_1 = \mathrm{Res}_{E'/\mathbb{Q}_p} H_0$, so ρ takes values in $H_1(\mathbb{Q}_p)$, and Serre's construction of μ then yields μ_i as above which can be checked to be well-defined independent of the choice of H_0 and so on via an elementary calculation (do the case $H = \mathrm{GL}_n$ first).

Note that there is a choice of sign that one has to make when defining μ_i; we follow Serre so, for example, the cyclotomic character gives rise to the identity map $\mathrm{GL}_1 \to \mathrm{GL}_1$.

Now the conjugacy class of μ_i arises as the base extension (via i) of a cocharacter $\nu_i : (\mathrm{GL}_1)_{\overline{\mathbb{Q}}_p} \to H$ over $\overline{\mathbb{Q}}_p$. Now any $i : \overline{\mathbb{Q}}_p \to C$ is an injection whose image contains k and hence induces an injection $j =$"i^{-1}": $k \to \overline{\mathbb{Q}}_p$. Another careful calculation, which again we omit, shows that the conjugacy class of ν_i depends only on j, so we may set $\nu_j := \nu_i$, a conjugacy class of maps $\mathrm{GL}_1 \to H$ over $\overline{\mathbb{Q}}_p$.

In applications, H will be related to an L-group as follows. If G is connected and reductive over k, and $\rho : \mathrm{Gal}(\overline{k}/k) \to {}^L G(\overline{\mathbb{Q}}_p)$ is an admissible representation, then, because G splits over a finite Galois extension k' of k, ρ will descend to a representation $\rho : \mathrm{Gal}(\overline{k}/k) \to \hat{G}(\overline{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(k'/k)$. The target group can be made into the $\overline{\mathbb{Q}}_p$-points of an algebraic group H over $\overline{\mathbb{Q}}_p$, and if the associated representation is Hodge-Tate then the preceding arguments associate a conjugacy class of maps $\nu_j : \mathrm{GL}_1 \to H$ to each $j : k \to \overline{\mathbb{Q}}_p$. If $\hat{T}$ is a torus in $\hat{G}$ as usual, then ν_j gives rise to an element of $X_*(\hat{T})/W$, with W the Weyl group of $G_{\overline{k}}$.

3. Global definitions, and the first conjectures.

3.1. **Algebraicity and arithmeticity.** Let G be a connected reductive group defined over a number field F. Fix an algebraic closure $\overline{F}$ of F and form the L-group ${}^L G = \hat{G} \rtimes \mathrm{Gal}(\overline{F}/F)$ as in the previous section. For each place v of F, fix an algebraic closure $\overline{F_v}$ of F_v, and an embedding $\overline{F} \hookrightarrow \overline{F_v}$. Nothing we do depends in any degree of seriousness on these choices—changing them will just change things "by an inner automorphism".

Let π be an automorphic representation of G. Then we may write $\pi = \otimes'_v \pi_v$, a restricted tensor product, where v runs over all places (finite and infinite) of F. Recall that in the previous section we defined notions of L-algebraic and C-algebraic for certain representations of real and complex groups. We now globalise these definitions.

Definition 3.1.1. We say that π is *L-algebraic* if π_v is L-algebraic for all infinite places v of F.

Definition 3.1.2. We say that π is *C-algebraic* if π_v is C-algebraic for all infinite places v of F.

Note that, for $G = \mathrm{GL}_n$, the notion of C-algebraic coincides (in the isobaric case) with Clozel's notion of algebraic used in [Clo90], although for GL_2 this choice of normalisation goes back to Hecke. Note also that restriction of scalars preserves both notions: if K/F is a finite extension of number fields and π is an automorphic representation of G/K then π is L-algebraic (resp. C-algebraic) when considered as a representation of $G(\mathbb{A}_K)$ if and only if π is L-algebraic (resp. C-algebraic) when considered as a representation of $\mathrm{Res}_{K/F}(G)(\mathbb{A}_F)$. Indeed, this is a local statement and we indicated the proof earlier.

As examples of these notions, we observe that for Hecke characters of number fields, our notions of L-algebraic and C-algebraic both coincide with the classical notion of being algebraic or of type A_0. For GL_2 the notions diverge: the trivial 1-dimensional representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is C-algebraic but not L-algebraic, whereas the representation $|\det|^{1/2}$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is L-algebraic but not C-algebraic. For

$\mathrm{GL}_3/\mathbb{Q}$ the notions of L-algebraic and C-algebraic coincide again; indeed they coincide for GL_n over a number field if n is odd, and differ by a non-trivial twist if n is even.

Definition 3.1.3. We say that π is *L-arithmetic* if there is a finite subset S of the places of F, containing all infinite places and all places where π is ramified, and a number field $E \subset \mathbb{C}$, such that for each $v \notin S$, the Satake parameter of π_v is defined over E.

Definition 3.1.4. We say that π is *C-arithmetic* if there is a finite subset S of the places of F, containing all infinite places and all places where π is ramified, and a number field $E \subset \mathbb{C}$, such that π_v is defined over E for all $v \notin S$.

Again we note that for K/F a finite extension and π an automorphic representation of G/K, π is L-arithmetic (resp. C-arithmetic) if and only if π considered as an automorphic representation of $\mathrm{Res}_{K/F}(G)$ is.

Let us consider some examples. An automorphic representation π of GL_n/F will be L-arithmetic if there is a number field such that all but finitely many of the Satake parameters attached to π have characteristic polynomials with coefficients in that number field. So, for example, the trivial 1-dimensional representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ would not be L-arithmetic, because the trace of the Satake parameter at a prime p is $\sqrt{p}+1/\sqrt{p} = (p+1)/\sqrt{p}$, and any subfield of $\mathbb{C}$ containing $(p+1)/\sqrt{p}$ for infinitely many primes p would also contain $\sqrt{p}$ for infinitely many primes p and hence cannot be a number field. However it would be C-arithmetic, because for all primes p, π_p is the base extension to $\mathbb{C}$ of a representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ on a vector space over $\mathbb{Q}$. Similarly, the representation $|\det|^{1/2}$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is L-arithmetic, because all Satake parameters are defined over $\mathbb{Q}$. However this representation is not C-arithmetic: each individual π_p is defined over a number field but there is no number field over which infinitely many of the π_p are defined, again because such a number field would have to contain the square root of infinitely many primes.

Now let π be an arbitrary automorphic representation for an arbitrary connected reductive group G.

Conjecture 3.1.5. *π is L-arithmetic if and only if it is L-algebraic.*

Conjecture 3.1.6. *π is C-arithmetic if and only if it is C-algebraic.*

These conjectures are seemingly completely out of reach. For the group GL_1 over a number field they are true; indeed in this case both conjectures say the same thing, the "algebraic implies arithmetic" direction being relatively standard, and the "arithmetic implies algebraic" direction being a non-trivial result in transcendence theory due to Waldschmidt in [Wal81]. We prove the conjectures for a general torus in section 4, for the most part by reducing to the case of GL_1. On the other hand, neither direction of either conjecture is known for the group $\mathrm{GL}_2/\mathbb{Q}$! Although perhaps nowadays conjectures of this form are part of the folklore, it is worth pointing out that as far as we know the first person to raise such conjectures explicitly was Clozel in [Clo90].

If one makes conjectures 3.1.5 and 3.1.6 for all groups G simultaneously, then they are in fact equivalent, by the results of section 5 below; for groups with a twisting element (see section 5 for this terminology) this follows from Propositions 5.2.2 and 5.2.3, and the general case reduces to this one by passage to z-extensions—see Proposition 5.2.9.

3.2. **Galois representations attached to automorphic representations.** We now fix a prime number p and turn to the notion of associating p-adic Galois representations to automorphic representations. Because automorphic representations are objects defined over $\mathbb{C}$ and p-adic Galois representations are defined over p-adic fields, we need a method of passing from one field to the other. We have already fixed an injection $\overline{\mathbb{Q}} \to \mathbb{C}$; now we fix once and for all a choice of algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and, reluctantly, an isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$ of "coefficient fields". Recall that our L-groups are defined over our fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$; our fixed inclusion $\overline{\mathbb{Q}} \to \mathbb{C}$ then induces, via ι, an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Ideally we should only be fixing an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, and all our constructions should only depend on the restriction of ι to $\overline{\mathbb{Q}}$, but of course we cannot prove this. Our choice of ι does affect matters, in the following way: if $f = \sum a_n q^n$ is one of the holomorphic cuspidal newforms for $\mathrm{GL}_2/\mathbb{Q}$ of level 1 and weight 24 then 13 splits into two prime ideals in the coefficient field of f, and a_{13} is in one of these prime ideals but not the other; hence f will be ordinary with respect to some choices of ι but not for others. For notational simplicity we drop ι from our notation but our conjectural association of p-adic Galois representations attached to automorphic representations will depend very much on this choice.

We now state two conjectures on the existence of Galois representations attached to L-algebraic automorphic representations, the second apparently stronger than the first. The first version is the more useful one when formulating conjectures about functoriality. Note that both conjectures depend implicitly on our choice of isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$ which we use to translate complex parameters to p-adic ones.

Conjecture 3.2.1. *If π is L-algebraic, then there is a finite subset S of the places of F, containing all infinite places, all places dividing p, and all places where π is ramified, and a continuous Galois representation $\rho_\pi = \rho_{\pi,\iota} : \mathrm{Gal}(\overline{F}/F) \to {}^L G(\overline{\mathbb{Q}}_p)$, which satisfies*

- *The composite of ρ_π and the natural projection ${}^L G(\overline{\mathbb{Q}}_p) \to \mathrm{Gal}(\overline{F}/F)$ is the identity map.*
- *If $v \notin S$, then $\rho_\pi|_{W_{F_v}}$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to $\iota(r_{\pi_v})$.*
- *If v is a finite place dividing p then $\rho_\pi|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ is de Rham, and the Hodge-Tate cocharacter of this representation can be explicitly read off from π via the recipe below.*
- *If v is a real place, let $c_v \in G_F$ denote a complex conjugation at v. Then $\rho_{\pi,\iota}(c_v)$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to the element $\iota(\alpha_v) = \iota(\lambda_v(i)\mu_v(i)r_{\pi_v}(j))$ of Lemma 2.3.2.*

Conjecture 3.2.2. *Assume that π is L-algebraic. Let S be the set of the places of F consisting of all infinite places, all places dividing p, and all places where π is ramified. Then there is a continuous Galois representation $\rho_{\pi,\iota} : \mathrm{Gal}(\overline{F}/F) \to {}^L G(\overline{\mathbb{Q}}_p)$, which satisfies*

- *The composite of $\rho_{\pi,\iota}$ and the natural projection ${}^L G(\overline{\mathbb{Q}}_p) \to \mathrm{Gal}(\overline{F}/F)$ is the identity map.*
- *If $v \notin S$, then $\rho_{\pi,\iota}|_{W_{F_v}}$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to $\iota(r_{\pi_v})$.*
- *If v is a finite place dividing p then $\rho_{\pi,\iota}|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ is de Rham, and the Hodge-Tate cocharacter associated to this representation is given by the*

recipe in the remark below. Furthermore, if π_v is unramified then $\rho_{\pi,\iota}|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ is crystalline.

- *If v is a real place, let $c_v \in G_F$ denote a complex conjugation at v. Then $\rho_{\pi,\iota}(c_v)$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to $\iota(\alpha_v) = \iota(\lambda_v(i)\mu_v(i)r_{\pi_v}(j)))$ of Lemma 2.3.2.*

Remark 3.2.3. The recipe for the Hodge-Tate cocharacter in the conjectures above is as follows. Say $j : F \to \overline{\mathbb{Q}}$ is an embedding of fields. We have fixed $\overline{\mathbb{Q}} \to \mathbb{C}$ and (via ι) $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, so j induces $F_v \to \overline{\mathbb{Q}}_p$ for some place $v|p$ and $F_w \to \mathbb{C}$ for some place $w|\infty$. The inclusion $F_w \to \mathbb{C}$ enables us to identify $\mathbb{C}$ as an algebraic closure of F_w and now attached to π_w and the identity $\sigma : \mathbb{C} \to \mathbb{C}$ we have constructed an element $\lambda_\sigma \in X^*(T)/W$. Our conjecture is that this element λ_σ is the Hodge-Tate cocharacter associated to the embedding $F_v \to \overline{\mathbb{Q}}_p$.

Remark 3.2.4. The representation $\rho_{\pi,\iota}$ is not necessarily unique up to $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugation. One rather artificial reason for this is that if π is a non-isobaric L-algebraic automorphic representation of $\mathrm{GL}_2/\mathbb{Q}$ such that π_ℓ is 1-dimensional for almost all v, then there are often many non-semisimple 2-dimensional Galois representations that one can associate to π (as well as a semisimple one). But there are other more subtle reasons too. For example if G is a torus over F then the admissible Galois representations into the L-group of G are parametrised by $H^1(F, \hat{G})$ (with the Galois group acting on $\hat{G}$ via the action used to form the L-group), and there may be non-zero elements of this group which restrict to zero in $H^1(F_v, \hat{G})$ for all places v of F. If this happens then there is more than one Galois representation that can be associated to the trivial 1-dimensional automorphic representation of G. We are grateful to Hendrik Lenstra and Bart de Smit for showing us an explicit example of a rank 3 torus over $\mathbb{Q}$ where this phenomenon occurs. If Γ is the group $(\mathbb{Z}/2\mathbb{Z})^2$ and Q is the quaternion group of order 8 then Q gives a non-zero element of $H^2(\Gamma, \pm 1)$ whose image in $H^2(\Gamma, \mathbb{C}^\times)$ is non-zero but whose restriction to $H^2(D, \mathbb{C}^\times)$ is zero for any cyclic subgroup D of Γ (consider the corresponding extension of Γ by $\mathbb{C}^\times$ to see these facts). We now "dimension shift". The rank four torus $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{C}^\times$ has no cohomology in degree greater than zero and has $\mathbb{C}^\times$ as a subgroup, so the quotient group T is a complex torus with an action of Γ and with the property that there's an element of $H^1(\Gamma, T)$ whose restriction to any cyclic subgroup is zero. Finally, Γ is isomorphic to $\mathrm{Gal}(\mathbb{Q}(\sqrt{13}, \sqrt{17})/\mathbb{Q})$ (a non-cyclic group all of whose decomposition groups are cyclic) and T with its Galois action can be realised as the complex points of the dual group of a torus over $\mathbb{Q}$, giving us our example: there is more than one Galois representation associated to the trivial 1-dimensional representation of this torus.

Remark 3.2.5. We have normalised local class field theory so that geometric Frobenius elements correspond to uniformisers, and defined our Weil groups accordingly. Had we normalised things the other way (associating arithmetic Frobenius to uniformisers) then our unramified local Langlands dictionary at good finite places is changed by a non-trivial involution. Had we made this choice initially, conjectures 3.2.1 and 3.2.2 need to be modified: one needs to change the Hodge-Tate cocharacter μ to $-\mu$. However these new conjectures are equivalent to the conjectures as stated, because the required Galois representation predicted by the new conjecture may be obtained directly from ρ_π by applying the Chevalley involution of ${}^LG$ (the Chevalley involution of $\hat{G}$ extends to ${}^LG$ and induces the identity map on the Galois group), or indirectly as $\rho_{\tilde{\pi}}$ where $\tilde{\pi}$ is the contragredient of π. We omit the formal proof that these constructions do the job, and we confess that we were not able to find a precise published reference for the statements at infinity that we need. The point is that we need to know how the local Langlands correspondence for real and complex reductive groups behaves under taking contragredients. The involution on the π side induced by contragredient corresponds on the Galois side to the involution on the local Weil representations induced by an involution of the Weil group of the reals/complexes sending $z \in \overline{k}^{\times} \cong \mathbb{C}^{\times}$ to z^{-1}. It also corresponds to the involution on the Weil representations induced by the Chevalley involution. Both these facts seem to be well-known to the experts but the proof seems not to be in the literature.

3.3. **Example: the groups $\mathrm{GL}_2/\mathbb{Q}$ and $\mathrm{PGL}_2/\mathbb{Q}$.** The following example illustrates the differences between the C- and L- notions in two situations, one where things can be "fixed by twisting" and one where they cannot. The proofs of the assertions made here only involve standard unravelling of definitions and we shall omit them.

Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$. For N a positive integer, let $K_0(N)$ denote the subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ consisting of matrices which are upper triangular modulo N. If $\mathrm{GL}_2^+(\mathbb{R})$ denotes the matrices in $\mathrm{GL}_2(\mathbb{R})$ with positive determinant then $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})K_0(N)\,\mathrm{GL}_2^+(\mathbb{R})$. Now let f be a modular form of weight $k \geq 2$ which is a normalised cuspidal eigenform for the subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, and let s denote a complex number. We think of f as a function on the upper half plane. Associated to f and s we define a function ϕ_s on $\mathrm{GL}_2(\mathbb{A})$ by writing an element of $\mathrm{GL}_2(\mathbb{A})$ as $\gamma\kappa u$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $\kappa \in K_0(N)$ and $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$, and defining $\phi_s(\gamma\kappa u) = (\det u)^{k-1+s}(ci+d)^{-k}f((ai+b)/(ci+d))$. This function is well-defined and is a cuspidal automorphic form, which generates an automorphic representation π_s of $\mathrm{GL}_2(\mathbb{A})$. The element s is just a twisting factor; if s is a generic complex number then π_s will not be algebraic or arithmetic for either of the "C" or "L" possibilities above.

First we consider the arithmetic side of the story. For p a prime not dividing N, let a_p is the coefficient of q^p in the q-expansion of f. It is well-known that the subfield of $\mathbb{C}$ generated by the a_p is a number field E. An elementary but long explicit calculation shows the following. If $\pi_{s,p}$ denotes the local component of π_s at p, then $\pi_{s,p}$ has a non-zero invariant vector under the group $\mathrm{GL}_2(\mathbb{Z}_p)$ and the action of the Hecke operators T_p and S_p on this 1-dimensional space are via the complex numbers $p^{2-k-s}a_p$ and p^{2-k-2s}. The Satake parameter associated to $\pi_{s,p}$ is the semisimple conjugacy class of $\mathrm{GL}_2(\mathbb{C})$ consisting of the semisimple elements with characteristic polynomial $X^2 - a_p p^{3/2-k-s}X + p^{2-k-2s}$. Hence π_s is L-arithmetic if $s \in \frac{1}{2} + \mathbb{Z}$. In fact one can go further. By the six exponentials theorem of transcendental number theory one sees easily that a complex number c with the property that p^c is algebraic for at least three prime numbers p must be rational. Hence if π_s is L-arithmetic then s is rational and (because a number field is only ramified at finitely many primes and hence cannot contain the tth root of infinitely many prime numbers for any $t > 1$) one can furthermore deduce that $2s \in \mathbb{Z}$. Next one observes that a_p must be non-zero for infinitely many primes $p \nmid N$ (because one can apply the Cebotarev density theorem to the mod $\ell > 2$ Galois representation associated to f and to the identity matrix) and deduce (again because a number field cannot contain the square root of infinitely many primes) that π_s is L-arithmetic iff $s \in \frac{1}{2} + \mathbb{Z}$. Now by Proposition 5.2.3 (whose proof uses nothing that we haven't already established) we see that π_s is C-arithmetic iff $s \in \mathbb{Z}$.

We now consider the algebraic side of things. If $\pi_{s,\infty}$ denotes the local component of π_s at infinity and we choose the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{gl}_2(\mathbb{C})$ spanned by $H := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $Z := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ then the infinitesimal character of $\pi_{s,\infty}$ (thought of as a Weil group orbit in $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$) sends H to $\pm(k-1)$ and Z to $2s+k-2$. The characters of the torus in $\mathrm{GL}_2(\mathbb{R})$ give rise to the lattice $X^*(T)$ in $\mathrm{Hom}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ consisting of linear maps that send H and Z to integers of the same parity. Hence π_s is C-algebraic iff $s \in \mathbb{Z}$ and L-algebraic iff $s \in \frac{1}{2} + \mathbb{Z}$. In particular π_s is L-algebraic iff it is L-arithmetic, and π_s is C-algebraic iff it is C-arithmetic.

We now play the same game for Maass forms. If f (a real analytic function on the upper half plane) is a cuspidal Maass form of level $\Gamma_0(N)$ which is an eigenform for the Hecke operators, and $s \in \mathbb{C}$ then one can define a function ϕ_s on $\mathrm{GL}_2(\mathbb{A})$ by writing an element of $\mathrm{GL}_2(\mathbb{A})$ as $\gamma\kappa u$ as above, writing $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and defining $\phi_s(\gamma\kappa u) = \det(u)^s f((ai+b)/(ci+d))$. If we now assume that f is the Maass form associated by Langlands and Tunnell to a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{SL}_2(\mathbb{C})$ with solvable image, and if $p \nmid N$ is prime and $a_p = \mathrm{tr}(\rho(\mathrm{Frob}_p))$, then the a_p generate a number field E (an abelian extension of $\mathbb{Q}$ in this case) and a similar explicit calculation, which again we omit, shows that π_s is L-arithmetic iff π_s is L-algebraic iff $s \in \mathbb{Z}$, and that π_s is C-arithmetic iff π_s is C-algebraic iff $s \in \frac{1}{2} + \mathbb{Z}$.

Note in particular that the answer in the Maass form case is different to the holomorphic case in the sense that $s \in \mathbb{Z}$ corresponded to the C-side in the holomorphic case and the L-side in the Maass form case.

An automorphic representation for $\mathrm{PGL}_2/\mathbb{Q}$ is just an automorphic representation for $\mathrm{GL}_2/\mathbb{Q}$ with trivial central character. One checks that the π_s corresponding to the holomorphic modular form has trivial central character iff $s = 1 - \frac{k}{2}$ (this is because the form was assumed to have trivial Dirichlet character) and, again because the form has trivial character, k must be even so in particular the π_s which descends to $\mathrm{PGL}_2/\mathbb{Q}$ is C-algebraic and C-arithmetic. However, the π_s corresponding to the Maass form with trivial character has trivial central character iff $s = 0$, which is L-algebraic and L-arithmetic. Hence, when applied to the group $\mathrm{PGL}_2/\mathbb{Q}$, our conjecture above says that there should be a Galois representation to $\mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$ associated to the Maass form but it says nothing about the holomorphic form. However, the holomorphic form is clearly algebraic in some sense and indeed there is a Galois representation associated to the holomorphic form—namely the Tate module of the elliptic curve. Note however that the determinant of the Tate module of an elliptic curve is the cyclotomic character, which is not the square of any 1-dimensional Galois representation (complex conjugation would have to map to an element of order 4) and hence no twist of the Tate module of an elliptic curve can take values in $\mathrm{SL}_2(\overline{\mathbb{Q}}_\ell)$. This explains why we restrict to L-algebraic representations for our general conjecture.

3.4. **Why C-algebraic?** Our conjecture above only attempts to associate Galois representations to L-algebraic automorphic representations. So why consider C-algebraic representations at all? For GL_n the issue is only one of twisting: π is L-algebraic iff $\pi.|\det(.)|^{(n-1)/2}$ is C-algebraic. Furthermore, for groups such as SL_2 in which half the sum of the positive roots is in $X^*(T)$, the notions of L-algebraic and C-algebraic coincide. On the other hand, as the previous example of $\mathrm{PGL}_2/\mathbb{Q}$ attempted to illustrate, one does not always have this luxury of being able to pass easily between L-algebraic and C-algebraic representations for a given group G. Furthermore a lot of naturally occurring representations are C-algebraic: for example any cohomological automorphic representation will always be C-algebraic (see Lemma 7.3.2 below) and, as the case of $\mathrm{PGL}_2/\mathbb{Q}$ illustrated, there may be natural candidates for Galois representations associated to these automorphic representations, but they may not take values in the L-group of G! In fact, all known examples of Galois representations attached to automorphic representations ultimately come from the cohomology of Shimura varieties (although in some cases the constructions also use congruence arguments), and this cohomology is naturally decomposed in terms of cohomological automorphic representations. Much of the rest of this paper is devoted to examining the relationship between C-algebraic and L-algebraic in greater detail.

4. The case of tori.

4.1. In this section we prove conjectures 3.1.5–3.2.2 when G is a torus over a number field F. That we can do this should not be considered surprising. Indeed, if $G = \mathrm{GL}_1$ then the results have been known for almost 30 years, and the general case can be reduced to the GL_1 case via base change and a p-adic version of the local and global Langlands conjectures for tori. Unfortunately we have not been able to find a reference which does what we want so we include some of the details here.

First note that if G is a torus then the Satake isomorphism preserves $\mathbb{Q}$-structures and hence the notions of C-arithmetic and L-arithmetic coincide and we can use the phrase "arithmetic" to denote either of these notions. Furthermore, half the sum of the positive roots is zero so the notions of C-algebraic and L-algebraic also coincide, and we can use the phrase "algebraic" to mean either of these two notions (and in the case $G = \mathrm{GL}_1/F$ this coincides with the classical definition, and with Weil's notion of being of type (A_0)).

Recall that to give a torus G/F is to give its character group, which (after choosing an $\overline{F}$) is a finite free $\mathbb{Z}$-module equipped with a continuous action of $\mathrm{Gal}(\overline{F}/F)$. Let $K \subset \overline{F}$ denote a finite Galois extension of F which splits G; then this action of $\mathrm{Gal}(\overline{F}/F)$ factors through $\mathrm{Gal}(K/F)$. An automorphic representation of G/F is just a continuous group homomorphism $G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}^\times$.

Let BC denote the usual base change map from automorphic representations of G/F to automorphic representations of G/K, induced by the norm map $N : G(\mathbb{A}_K) \to G(\mathbb{A}_F)$.

Lemma 4.1.1. *If π is an automorphic representation of G/F then π is algebraic if and only if $BC(\pi)$ is.*

Proof. This is a local statement, and if we translate it over to a statement about representations of Weil groups then it says that if k is an algebraic closure of $\mathbb{R}$ then $r : W_{\mathbb{R}} \to {}^LG(\mathbb{C})$ is algebraic iff its restriction to W_k is, which is clear because our definition of algebraicity of r only depended on the restriction of r to W_k. $\square$

Now let T denote a torus over a local field k, and assume T splits over an unramified extension of k. The topological group $T(k)$ has a unique maximal compact subgroup U. Let χ be a continuous group homomorphism $T(k) \to \mathbb{C}^\times$ with U in its kernel.

Lemma 4.1.2. *For E a subfield of $\mathbb{C}$, the following are equivalent:*
(i) χ is defined over E
(ii) The Satake parameter of χ is defined over E
(iii) The image of χ is contained in $E^\times$.

Proof. The Satake isomorphism is the identity isomorphism $\mathbb{C}[T(k)/U] = \mathbb{C}[T(k)/U]$, which induces the identity isomorphism $\mathbb{Q}[T(k)/U] = \mathbb{Q}[T(k)/U]$, so (i) and (ii) are equivalent. The equivalence of (i) and (iii) follows from the statement that $\chi : T(k)/U \to \mathbb{C}^\times$ is $E^\times$-valued if and only if the induced ring homomorphism $\mathbb{Q}[T(k)/U] \to \mathbb{C}$ is E-valued. □

If k_1/k is a finite extension of local fields and if T/k is a torus then we also use the notation BC to denote the map $\mathrm{Hom}(T(k), \mathbb{C}^\times) \to \mathrm{Hom}(T(k_1), \mathbb{C}^\times)$ induced by the norm map $N : T(k_1) \to T(k)$. Now suppose again that T is an unramified torus over k and $\chi : T(k) \to \mathbb{C}^\times$ is an unramified character.

Corollary 4.1.3. *If χ is defined over E and if k_1/k is a finite extension, then* $\mathrm{BC}(\chi)$ *is defined over E.*

Proof. The image of $\mathrm{BC}(\chi)$ is contained in the image of χ. □

We now go back to the global situation. Let π denote an automorphic representation of G/F, with G a torus, and let $\mathrm{BC}(\pi)$ denote its base change to G/K, where K is a finite Galois extension of F which splits G.

Corollary 4.1.4. *If π is arithmetic then* $\mathrm{BC}(\pi)$ *is arithmetic.*

Proof. Immediate from the previous corollary. □

Theorem 4.1.5. *If G is a split torus over a number field, then the notions of arithmetic and algebraic coincide.*

Proof. That algebraic implies arithmetic is standard; the other implication is Théorème 5.1 of [Wal82] (which uses a non-trivial result in transcendence theory). □

Corollary 4.1.6. *If G is a torus over a number field F and π is an automorphic representation of G, and π is arithmetic, then π is algebraic.*

Proof. If π is arithmetic then its base change to K (a splitting field for G) is arithmetic (by Corollary 4.1.4), and hence algebraic by the previous theorem. Hence π is algebraic by Lemma 4.1.1. □

To show that algebraic automorphic representations for G are arithmetic, we give a re-interpretation of what it means for an automorphic representation of a torus to be algebraic; we are grateful to Ambrus Pál for pointing out to us that such a re-interpretation should exist. First some notation. Let $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$. Let Σ denote the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$. Recall that because we have fixed an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, each $\sigma \in \Sigma$ can be regarded as an embedding $F \to \mathbb{C}$, and hence induces maps $F_\infty \to \mathbb{C}$ and $\sigma_\infty : G(F_\infty) \to G_\sigma(\mathbb{C})$, where G_σ is the group over $\overline{\mathbb{Q}}$ induced from G via base extension via σ.

Proposition 4.1.7. *The representation π is algebraic if and only if for each $\sigma \in \Sigma$ there is an algebraic character $\lambda_\sigma : G_\sigma \to \mathrm{GL}_1/\overline{\mathbb{Q}}$ such that π agrees with $\prod_{\sigma\in\Sigma} \lambda_\sigma \circ \sigma_\infty$ on $G(F_\infty)^0$ (the identity component of the Lie group $G(F_\infty)$).*

Proof. This statement is local at infinity, and can be checked by "brute force", explicitly working out what the local Langlands correspondence for tori over the reals and complexes is and noting that it is true in every case. □

Corollary 4.1.8. *If π is an algebraic automorphic representation of G/F, and if we write $\pi = \pi_f \times \pi_\infty$, with $\pi_\infty : G(F_\infty) \to \mathbb{C}^\times$, then $\pi_\infty(G(F))$ is contained within a number field.*

Proof. By the preceding proposition we know that $\pi_\infty|_{G(F)}$ is the product of a character of order at most 2 by a continuous group homomorphism $G(F) \to \mathbb{C}^\times$ which is the product of maps $\phi_\sigma : G(F) \stackrel{\sigma}{\to} G_\sigma(\mathbb{C}) \to \mathbb{C}^*$ given by composing an algebraic character with an embedding $\sigma : F \hookrightarrow \mathbb{C}$. Hence it suffices to prove that $\phi(G(F))$ is contained within a number field for such a ϕ. However both T and $\mathbb{G}_m$ are defined over F, so the character descends to some number field L, which we may assume splits T and contains the images of all embeddings $F \hookrightarrow \mathbb{C}$. But then $\phi(F) \subset \phi(L) \subset L^\times$, as required. □

Theorem 4.1.9. *The notions of arithmetic and algebraic coincide for automorphic representations of tori over number fields.*

Proof. Let G/F be a torus over a number field, and let π be an automorphic representation of G. By Corollary 4.1.6 we know that if π is arithmetic then π is algebraic, so we only have to prove the converse. We will make repeated use of the trivial observation (already used above) that if X is a finite index subgroup of an abelian group Y, then the image of a character of Y is contained in a number field if and only if the image of its restriction to X is contained in a (possibly smaller) number field.

Let $\pi = \otimes_v \pi_v$ be algebraic. If K is a finite Galois extension F splitting G then $BC_{K/F}(\pi)$ is algebraic by Lemma 4.1.1 and hence arithmetic by Theorem 4.1.5. Hence there is a number field E and some finite set S_K of places of K, containing all the infinite places, such that for $w \notin S_K$, $\mathrm{BC}(\pi)_w$ is defined over E, and hence $\mathrm{BC}(\pi)_w$ has image in $E^\times$. By increasing S_K if necessary, we can assume that S_K is precisely the set of places of K lying above a finite set S of places of F.

Let $N : G(\mathbb{A}_K) \to G(\mathbb{A}_F)$ denote the norm map. Standard results from global class field theory (see for example p.244 of [Lan97] for the crucial argument) imply that $G(F)N(G(\mathbb{A}_K))$ is a closed and open subgroup of finite index in $G(\mathbb{A}_F)$. Hence if $\mathbb{A}_F^S$ denotes the restricted product of the completions of F at places not in S, and $\mathbb{A}_K^{S_K}$ denotes the analogous product for K, then $G(F)N(G(\mathbb{A}_K^{S_K}))$ has finite index in $G(\mathbb{A}_F^S)$. Let $\pi^S : G(\mathbb{A}_F^S) \to \mathbb{C}^\times$ denote the restriction of π to $G(\mathbb{A}_F^S)$. Then $\pi = \pi^S.\prod_{v\in S, v\nmid\infty} \pi_v.\pi_\infty$. We know that π is trivial on $G(F)$ (by definition) and that π_∞ sends $G(F)$ to a number field (by Corollary 4.1.8). We also know that $\pi_v(G(F_v))$ (and thus $\pi_v(G(F))$ is contained within a number field for each finite place $v \in S$ (because $BC_{K/F}(\pi)$ is arithmetic, we know that $\pi_v(N(K_w))$ is contained in a number field, where $w|v$ is a place of K, and $N(K_w)$ has finite index in F_v). Hence $\pi^S(G(F))$ is contained within a number field. Then since $G(F)N(G(\mathbb{A}_K^{S_K}))$ has finite index in $G(\mathbb{A}_F^S)$, we deduce that $\pi(G(\mathbb{A}_F^S))$ is contained within a number field, and hence π is arithmetic, as required. □

We now need to prove Conjecture 3.2.2 (which of course implies Conjecture 3.2.1). This follows straightforwardly from Langlands' proof of the Langlands correspondence for tori, and the usual method for associating Galois representations to algebraic representations of $\mathbb{G}_m$. Take π, G, F and K as above, and again let Σ denote the field embeddings $F \to \overline{\mathbb{Q}}$, noting now that because of our fixed embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ we can also interpret each element of Σ as a field embedding $F \to \overline{\mathbb{Q}}_p$.

The first step is to associate a "p-adic automorphic representation"—a continuous group homomorphism $\pi_p : G(F)\backslash G(\mathbb{A}_F) \to \overline{\mathbb{Q}}_p$—to π, which we do by mimicking the standard construction in the split case. For $\sigma \in \Sigma$ recall that σ_∞ is the induced map $G(F_\infty) \to G_\sigma(\mathbb{C})$. By Proposition 4.1.7 there are characters $\lambda_\sigma \in X^*(G_\sigma)$ (regarded here as maps $G_\sigma(\mathbb{C}) \to \mathbb{C}^\times$) for each $\sigma \in \Sigma$ with the property that

$$\pi|_{G(F_\infty)^0} = \prod_{\sigma\in\Sigma} \lambda_\sigma \circ \sigma_\infty.$$

The right hand side of the above equation can be regarded as a character of $G(F_\infty)$ and hence as a character λ_∞ of $G(\mathbb{A}_F)$, trivial at the finite places. Define $\pi^{\mathrm{alg}} = \pi/\lambda_\infty$, a continuous group homomorphism $G(\mathbb{A}_F) \to \mathbb{C}^\times$ trivial on $G(F_\infty)^0$ but typically non-trivial on $G(F)$. However $\pi^{\mathrm{alg}}(G(F))$ is contained within a number field by Corollary 4.1.8, and it is now easy to check that the image of π^{alg} is contained within $\overline{\mathbb{Q}}$. We now regard π^{alg} as taking values in $\overline{\mathbb{Q}}_p$ via our fixed embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$.

Now, let $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and note that every $\sigma : F \to \overline{\mathbb{Q}}_p$ in Σ induces a map $F_p \to \overline{\mathbb{Q}}_p$ and hence a map $\sigma_p : G(F_p) \to G_\sigma(\overline{\mathbb{Q}}_p)$. Each λ_σ can be regarded as a map $G_\sigma(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p^\times$, and hence the product

$$\lambda_p := \prod_\sigma \lambda_\sigma \circ \sigma_p$$

is a continuous group homomorphism $G(F_p) \to \overline{\mathbb{Q}}_p^\times$ and can also be regarded as a continuous group homomorphism $G(\mathbb{A}_F) \to \overline{\mathbb{Q}}_p^\times$, trivial at all places other than those above p. The crucial point, which is easy to check, is that the product $\pi_p := \pi^{\mathrm{alg}}\lambda_p$ is a continuous group homomorphism $G(\mathbb{A}_F) \to \overline{\mathbb{Q}}_p^\times$ which is trivial on $G(F)$.

Now, in Theorem 2(b) of [Lan97], Langlands proves that there is a natural surjection with finite kernel from the set of $\widehat{G}(\mathbb{C})$-conjugacy classes of continuous homomorphisms from the Weil group W_F to ${}^LG(\mathbb{C})$ to the set of continuous homomorphisms from $G(F)\backslash G(\mathbb{A}_F)$ to $\mathbb{C}^\times$ (that is, the set of automorphic representations of G/F), compatible with the local Langlands correspondence at every place. His proof starts by establishing a natural surjection from the analogous sets with the continuity conditions removed, and then checking that continuity on one side is equivalent to continuity on the other. However, $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ as abstract fields, and one can check that the calculations on pages 243ff make no use of any particular features of the topology of $\mathbb{C}^\times$, and hence apply equally well to continuous homomorphisms $W_F \to {}^LG(\overline{\mathbb{Q}}_p)$ and continuous characters $G(F)\backslash G(\mathbb{A}_F) \to \overline{\mathbb{Q}}_p^\times$. Thus π_p gives a continuous homomorphism (or perhaps several, in which case we simply choose one)

$$r_\pi : W_F \to {}^LG(\overline{\mathbb{Q}}_p),$$

and by construction we see that for each finite place $v \nmid p$ at which $\pi_v = (\pi_p)_v$ is unramified, $r_\pi|_{W_{F_v}}$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to r_{π_v}. Again, by construction the composite of r_π and the natural projection ${}^LG(\overline{\mathbb{Q}}_p) \to \mathrm{Gal}(\overline{F}/F)$ is just the natural surjection $W_F \to \mathrm{Gal}(\overline{F}/F)$.

Lemma 4.1.10. *The representation r_π of W_F factors through the natural surjection $W_F \to \mathrm{Gal}(\overline{F}/F)$.*

Proof. The kernel of the natural surjection $W_F \to \mathrm{Gal}(\overline{F}/F)$ is the connected component of the identity in W_F; but r_π must vanish on this, because ${}^LG(\overline{\mathbb{Q}}_p)$ is totally disconnected. □

We let ρ_π denote the representation of $\mathrm{Gal}(\overline{F}/F)$ determined by r_π.

Lemma 4.1.11. *The representation ρ_π satisfies all the properties required in the statement of Conjecture 3.2.2.*

Proof. We need to check the claimed properties at places dividing p and at real places. For the former, we must firstly check that ρ_π is de Rham with the correct Hodge-Tate weights. However, it is sufficient to check this after restriction to any finite extension of F, and in particular we may choose an extension which splits G. The evident compatibility of the construction of ρ_π with base change then easily reduces us to the split case, which is standard. Similarly, the property of being crystalline may be checked over any unramified extension, and if π_v is unramified then by definition G splits over an unramified extension of F_v, and we may again reduce to the split case.

Suppose now that v is a real place of F. Recall that the natural surjection $W_{F_v} \to \mathrm{Gal}(\overline{F_v}/F_v)$ sends j to complex conjugation, so we need to determine $r_\pi|_{W_{F_v}}(j)$. Let $\sigma_v : F \hookrightarrow \mathbb{C}$ denote the embedding corresponding to v (it is unique because v is a real place) and let λ_v denote the character λ_{σ_v} of G_{σ_v}. Let χ_v denote the map $G(F_v) \to \mathbb{C}^\times$ induced by σ_v and λ_v. Then $(\pi_p)_v = \pi_v/\chi_v$. Applying local Langlands we see that the cohomology class in $H^1(W_{F_v}, \widehat{G})$ associated to $(\pi_p)_v$ is the difference of those associated to π_v and χ_v (because local Langlands is an isomorphism of groups in this abelian setting). Furthermore, one can check (either by the construction of the local Langlands correspondence for real tori in section 9.4 of [Bor79], or an explicit case-by-case check) that the cohomology class attached to χ_v is represented by a cocycle which sends $j \in W_{F_v}$ to the element $\lambda_v(-1)$ of $\widehat{G}$ (where we now view λ_v as a cocharacter $\mathbb{C}^\times \to \widehat{G}$). Our assertion about $r_\pi|_{W_{F_v}}(j)$ now follows immediately from an explicit calculation on cocycles. □

5. Twisting and Gross' η.

5.1. **Algebraicity and arithmeticity under z-extensions.** We begin by recalling the notion of a z-extension, explicitly formalised by Kottwitz but used earlier by Langlands. The next definition and proposition make sense for a connected reductive group over an arbitrary field F of characteristic zero, but we will quickly specialise to the case of F a number field afterwards.

Definition 5.1.1. We say that a central extension

$$1 \to X \to G' \to G \to 1$$

of algebraic groups over F is a *z-extension* if G' is connected and reductive, the derived subgroup of G' is simply connected, and X is isomorphic to a product of tori of the form $\mathrm{Res}_{M_i/F}\,\mathbb{G}_m$, M_i a finite extension of F.

Note that one can (after making compatible choices of maximal compact subgroups at infinity, and making use of Shapiro's lemma and Hilbert 90 to see that various H^1 groups with coefficients in X vanish) identify automorphic representations on G with automorphic representations on G' which are trivial on X. Note also the following standard result, due to Langlands:

Proposition 5.1.2. *A connected reductive group over a field of characteristic zero admits a z-extension.*

Proof. This is Proposition V.3.1 of [DMOS82]. □

We abuse notation slightly and speak about the z-extension $G' \to G$ interchangeably with the z-extension $1 \to X \to G' \to G \to 1$.

Lemma 5.1.3. *If $G' \to G$ is a z-extension, and $G'' \to G'$ is a z-extension, then so is $G'' \to G$.*

Proof. This is presumably standard, but for lack of a reference we sketch a proof. Let Y be the kernel of $G'' \to G$, and let X, X' denote the kernels of $G' \to G$ and $G'' \to G'$, so we have an exact sequence $0 \to X' \to Y \to X \to 0$ and hence Y must be connected. Next we note that Y must be in the centre Z'' of G'', for if it were not then the image of Y in G''/Z'' would be a homomorphic image of X and a positive-dimensional component of the centre of a group which has finite centre, a contradiction. In particular Y must be a central torus in G''. It remains to show that Y is a product of tori which are restrictions of scalars of $\mathbb{G}_m$, which we do following an argument which Ben Webster showed us at `mathoverflow.net`. By taking character groups, it suffices to show that if Γ is a finite group and Λ, Λ' are two Γ-modules each of which are of the form $\oplus_i \operatorname{Ind}_{\Delta_i}^{\Gamma} \mathbb{Z}$ (here the Δ_i are subgroups of G), then any $\mathbb{Z}[\Gamma]$-extension of Λ by Λ' is also of this form. It suffices then to show that the extension splits, which we can do thus: Λ and Λ' are both integral representations of G coming from the G-action on finite sets. Hence $P := \operatorname{Hom}_{\mathbf{Z}}(\Lambda, \Lambda')$ with its G-action is too (take the product of the sets). But $\operatorname{Ext}^1_{\mathbf{Z}[\Gamma]}(\Lambda, \Lambda') = H^1(\Gamma, P)$ which is zero by Shapiro's Lemma. □

We now go back to the case of a connected reductive group G over a number field F. If $\widetilde{G}$ is a z-extension of G then the induced map $\widetilde{G}(\mathbb{A}_F) \to G(\mathbb{A}_F)$ is a surjection, and if π is an automorphic representation of $G(\mathbb{A}_F)$ then the induced representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbb{A}_F)$ is also an automorphic representation. Furthermore, we have the following compatibilities between π and $\widetilde{\pi}$.

Lemma 5.1.4. *π is L-algebraic (resp. C-algebraic, resp. L-arithmetic, resp. C-arithmetic) if and only if π' is.*

Proof. Let us start with the L- and C-algebraicity assertions. These assertions follow from purely local assertions at infinity: one needs to check that if k is an archimedean local field (a completion of F in the application), and if π is a representation of $G(k)$, with π' the induced representation of $G'(k)$, then π is L-algebraic (resp. C-algebraic) if and only if π' is. These statements can easily be checked using infinitesimal characters. Indeed a straightforward calculation (using an explicit description of the Harish-Chandra isomorphism) shows that if T' is a maximal torus in G' over the complexes (where we base change G' to the complexes via the map $k \to \mathbb{C}$ induced from $\sigma : \overline{k} \to \mathbb{C}$), and if the image of T' in G is T (a maximal torus of G), and if λ_σ and λ'_σ are the elements of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ and $X^*(T') \otimes_{\mathbb{Z}} \mathbb{C}$ corresponding to π and π' as in § 2.3, then the natural map $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \to X^*(T') \otimes_{\mathbb{Z}} \mathbb{C}$ sends λ_σ to λ'_σ and both results follow easily.

It remains to prove the arithmeticity statements. Again these statements follow from purely local assertions. Let k denote a non-archimedean local field (a non-archimedean completion of F at which everything is unramified in the application) and let π be an unramified representation of $G(k)$, with π' the corresponding representation of $G'(k)$. Assume π' is also unramified. The C-arithmeticity assertion of the Proposition follows from the assertion that π is defined over a subfield E of $\mathbb{C}$ iff π' is; this is however immediate from Lemma 2.2.2. The L-arithmeticity statement follows from the assertion that the Satake parameter of π is defined over E iff the Satake parameter of π' is defined over E, which is then what remains to be proved. So let T' be the centralizer of a maximal split torus in G' over k, and let T be its image in G. Then $T'(k) \to T(k)$ is a surjection (because the kernel of $T' \to T$ is the kernel of the z-extension $G' \to G$). As noted in the proof of Lemma 2.2.4, Theorem 2.9 of [BZ77] shows that the W_d-orbit of complex characters of $T(k)$ determined by the Satake isomorphism applied to π are precisely the characters for which π occurs as a subquotient of the corresponding induced representations, and the analogous assertion also holds for π'. It follows that the orbit of characters of $T'(k)$ corresponding to π' is precisely the orbit induced from the characters of $T(k)$ via the surjection $T'(k) \to T(k)$. This implies that the Satake parameter of π' (thought of as a character of $\mathbb{Q}[X_*(T'_d)]^{W_d}$) is induced from the Satake parameter of π via a map between the corresponding unramified Hecke algebras which is in fact the obvious map, and our assertion now follows easily. □

5.2. **Twisting elements.** We now explain the relationship between L-algebraic and C-algebraic automorphic representations for a connected reductive group G over a number field F. In particular, we examine the general question of when L-algebraic representations can be twisted to C-algebraic representations, following an idea of Gross (see [Gro99]). We show that in general it is always possible to replace G by a cover for which this twisting is possible, and in this way one can formulate general conjectures about the association of Galois representations to C-algebraic (and in particular cohomological by Lemma 7.3.2 below) automorphic representations. As usual, let X^* denote the character group in the based root datum for G, with its Galois action. Let us stress that we always equip X^* with the Galois action coming from the construction used to define the L-group (which might well not be the same as the "usual" Galois action on $X^*(T)$ induced by the Galois action on $T(\overline{F})$, if T is a maximal torus in G which happens to be defined over F).

Definition 5.2.1. We say that an element $\theta \in X^*$ is a *twisting element* if θ is $\mathrm{Gal}(\overline{F}/F)$-stable and $\langle \theta, \alpha^\vee \rangle = 1$ for all simple coroots $\alpha^\vee$.

For some groups G there are no twisting elements; for example, if $G = \mathrm{PGL}_2$. On the other hand, if G is semi-simple and simply-connected then half the sum of the positive roots is a twisting element. Another case where twisting elements exist (see Remark 5.2.5) are groups G that are split and have simply connected derived subgroup, for example $G = \mathrm{GL}_n$, although in this case half the sum of the positive roots might not be in X^*.

If Q is the quotient of G by its derived subgroup, then $X^*(Q) \subseteq X^*$ and the arguments in section II.1.18 of [Jan03] show that $X^*(Q) = (X^*)^W$, where W is the Weyl group of $G_{\overline{F}}$. Furthermore, $X^*(Q)$ is $\mathrm{Gal}(\overline{F}/F)$-stable, and the induced action of $\mathrm{Gal}(\overline{F}/F)$ on $X^*(Q)$ is precisely the usual action, induced by the Galois action on $Q(\overline{F})$.

Now let δ denote half the sum of the positive roots of G. If $\delta \in X^*$ then δ is a twisting element; but in general we only have $\delta \in \frac{1}{2}X^*$. Let S' denote the maximal split torus quotient of G, so that

$$X^*(S') = (X^*)^{W,\mathrm{Gal}(\overline{F}/F)}.$$

Then if θ is a twisting element, we see that

$$\theta - \delta \in \frac{1}{2}X^*(S').$$

Thus we have a character $|\cdot|^{\theta-\delta}$ of $G(F)\backslash G(\mathbb{A}_F)$, defined as the composite

$$G(\mathbb{A}_F) \longrightarrow S'(\mathbb{A}_F) \xrightarrow{2(\theta-\delta)} \mathbb{A}_F^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0}^\times \xrightarrow{x\mapsto\sqrt{x}} \mathbb{R}_{>0}^\times$$

The main motivation behind the notion of twisting elements is the following two propositions.

Proposition 5.2.2. *If θ is a twisting element, then an automorphic representation π is C-algebraic if and only if $\pi \otimes |\cdot|^{\theta-\delta}$ is L-algebraic.*

Proof. This is a consequence of condition 10.3(2) of [Bor79], although formally one has to "reverse-engineer" the construction of (using the notation of §10.2 of loc. cit.) $\alpha \mapsto \pi_\alpha$. We sketch the argument using the notation there. The question is local at each infinite place, so let k denote a completion of F at an infinite place. Choose (by Proposition 5.1.2) a z-extension $\widetilde{G}$ of G. The character $|\cdot|^{\theta-\delta}$ induces a character of $G(k)$ and of $\widetilde{G}(k)$. If Q denotes the maximal torus quotient of $\widetilde{G}$ then this character can be extended to a character of $Q(k)$. The associated element of $H^1(W_k, \widehat{Q})$ is the image of an element $\alpha \in H^1(W_k, Z_L)$, with Z_L the centre of $\widehat{G}$, and one checks easily that the character π_α of $G(k)$ in §10.2 of [Bor79] coincides with $|\cdot|^{\theta-\delta}$. If T_0 is a maximal torus in $\widetilde{G}$, and T is its image in G, then the restriction of α to $\overline{k}^\times$ is a Z_L-valued character which, when considered as a $\widehat{T}_0$-valued character of $\overline{k}^\times$, has image in $\widehat{T}$ and which (via an easy diagram chase) coincides with the restriction to $\overline{k}^\times$ of the cohomology class associated via local Langlands to the restriction of $|\cdot|^{\theta-\delta}$ to $T(k)$. Hence a on $\overline{k}^\times$ is the composite of the norm map down to $\mathbb{R}_{>0}$, the square root map, and the cocharacter of $\hat{T}$ associated to $2(\theta-\delta)$. Twisting π by $|\cdot|^{\theta-\delta}$ corresponds to twisting r_π by a by 10.3(2) of [Bor79] and the result follows easily. $\square$

Proposition 5.2.3. *If θ is a twisting element, then π is C-arithmetic if and only if $\pi \otimes |\cdot|^{\theta-\delta}$ is L-arithmetic.*

Proof. Again this is a local issue: by Definitions 3.1.3 and 3.1.4 it suffices to check that if k (a completion of F) is a non-archimedean local field, if χ denotes the restriction of $|\cdot|^{\theta-\delta}$ to $G(k)$ and if π is an unramified representation of $G(k)$, then π is defined over a subfield E of $\mathbb{C}$ iff $\pi \otimes \chi$ has Satake parameter defined over the same subfield E. This is relatively easy to check: we sketch the details (using the notation of section 2.2). Let T be a maximal torus of G/k, with maximal compact subgroup ${}^oT$. Then χ induces an automorphism i of $H_{\mathbb{C}}(T(k), {}^oT)$ sending $[{}^oTt{}^oT]$ to $\chi(t)[{}^oTt{}^oT]$, and i commutes with the action of the Weyl group W_d and hence induces an automorphism i of the W_d-invariants of this complex Hecke algebra. If

m_π is the complex character of $H(T(k),{}^oT)^{W_d}$ associated to π and $m_{\pi\otimes\chi}$ is the character associated to $\pi\otimes\chi$ then one checks easily that $m_{\pi\otimes\chi}=m_\pi\circ i$. The other observation we need is that if K is a hyperspecial maximal compact subgroup of $G(k)$ and if S denotes the Satake isomorphism $S: H_{\mathbb{C}}(G(k),K)\to H_{\mathbb{C}}(T(k),{}^oT)^{W_d}$ then $i\circ S$ maps $H_{\mathbb{Q}}(G(k),K)$ into $H_{\mathbb{Q}}(T(k),{}^oT)$ (this follows immediately from formula (19) of section 4.2 of [Car79]), and hence into $H_{\mathbb{Q}}(T(k),{}^oT)^{W_d}$, and an injection between $\mathbb{Q}$-vector spaces which becomes an isomorphism after tensoring with $\mathbb{C}$ must itself be an isomorphism. Hence $i\circ S: H_{\mathbb{Q}}(G(k),K)\cong H_{\mathbb{Q}}(T(k),{}^oT)^{W_d}$ and now composing with m_π the result follows easily. □

Thus for groups with a twisting element, our L-notions and C-notions can be twisted into each other. What can one do when G has no twisting element? We know from Proposition 5.1.2 that G has a z-extension and hence a central cover with simply-connected derived subgroup. This cover may not have a twisting element either. But we shall now see that a rather simple z-extension of this cover does have one.

Proposition 5.2.4. *Suppose that G has simply connected derived subgroup. Then there is a z-extension*

$$1\to\mathbb{G}_m\to\widetilde{G}\to G\to 1$$

such that $\widetilde{G}$ has a twisting element.

Proof. We give an explicit construction that gives little away; a remark after the proof will indicate where the construction came from.

Fix a Borel B containing a maximal torus T in $G_{\overline{F}}$. Let $G^{\rm der}$ denote the derived subgroup of G, and define $B_{\rm der}:=B\cap(G^{\rm der})_{\overline{F}}$ and $T_{\rm der}:=T\cap(G^{\rm der})_{\overline{F}}$, a Borel and a maximal torus in $(G^{\rm der})_{\overline{F}}$. Now $G^{\rm der}$ is assumed simply-connected, so half the sum of the positive roots (with respect to $B_{\rm der}$) is a character $\overline{\eta}$ of $T_{\rm der}$. Then $\overline{\eta}$ induces a character d of the centre $Z_{\rm der}:=Z(G^{\rm der})$ of $G^{\rm der}$, defined over $\overline{F}$. One checks that d is independent of the choice of Borel and torus chosen, and furthermore (via a diagram-chase) that d on $\overline{F}$-points commutes with the actions of Galois and hence d is defined over F. If $Z:=Z(G)$ is the centre of G then it is well-known that the map $G^{\rm der}\times Z\to G$ induced by the inclusions is a central isogeny over F with kernel $Z_{\rm der}$ (embedded via $z\mapsto(z,z^{-1})$ say). Now define $\widetilde{G}$, a connected reductive group over F, by

$$\widetilde{G}:=(G^{\rm der}\times Z\times\mathbb{G}_m)/Z_{\rm der}$$

with $Z_{\rm der}$ embedded in the right hand side via the map sending z to $(z,z^{-1},d(z))$. An easy diagram chase shows that there is an exact sequence

$$0\to\mathbb{G}_m\to\widetilde{G}\to G\to 0$$

making $\widetilde{G}$ a z-extension of G. Let $\widetilde{T}$ denote the pre-image of T in $\widetilde{G}_{\overline{F}}$.

We need to check that $\widetilde{G}$ has a twisting element. Define $\theta:(T_{\rm der}\times Z_{\overline{F}}\times\mathbb{G}_{m\overline{F}})\to\mathbb{G}_{m\overline{F}}$ by $\theta(a,b,c):=\overline{\eta}(a)/c$. Then θ is trivial on $(Z_{\rm der})_{\overline{F}}$ embedded as above, so θ descends to a character of the image of $T_{\rm der}\times Z_{\overline{F}}\times\mathbb{G}_{m\overline{F}}$ in $\widetilde{G}$, which is exactly $\widetilde{T}$, the pre-image of T in $\widetilde{G}_{\overline{F}}$. Furthermore, θ restricts to $\overline{\eta}$ on $T_{\rm der}$, from which it follows easily that θ pairs to 1 with every simple coroot.

It remains to check that $\theta\in X^*(\widetilde{T})$ is Galois-stable, with respect to the Galois action on $X^*(\widetilde{T})$ defined using the L-group formalism with respect to the Borel subgroup $\widetilde{B}$, the pre-image of B in $\widetilde{G}_{\overline{F}}$, and this is just an explicit diagram chase, which comes down to the fact that $\overline{\eta}$ is Galois-stable in $X^*(T_{\mathrm{der}})$. □

Remark 5.2.5. What is going on here is that if Q denotes the quotient G/G^{der} then we have a short exact sequence

$$0 \to T_{\mathrm{der}} \to T \to Q_{\overline{F}} \to 0$$

over $\overline{F}$, and hence a short exact sequence of character groups

$$0 \to X^*(Q_{\overline{F}}) \to X^*(T) \to X^*(T_{\mathrm{der}}) \to 0.$$

If we give $X^*(Q_{\overline{F}})$ the Galois action coming from the fact that Q is defined over F, then this becomes an exact sequence of Galois modules. The element $\overline{\eta}$ is a Galois-stable element of $X^*(T_{\mathrm{der}})$, and G has a twisting element if and only if $\overline{\eta}$ lifts to a Galois-stable element of $X^*(T)$ (and conversely, any twisting element in $X^*(T)$ must map to $\overline{\eta} \in X^*(T_{\mathrm{der}})$). Note in particular, as we asserted earlier, that if G splits and has simply-connected derived subgroup, then Galois acts trivially on everything, so any lift of $\overline{\eta}$ to $X^*(T)$ is a twisting element for G. The obstruction to this lift existing in general is the image ζ of $\overline{\eta}$ in $H^1(\mathrm{Gal}(\overline{F}/F), X^*(Q_{\overline{F}}))$, and ζ gives rise to an extension

$$0 \to X^*(Q_{\overline{F}}) \to E \to \mathbb{Z} \to 0$$

of Galois modules, which gives an extension of Q by $\mathbb{G}_m$ over F and hence an extension of G by $\mathbb{G}_m$ over F; an easy diagram-chase shows that this is the extension that we have constructed.

In particular, if we replace F by a finite extension K where G splits, then $\overline{\eta}$ does lift to $X^*(T)$, so $\zeta = 0$ and we deduce that over K, $\widetilde{G}$ is just a product $G \times \mathrm{GL}_1$. In particular, the dual group $(\widetilde{G})\hat{}$ is just $\widehat{G} \times \mathrm{GL}_1$ over $\overline{\mathbb{Q}}$. The next lemma takes this observation further.

Lemma 5.2.6. *With notation as in Proposition 5.2.4, we have*

$${}^L\widetilde{G} \cong (\hat{G} \times \mathbb{G}_m) \rtimes \mathrm{Gal}(\overline{F}/F),$$

where $g \in \mathrm{Gal}(\overline{F}/F)$ acts on $\hat{G} \times \mathbb{G}_m$ by

$$g(x, \lambda) = \Big(\big(\tilde{\zeta}(g)\big)(\lambda).gx, t\Big)$$

where $\tilde{\zeta}$ is a cocycle representing the element $\zeta \in H^1(\mathrm{Gal}(\overline{F}/F), X^*(Q)) = H^1(\mathrm{Gal}(\overline{F}/F), X_*(\hat{Q})$ from Remark 5.2.5.

Proof. This is mostly just an unravelling of definitions. The map $\widetilde{T} \to T$ and the map $\theta : \widetilde{T} \to \mathbb{G}_m$ induce an isomorphism $\widetilde{T} = T \times \mathbb{G}_m$ and hence isomorphisms $X^*(\widetilde{T}) = X^*(T) \oplus \mathbb{Z}$ and $X_*(\widetilde{T}) = X_*(T) \oplus \mathbb{Z}$, which are direct sums as Galois modules but not in general as based root data, for the simple coroots in $X_*(\widetilde{T})$ are of the form $(\alpha^\vee, 1)$ with $\alpha \in X_*(T)$ a simple coroot. So let us instead decompose $\widetilde{T}$ as $T \times \mathbb{G}_m$ in a way which is less ideal from the point of view of the Galois action but better from the point of view of the root data. We achieve this thus: we have $\overline{\eta} \in X^*(T_{\mathrm{der}})$, half the sum of the positive roots. Lift $\overline{\eta}$ arbitrarily to $\eta \in X^*(T)$, which may not be fixed by Galois if G has no splitting element. Then η induces a character of $\widetilde{T}$ also denoted η. Now define a character $\sigma : \widetilde{T} \to \mathrm{GL}_1$ by $\sigma = \eta/\theta$. Because $\theta = \eta = \overline{\eta}$ on T_{der}, we see that $T_{\mathrm{der}} \subseteq \ker(\sigma)$. Furthermore the composite of the natural inclusion $\mathbb{G}_m \to \widetilde{T}$ and σ is the identity on $\mathbb{G}_m$, and hence σ and the projection $\widetilde{T} \to T$ induce an isomorphism $\widetilde{T} = T \times \mathbb{G}_m$, which induces an isomorphism between the based root datum of $\widetilde{G}$ and the direct sum of the based root data for $\widetilde{G}$ and $\mathbb{G}_m$. We fix this identification for the rest of this argument. This direct sum is not in general Galois-stable; an elementary calculation shows that for $(\alpha, n) \in X^*(\widetilde{T}) = X^*(T) \oplus \mathbb{Z}$ and $g \in \mathrm{Gal}(\overline{F}/F)$ we have $g(\alpha, n) = (g\alpha + n(g\eta - \eta), n)$. Note that $g \mapsto g\eta - \eta$ is a cocycle representing ζ. It suffices now to find an action of $\mathrm{Gal}(\overline{F}/F)$ on $\hat{G} \times \mathrm{GL}_1$ which induces the above action on the cocharacter groups $X_*(\hat{T}) \oplus \mathbb{Z} = X^*(T) \oplus \mathbb{Z}$, but the formula in the statement of the lemma is easily checked to give an action (note that the image of $\tilde{\zeta}$ is in $\hat{Q}$, which is central in $\hat{G}$) which does this. □

Corollary 5.2.7. *Any connected reductive group over a number field has a z-extension which has a twisting element.*

Proof. This follows from Proposition 5.1.2, Proposition 5.2.4 and Lemma 5.1.3. □

Remark 5.2.8. Once can now (having chosen a z-extension and a twisting element for the covering group as above) use Conjecture 3.2.1 to formulate a conjecture associating Galois representations to C-algebraic automorphic representations (for example, to cohomological representations by Lemma 7.3.2 below) for an arbitrary connected reductive group G over a number field. One uses Lemma 5.1.4 to pull the C-algebraic representations back to a C-algebraic representation on a group with a twisting element, twists them so they become L-algebraic, and then uses Conjecture 3.2.1 on this bigger group. For an explicit example of this, see section 8.3.

We end this section by showing that the results in it imply the equivalence of Conjectures 3.1.5 and 3.1.6 (made for all groups simultaneously).

Proposition 5.2.9. *Let G be a connected reductive group over a number field. If Conjecture 3.1.5 is true for all z-extensions of G then Conjecture 3.1.6 is true for G. Similarly if Conjecture 3.1.6 is true for all z-extensions of G then Conjecture 3.1.5 is true for G.*

Proof. We prove the first assertion; the second one is similar. Say G is connected and reductive, and π is C-arithmetic (resp. C-algebraic). By Corollary 5.2.7 there is a z-extension G' of G with a twisting element θ. Let π' be the pullback of π to G'. Then π' is C-arithmetic (resp. C-algebraic) by Lemma 5.1.4. By Proposition 5.2.3 (resp. Proposition 5.2.2) $\pi' \otimes |.|^{\theta-\delta}$ is L-arithmetic (resp. L-algebraic). Applying Conjecture 3.1.5 to G' we deduce that $\pi' \otimes |.|^{\theta-\delta}$ is L-algebraic (resp. L-arithmetic). Running the argument backwards now shows us that π is C-algebraic (resp. C-arithmetic). □

6. Functoriality.

6.1. Suppose that G, G' are two connected reductive groups over F, and that we have an L-group homomorphism

$$r : {}^L G \to {}^L G',$$

i.e. a homomorphism of algebraic groups over $\overline{\mathbb{Q}}$ which respects the projections to $\mathrm{Gal}(\overline{F}/F)$. Assume that G' is quasi-split over F. Then we have the following weak version of Langlands' functoriality conjecture (note that we are only demanding compatibility with the local correspondence at a subset of the unramified places, and at infinity).

Conjecture 6.1.1. *If π is an automorphic representation of G, then there is an automorphic representation π' of G', called a* functorial transfer *of π, such that*

- *For all infinite places v, and for all finite places v at which π and G' are unramified, $r_{\pi'_v}$ is $\hat{G}'(\mathbb{C})$-conjugate to $r \circ r_{\pi_v}$.*

A trivial consequence of the definitions is

Lemma 6.1.2. *If π is L-algebraic, then any functorial transfer of π is L-algebraic.*

We also have the only slightly less trivial

Lemma 6.1.3. *If π is L-arithmetic, then any functorial transfer of π is L-arithmetic.*

Proof. This result follows from a purely local assertion. If v is a finite place where G and G' are unramified and if k is the completion of F at v then the morphism r of L-groups induces a morphism $\hat{G} \to \hat{G}'$ which commutes with the action of the Frobenius at v. If T_d (resp. T'_d) denotes a maximal k-split torus in G/k (resp. G'/k) with centralizer T (resp. T') then the map $\hat{G} \to \hat{G}'$ induces a map $\hat{T} \to \hat{T}'$ (well-defined up to restricted Weyl group actions) which commutes with the action of Frobenius, and hence maps $\hat{T}_d \to \hat{T}'_d$ and $X^*(\hat{T}'_d) = X_*(T'_d) \to X^*(\hat{T}_d) = X_*(T_d)$. Now looking at the explicit definition of the Satake isomorphism in Proposition 6.7 of [Bor79] we see, after unravelling, that the map $\mathbb{Q}[X_*(T'_d)]^{W'_d} \to \mathbb{Q}[X_*(T_d)]^{W_d}$ induced from $X_*(T'_d) \to X_*(T_d)$ above has the property that, after tensoring up to $\mathbb{C}$ and taking spectra, it sends the point in $\mathrm{Spec}(\mathbb{C}[X_*(T_d)]^{W_d})$ corresponding to r_{π_v} to the point in $\mathrm{Spec}(\mathbb{C}[X_*(T'_d)]^{W'_d})$ corresponding to $r \circ r_{\pi_v}$. We now deduce that if the Satake parameter of π_v is defined over a subfield E of $\mathbb{C}$ then so is the Satake parameter of π'_v (because the homomorphism $\mathbb{Q}[X_*(T'_d)]^{W'_d} \to \mathbb{C}$ corresponding to π'_v factors through $\mathbb{Q}[X_*(T_d)]^{W_d}$ and hence through E) and the result follows. □

In addition,

Proposition 6.1.4. *If Conjecture 3.2.1 holds for π, then it holds for any functorial transfer of π.*

Proof. With notation as above, one easily checks that $\rho_{\pi',\iota} := r \circ \rho_{\pi,\iota}$ satisfies all the conditions of Conjecture 3.2.1. □

Note that functoriality relies on things normalised in Langlands' canonical way; the natural analogues of the results above in the C-algebraic and C-arithmetic cases are not true in general, because a morphism of algebraic groups does not send half the sum of the positive roots to half the sum of the positive roots in general.

7. Reality checks.

7.1. By Proposition 2 of [Lan79] any automorphic representation π on G is a subquotient of an induction $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \sigma$, where P is a parabolic subgroup of G with Levi quotient M, and σ is a cuspidal representation of M. If π' is another automorphic subquotient of $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \sigma$, then π_v and π'_v are equal for all but finitely many places, so π is C-arithmetic (respectively L-arithmetic) if and only if π' is C-arithmetic (respectively L-arithmetic). The following lemma shows that π is

C-algebraic (respectively L-algebraic) if and only if π' is C-algebraic (respectively L-algebraic).

Lemma 7.1.1. *Suppose that π and π' are subquotients of a common induction $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\sigma$. Then π and π' have the same infinitesimal character.*

Proof. This is immediate from the calculation of the infinitesimal character of an induction - see for example Proposition 8.22 of [Kna01]. □

Furthermore, we can check the compatibility of Conjecture 3.2.1 for π and π' (note that we cannot check the compatibility for Conjecture 3.2.2 because π and π' may be ramified at different places).

Proposition 7.1.2. *Suppose that π and π' are subquotients of a common induction $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\sigma$. Suppose that π is L-algebraic. If Conjecture 3.2.1 is valid for π then it is valid for π'.*

Proof. Suppose that Conjecture 3.2.1 is valid for π. We wish to show that $\rho_{\pi',\iota} := \rho_{\pi,\iota}$ satisfies all the conditions in Conjecture 3.2.1. Since for all but finitely many places π_v and π'_v are unramified and isomorphic, the first two conditions are certainly satisfied. The third condition is satisfied by Lemma 7.1.1. It remains to check that if v is a real place, then (with obvious notation) $\lambda_v(i)\mu_v(i)r_{\pi_v}(j)$ and $\lambda'_v(i)\mu'_v(i)r_{\pi'_v}(j)$ are $\hat{G}(\mathbb{C})$-conjugate. As explained to us by David Vogan, it follows from the results of [ABV92] (specifically from Theorem 1.24 and Proposition 6.16) that $\lambda_v(-1)r_{\pi_v}(j)$ and $\lambda'_v(-1)r_{\pi'_v}(j)$ are $\hat{G}(\mathbb{C})$-conjugate. It is easy to check that these are $\hat{G}(\mathbb{C})$-conjugate to $\lambda_v(i)\mu_v(i)r_{\pi_v}(j)$ and $\lambda'_v(i)\mu'_v(i)r_{\pi'_v}(j)$ respectively (for example, via $r_{\pi_v}(e^{i\pi/4})$, $r_{\pi'_v}(e^{i\pi/4})$), as required. □

More generally, if π, π' are nearly equivalent (that is, π_v and π'_v are isomorphic for all but finitely many v), then π is C-arithmetic (respectively L-arithmetic) if and only if π' is C-arithmetic (respectively L-arithmetic). We would like to be able to prove as above that π and π' have the same infinitesimal character, and we would like to obtain the analogue of Proposition 7.1.2. Unfortunately, these seem in general to be well beyond the reach of current techniques. However, we can prove these results for GL_n, and we can then deduce them for general groups under the assumption of functoriality.

Proposition 7.1.3. *If $G = \mathrm{GL}_n$ and π, π' are nearly equivalent, then π, π' have the same infinitesimal character. Suppose further that π is L-algebraic. If Conjecture 3.2.1 is valid for π then it is valid for π'.*

Proof. By the strong multiplicity one theorem for isobaric representations (Theorem 4.4 of [JS81]), π, π' are both subquotients of a common induction $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\sigma$. The result follows from Lemma 7.1.1 and Proposition 7.1.2. □

Proposition 7.1.4. *Let G be arbitrary. Assume Conjecture 6.1.1. If π and π' are nearly equivalent automorphic representations of G, then for any infinite place v, π_v and π'_v have the same infinitesimal characters. Suppose further that π is L-algebraic. If Conjecture 3.2.1 is valid for π then it is valid for π'.*

Proof. To begin with, note that for each infinite place v we have a natural injection ${}^LG_v \to {}^LG$, where G_v is the base change of G to F_v. Since $r_{\pi_v}|_{\mathbb{C}^\times}$ is valued in

${}^LG_v$, as is $c_v := \lambda_v(i)\mu_v(i)r_{\pi_v}(j)$, we see that their $\hat{G}(\mathbb{C})$-conjugacy classes in ${}^LG$ are determined by their $\hat{G}(\mathbb{C})$-conjugacy classes in ${}^LG_v$.

Now, the $\hat{G}(\mathbb{C})$-conjugacy classes of semisimple elements of ${}^LG_v(\mathbb{C})$ are determined by the knowledge of the conjugacy classes of their images under all representations of ${}^LG_v(\mathbb{C})$. To see this, note that since the formation of the L-group is independent of the choice of inner form, it suffices to check this in the case where G_v is quasi-split; but the result then follows immediately from Proposition 6.7 of [Bor79].

Let $r : {}^LG \to \mathrm{GL}_n \times \mathrm{Gal}(\overline{F}/F)$ be a homomorphism of L-groups. Then by Conjecture 6.1.1, there are automorphic representations Π, Π' on GL_n which are functorial transfers of π, π' respectively. By Proposition 7.1.3, Π and Π' have the same infinitesimal characters. Thus for each infinite place v, $r_{\Pi_v}|_{\mathbb{C}^\times}$ and $r_{\Pi'_v}|_{\mathbb{C}^\times}$ are conjugate, i.e. $r \circ r_{\pi,\iota}|_{\mathbb{C}^\times}$ and $r \circ r_{\pi',\iota}|_{\mathbb{C}^\times}$ are conjugate. Since this is true for all r, we see that $r_{\pi,\iota}|_{\mathbb{C}^\times}$ and $r_{\pi',\iota}|_{\mathbb{C}^\times}$ are conjugate, whence π and π' have the same infinitesimal character.

As in the proof of Proposition 7.1.2, it remains to check that if v is a real place of F, then $c_v := \lambda_v(i)\mu_v(i)r_{\pi_v}(j)$ and $c'_v := \lambda'_v(i)\mu'_v(i)r_{\pi'_v}(j)$ are $\hat{G}(\mathbb{C})$-conjugate. By a similar argument to that used in the first half of this proof, we see that if $r : {}^LG \to \mathrm{GL}_n \times \mathrm{Gal}(\overline{F}/F)$ is a homomorphism of L-groups, then $r(c_v)$ and $r(c'_v)$ are conjugate in $\mathrm{GL}_n(\mathbb{C})$. Now, c_v and c'_v are both semisimple; for example, $c_v = \lambda_v(i)\mu_v(i)r_{\pi_v}(j) = \mu_v(-1)r_{\pi_v}(ij)$, a product of commuting semisimple elements. Thus c_v and c'_v are $\hat{G}(\mathbb{C})$-conjugate. □

7.2. **Cohomological representations.**

7.3. Cohomological automorphic representations provide a good testing ground for our conjectures. It follows easily (see below) that any cohomological representation is C-algebraic, and one can often show that they are C-arithmetic, too. In the case $G = \mathrm{GL}_n$ these arguments are due to Clozel, who also shows that for GL_n any regular C-algebraic representation is cohomological after possibly twisting by a quadratic character (see Lemme 3.14 of [Clo90]).

Let v be an infinite place of F, and let K_v be the fixed choice of a maximal compact subgroup of $G(F_v)$ used in the definition of automorphic forms on G. Let $\mathfrak{g}_v$ be the complexification of the Lie algebra of $G(F_v)$. Recall that π_v may be thought of as a $(\mathfrak{g}_v, K_v)$-module, with underlying $\mathbb{C}$-vector space V_v, say.

Definition 7.3.1. We say that π_v is *cohomological* if there is an algebraic complex representation U of $G(F_v)$ and a non-negative integer i such that

$$H^i(\mathfrak{g}_v, K_v; U \otimes V_v) \neq 0.$$

We say that π is cohomological if π_v is cohomological for all archimedean places v.

Lemma 7.3.2. *If π is cohomological, then it is C-algebraic.*

Proof. By Corollary 4.2 of [BW00], if π is cohomological then for each archimedean place v there is a continuous finite-dimensional representation U_v of $G(F_v)$ such that π_v and U_v have the same infinitesimal characters. The result then follows from Lemma 7.3.3 below. □

Lemma 7.3.3. *If v is an archimedean place of F and U is a continuous finite dimensional representation of $G(F_v)$ with infinitesimal character χ_v, identified with an element of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ as in section 2.3, then $\chi_v - \delta \in X^*(T)$.*

Proof. This follows almost at once from the definition of the Harish-Chandra isomorphism; see for example (5.43) in [Kna02]. □

We note that a cuspidal cohomological unitary automorphic representation π is also C-arithmetic, at least when π_∞ is cohomological for the trivial representation; for we can restrict scalars down to $\mathbb{Q}$ and then follow the argument in §2.3 of [BR94]. This argument presumably works in some greater generality.

8. Relationship with theorems/conjectures in the literature.

8.1. In [Clo90], Clozel makes a number of conjectures about certain C-algebraic automorphic representations for GL_n. We now examine the compatibility of these conjectures with those of this paper. Clozel calls an automorphic representation of GL_n *algebraic* if it is C-algebraic and isobaric; his principal reason for restricting to isobaric representations is that he wishes to construct a Tannakian category of automorphic representations.

Let $\pi = \otimes' \pi_v$ be an algebraic (in Clozel's sense) representation of GL_n over F. Then Clozel conjectures (see conjectures 3.7 and 4.5 of [Clo90]) that

Conjecture 8.1.1. *Let $\pi_f = \otimes'_{v\nmid\infty}\pi_v$. Then there is a number field $E \subset \mathbb{C}$ such that π_f is defined over E (that is, such that $\pi_f \otimes_{\mathbb{C},\sigma} \mathbb{C} \cong \pi_f$ for all automorphisms σ of $\mathbb{C}$ which fix E pointwise). In addition, Conjecture 3.2.2 holds for $\pi \otimes |\cdot|^{(n-1)/2}$.*

(In fact, Clozel conjectures much more than this—he conjectures that there is a motive whose local L-factors agree with those of $\pi \otimes |\cdot|^{(n-1)/2}$ at all finite places; the required Galois representation is then obtained as the p-adic realisation of this motive.)

By Proposition 7.1.3 we see, since any automorphic representation of GL_n is nearly equivalent to an isobaric one, that Conjecture 8.1.1 implies Conjecture 3.2.1 for GL_n, and in fact an examination of the proof shows that it implies Conjecture 3.2.2. We claim that it also implies that π is C-arithmetic; in fact, this follows at once from Proposition 3.1(iii) of [Clo90]. Thus for GL_n our conjectures follow from those of Clozel.

The reason that our conjectures are weaker than Clozel's conjectures is that for groups other than GL_n we do not have as good an understanding of the local Langlands correspondence, which makes (for example) the formulation of a definition of L-arithmetic using all (possibly ramified) finite places problematic.

8.2. In [Gro99] Gross presents a conjecture which assigns a Galois representation to an automorphic representation on a group G with the property that any arithmetic subgroup is finite (in fact Gross gives six conditions equivalent to this in Proposition 1.4 of [Gro99]). We now discuss the relationship of this conjecture to our conjectures. The group G splits over a CM field L, and is assumed to have a twisting element η in the sense of Definition 5.2.1. In fact, Gross has informed us that one should in addition assume that the group G is semisimple and simply connected, so we make this assumption from now on. This assumption in fact implies that $\eta = \delta$ in the below, but for those who want to be more optimistic than Gross we have kept the two notations distinct in the below.

Let V be an absolutely irreducible representation of G over $\mathbb{Q}$ with trivial central character. Let S be a finite set of primes of size at least 2, containing all primes at which G is ramified. For each $l \notin S$ we let K_l be a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_l)$, and for each $l \in S$ we let K_l be the connected component of an Iwahori subgroup of $G(\mathbb{Q}_l)$. Let K be the product of the K_l. Then $M(V, K)$ is the space of algebraic modular forms given by

$$M(V,K) = \{f : G(\mathbb{A}_{\mathbb{Q}})/(G(\mathbb{R})_+ \times K) \to V : f(\gamma g) = \gamma f(g) \text{ for all } \gamma \in G(\mathbb{Q}), g \in G(\mathbb{A}_{\mathbb{Q}})\}.$$

Let T_S be the unramified Hecke algebra, the product of the unramified Hecke algebras T_l for each $l \notin S$. Let T_K be the full Hecke algebra, the product of T_S and the product of the Iwahori Hecke algebras T_l at places l in S. Let A be $T_K \otimes \mathbb{Q}[\pi_0(G(\mathbb{R}))]$. This acts on $M(V, K)$ (see section 6 of [Gro99]), and we let N be a simple A-submodule of $M(V, K)$. We assume that N gives the Steinberg character on T_l for all $l \in S$ (see section 12 of [Gro99]), and if V is trivial and $\prod_{l \in S} G(\mathbb{Q}_l)$ is compact, we exclude the case that N is trivial.

By Proposition 12.3 of [Gro99], $\mathrm{End}_A(N)$ is a CM field, and by (7.4) of [Gro99], $\pi_0(G(\mathbb{R}))$ acts on N through a character

$$\phi_\infty : \pi_0(G(\mathbb{R})) \to \{\pm 1\} \subset E^\times.$$

By Proposition 8.5 of [Gro99], the simple submodules of $N \otimes \mathbb{C}$ may be identified (compatibly with the actions of H_K) with irreducible automorphic representations $\pi = \pi_f \otimes \pi_\infty$ with $\pi_\infty \xrightarrow{\sim} V \otimes \mathbb{C}$, and π_l Steinberg for all $l \in S$. For all $l \notin S$, the unramified local Langlands correspondence (i.e. the Satake isomorphism) identifies the character of T_l on N with a homomorphism $r_{N,l} : W_{\mathbb{Q}_l} \to {}^L G(\mathbb{C})$, and π_l corresponds to this parameter under the local Langlands correspondence. Fix such a representation π.

Gross then makes the following conjecture (see Conjecture 17.2 as well as (15.3) and (16.8) of [Gro99], and note that while Gross normalises local class field theory so that an arithmetic Frobenius element corresponds to a uniformiser, this is equivalent to our formulation by Remark 3.2.5):

Conjecture 8.2.1. *If p is a prime, and $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, then there is a continuous Galois representation*

$$\rho_{N,\iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}_p)$$

satisfying

- *If $l \notin S$, then $\rho_{N,\iota}|_{W_{\mathbb{Q}_l}}$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to $\iota(r_{N,l}) \otimes |\cdot|^{\eta-\delta}$.*
- *If s_∞ is a complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\rho_{N,\iota}(s_\infty)$ is $\hat{G}(\overline{\mathbb{Q}}_p)$-conjugate to $(\iota(\eta(-1)\phi_\infty(-1)), s_\infty)$.*

This conjecture follows from Conjecture 3.2.2. The representation π is C-algebraic, so by Proposition 5.2.2 $\pi \otimes |\cdot|^{\eta-\delta}$ is L-algebraic. Applying Conjecture 3.2.2 gives everything in Conjecture 8.2.1 (for the description of complex conjugation, see [Gro]).

Note that Gross in fact conjectures something slightly stronger; he shows that π is C-algebraic, and in fact that π is defined over E, and conjectures that for any place $\lambda | p$ of E there is a natural Galois representation $\rho_{N,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(E_\lambda)$. As Gross has explained to us, this rationality conjecture should follow from the hypothesis that π is Steinberg at two places, together with local-global compatibility for the Galois representations at these places.

8.3. We now discuss an example drawn from [CHT08]. Let F be a totally real field, and let E be a quadratic totally imaginary extension of F. Let G be an n-dimensional unitary group over F which splits over E, and which is compact (that is, isomorphic to $U(n)$) at all infinite places. Assume that n is even. Then the dual group of G is GL_n, and if we let $\mathrm{Gal}(E/F) = \{1, c\}$, then the L-group of G is given by

$$^L G = \mathrm{GL}_n \rtimes \mathrm{Gal}(\overline{F}/F)$$

where $\mathrm{Gal}(\overline{F}/F)$ acts on GL_n via its projection to $\mathrm{Gal}(E/F) = \{1, c\}$, with

$$x^c := \Phi_n {}^t x^{-1} \Phi_n^{-1}$$

where Φ_n is an anti-diagonal matrix with alternating entries 1, -1. Note that because n is even, ${}^t\Phi_n = -\Phi_n = \Phi_n^{-1}$.

With the usual notation for characters of GL_n, a choice of element η as in the proof of Proposition 5.2.4 is given by $\eta = (n-1, n-2, \ldots, 0)$. Then $c(\eta) - \eta = (1-n, 1-n, \ldots, 1-n)$, so that by Lemma 5.2.6 we have

$$^L\tilde{G} = (\mathrm{GL}_n \times \mathbb{G}_m) \rtimes \mathrm{Gal}(E/F)$$

with $\mathrm{Gal}(E/F)$ acting by

$$(g, \mu)^c = (\mu^{1-n} \Phi_n {}^t g^{-1} \Phi_n^{-1}, \mu).$$

In Section 1 of [CHT08] there is a definition of a group $\mathcal{G}_n$. This group is also a semidirect product $(\mathrm{GL}_n \times \mathbb{G}_m) \rtimes \mathrm{Gal}(E/F)$, but with $\mathrm{Gal}(E/F)$ acting by

$$(g, \mu)^c = (\mu {}^t g^{-1}, \mu).$$

We claim that there is an isomorphism $j : \mathcal{G}_n \to {}^L\tilde{G}$ given by

$$j(g, \mu) = (\mu^{-n/2} g, \mu),$$
$$j(c) = (\Phi_n, -1)c.$$

The key calculation is to check that

$$j(c(g, \mu)c) = j(c)j(g, \mu)j(c).$$

We have

$$\begin{aligned} j(c(g, \mu)c) &= j(\mu {}^t g^{-1}, \mu) \\ &= (\mu^{1-n/2}({}^t g^{-1}), \mu), \end{aligned}$$

and

$$\begin{aligned} j(c)j(g, \mu)j(c) &= (\Phi_n, -1)c(\mu^{-n/2} g, \mu)(\Phi_n, -1)c \\ &= (\Phi_n, -1)((-\mu)^{1-n} \Phi_n {}^t(\mu^{-n/2} g \Phi_n)^{-1} \Phi_n^{-1}, -\mu) \\ &= ((-\mu)^{1-n} \Phi_n^2 {}^t(\mu^{-n/2} g \Phi_n)^{-1} \Phi_n^{-1}, \mu) \\ &= (\mu^{1-n} \mu^{n/2} ({}^t g^{-1}) {}^t\Phi_n^{-1} \Phi_n^{-1}, \mu) \\ &= (\mu^{1-n/2}({}^t g^{-1}), \mu), \end{aligned}$$

as required.

Note that j^{-1} is given by

$$j^{-1}(g, \mu) = (\mu^{n/2} g, \mu),$$
$$j^{-1}(c) = ((-1)^{n/2+1} \Phi_n, -1)c.$$

Suppose now that π is a cuspidal automorphic representation of G. By the assumption that G is compact at infinity, π is automatically cohomological, and thus C-algebraic. Since n is even, it is easy to check that G does not have a twisting element, so we cannot twist π to be L-algebraic. However, $\tilde{G}$ does have a twisting element, so we expect that (after choosing a twisting element) there should be a Galois representation associated to π, valued in ${}^L\tilde{G}$, or (via j^{-1}) in $\mathcal{G}_n$. With the usual notation for characters of GL_n, we choose the twisting element $\tilde{\eta}$ for $\tilde{G}$ given by

$$\tilde{\eta} = ((n-1+n/2, n-2+n/2, \ldots, n/2), -1).$$

We will refer to the automorphic representation on $\tilde{G}$ corresponding to π as π without fear of confusion, and let $\pi' = \pi \otimes |\cdot|^{\tilde{\eta}-\delta}$. Then π' is L-algebraic. Fix $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$; we wish to consider the conjectural Galois representation $j^{-1} \circ \rho_{\pi',\iota} : \mathrm{Gal}(\overline{F}/F) \to \mathcal{G}_n(\overline{\mathbb{Q}}_p)$.

Suppose that v is a place of F which splits in E, and for which π_v is unramified. Then G_v is isomorphic to GL_n, and the obvious map $j^{-1} \circ \rho_{\pi',\iota}|_{W_v} : W_v \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ should (by the form of $\tilde{\eta} - \delta$) be $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$-conjugate to $\iota(r_{\pi_v} \otimes |\cdot|^{(n-1)/2})$. Note that the set of such places is dense in F, so by the Cebotarev density theorem this is already enough information to determine the representation $j^{-1} \circ \rho_{\pi',\iota}$ up to $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$-conjugacy.

Suppose that v is an infinite place of F. Since G_v is compact, π_v is a finite-dimensional algebraic representation, of highest weight s_v, say. Then the Hodge-Tate weights of $j^{-1} \circ \rho_{\pi',\iota}$ with respect to v should be $s_v + (n-1, \ldots, 0)$. In order to specify complex conjugation, we need to consider the $(\mathrm{GL}_n \times \mathrm{GL}_1)$-conjugacy classes of elements $(g, \mu)c$ of $\mathcal{G}_n$ of order 2. It is easy to see that there are precisely two conjugacy classes: one with $\mu = 1$, and one with $\mu = -1$. We know (from the explicit description of the local Langlands correspondence for discrete series representations, cf. section 11 of [Bor79]) that (up to conjugacy) $r_{\pi_v}(z) = z^{s_v+\delta}\overline{z}^{-s_v-\delta}$ for all $z \in \mathbb{C}^\times$, and $r_{\pi_v}(j) = \Phi_n c$. Thus if c_v is a complex conjugation at v, we see that Conjecture 3.2.1 predicts that $j^{-1} \circ \rho_{\pi',\iota}(c_v)$ is in the conjugacy class of elements of order 2 of the form $(g, -1)c$.

Now, under certain mild hypotheses on G and π, we note that a Galois representation satisfying the above properties is proved to exist in [CHT08] and [Tay08]. Specifically, everything except the form of complex conjugation follows from Proposition 3.4.4 of [CHT08] (although see also Theorems 4.4.2 and Theorems 4.4.3 of [CHT08] for related results on GL_n whose notation may be easier to compare to the notation used in this paper), and the form of complex conjugation follows from Theorem 4.1 of [Tay08]. [Note when comparing the unramified places that by definition the local Langlands correspondence $\mathrm{rec}(\pi_v)$ used in [CHT08] is our r_{π_v}.]

Note that we assumed that n is even in the above discussion. In the case that n is odd, the group G does have a twisting element (for example δ), and one can check that the Galois representation to $\mathcal{G}_n$ constructed in [CHT08] gives (after twisting by $\epsilon^{(n-1)/2}$, with ϵ the p-adic cyclotomic character) a Galois representation to ${}^LG$, consistently with Conjecture 3.2.2. Because of this, when n is odd the construction of $\tilde{G}$ is not necessary and our conjecture is still consistent with their theorem.

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