TWISTED ACYCLICITY OF A CIRCLE AND LINK SIGNATURES

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ABSTRACT. Homology of the circle with non-trivial local coefficients is trivial. From this well-known fact we deduce geometric corollaries concerning links of codimension two. In particular, the Murasugi-Tristram signatures are extended to invariants of links formed of arbitrary oriented closed codimension two submanifolds of an odd-dimensional sphere. The novelty is that the submanifolds are not assumed to be disjoint, but are transversal to each other, and the signatures are parametrized by points of the whole torus. Murasugi-Tristram inequalities and their generalizations are also extended to this setup.

1. Introduction.

The goal of this paper is to simplify and generalize a part of classical link theory based on various signatures of links (defined by Trotter [19] Murasugi [10],[11], Tristram [18], Levine [7] [8], Smolinsky [17], Florens [2] and Cimasoni and Florens [1]). This part is known for its relations to topology of 4-dimensional manifolds, see [18], [20], [21] [4], [6] and applications in topology of real algebraic curves [12], [13] and [2].

Similarity of the signatures to the new invariants [15], [14], which were defined in the new frameworks of link homology theories and had spectacular applications [15], [9], [16] to problems on classical link cobordisms, gives a new reason to revisit the old theory.

There are two ways to introduce the signatures: the original 3-dimensional, via Seifert surface and Seifert form, and 4-dimensional, via the intersection form of the cyclic coverings of 4-ball branched over surfaces. I believe, this paper clearly demonstrates advantages of the latter, 4-dimensional approach, which provides more conceptual definitions, easily working in the situations hardly available for the Seifert form approach.

In the generalization considered here the classical links are replaced by collections of transversal to each other oriented submanifolds of codimension two.

Technically the work is based on a systematic use of twisted homology and the intersection forms in the twisted homology. Only the simplest kinds of twisted homology is used, the one with coefficients in $\mathbb{C}$, see Appendix.

1.1. Twisted acyclicity of a circle.

A key property of twisted homology, which makes the whole story possible, is the following well-known fact, which I call *twisted acyclicity of a circle*:

Twisted homology of a circle with coefficients in $\mathbb{C}$ *and non-trivial monodromy vanishes.*

This implies that the twisted homology of this kind completely ignores parts of the space formed by circles along which the monodromy of the coefficient system is non-trivial (for precise and detailed formulation see Section Appendix B).

1.2. How the acyclicity works. In particular, twisted acyclicity of a circle implies that the complement of a tubular neighborhood of a link looks like a closed manifold, because the boundary, being fibered to circles, is invisible for the twisted homology.

Moreover, the same holds true for a collection of pairwise transversal generically immersed closed manifolds of codimension 2 in arbitrary closed manifold, provided the monodromy around each manifold is non-trivial. The twisted homology does not feel the intersection of the submanifolds as a singularity.

The complement of a cobordism between such immersed links looks (again, from the point of view of twisted homology) like a compact cobordism between closed manifolds.

This, together with classical results about signatures of manifolds and relations between twisted homology and homology with constant coefficients, allows us to deal with a link of codimension two as if it was a single closed manifold.

1.3. Organization of the paper. I cannot assume the twisted homology well-known to the reader, and review the material related to it. Of course, the material on non-twisted homology is not reviewed. The review is limited to a very special twisted homology, the one with complex coefficients. More general twisted homology is not needed here.

The review is postponed to appendices. The reader somehow familiar with twisted homology may visit this section when needed. The experts are invited to look through appendices, too.

We begin in Section 2 with a detailed exposition restricted to the classical links. Section 3 is devoted to higher dimensional generalization, including motivation for our choice of the objects. Section 4 is devoted to *span inequalities*, that is, restrictions on homology of submanifolds of the ball, which span a given link contained in the boundary of the ball. Section 5 is devoted to *slice inequalities*, which are restrictions on homology of a link with given transversal intersection with a sphere of codimension one.

2. In the classical dimension.

2.1. Classical knots and links. Recall that a *classical knot* is a smooth simple closed curve in the 3-sphere S^3. This is how one usually defines classical knots. However it is not the curve per se that is really considered in the classical knot theory, but rather its placement in S^3. Classical knots incarnate the idea of knottedness: both the curve and S^3 are topologically standard, but the position of the curve in S^3 may be arbitrarily complicated topologically. Therefore a classical knot is rather a pair (S^3, K), where K is a smooth submanifold of S^3 diffeomorphic to S^1.

A *classical link* is a pair (S^3, L), where L is a smooth closed one-dimensional submanifold of S^3. If L is connected, then this is a knot.

2.2. Twisted homology of a classical link exterior. An *exterior* of a classical link (S^3, L) is the complement of an open tubular neighborhood of L. This is a compact 3-manifold with boundary. The boundary is the boundary of the tubular neighborhood of L. Hence, this is the total space of a locally trivial fibration over L with fiber S^1. An exterior $X(L)$ is a deformation retract of the complement

$S^3 \smallsetminus L$. It's a nice replacement of $S^3 \smallsetminus L$, because $\operatorname{Int} X(L)$ is homeomorphic to $S^3 \smallsetminus L$, but $X(L)$ is compact manifold and has a nice boundary.

If L consists of m connected components, $L = K_1 \cup \dots \cup K_m$, then by the Alexander duality $H_0(X(L)) = \mathbb{Z}$, $H_1(X(L)) = \mathbb{Z}^m$, $H_2(X(L)) = \mathbb{Z}^{m-1}$ and $H_i(X(L)) = 0$ for $i \neq 0, 1, 2$. The group $H_1(X(L))$ is dual to $H_1(L)$ with respect to the Alexander linking pairing $H_1(L) \times H_1(X(L)) \to \mathbb{Z}$. Hence a basis of $H_1(L)$ defines a dual basis in $H_1(X(L))$. An orientation of L determines a basis $[K_1]$, …, $[K_m]$ of $H_1(L)$, and the dual basis of $H_1(X(L))$, which is realized by meridians M_1, …, M_m positively linked to K_1, …, K_m, respectively. (The meridians are fibers of a tubular fibration $\partial X(L) \to L$ over points chosen on the corresponding components.)

Therefore, if L is oriented, then a local coefficient system on $X(L)$ with fiber $\mathbb{C}$ is defined by an m-tuple of complex numbers $(\zeta_1, \dots, \zeta_m)$, the images under the monodromy homomorphism $H_1(X(L)) \to \mathbb{C}^\times$ of the generators $[M_1]$, …, $[M_m]$ of $H_1(X(L))$.

Thus for an oriented classical knot L consisting of m connected components, local coefficient systems on $X(L)$ with fiber $\mathbb{C}$ are parametrized by $(\mathbb{C}^\times)^m$.

2.3. Link signatures. Let $L = K_1 \cup \dots \cup K_m \subset S^3$ be a classical link, $\zeta_i \in \mathbb{C}$, $|\zeta_i| = 1$, $\zeta = (\zeta_1, \dots, \zeta_m) \in (S^1)^m$ and $\mu : H_1(S^3 \smallsetminus L) \to \mathbb{C}^\times$ takes to ζ_i a meridian of K_i positively linked with K_i.

Let $F_1, \dots F_m \subset D^4$ be smooth oriented surfaces transversal to each other with $\partial F_i = F_i \cap \partial D^4 = K_i$. Extend the tubular neighborhood of L involved in the construction of $X(L)$ to a collection of tubular neighborhoods N_1, …, N_m of F_1, …, F_m, respectively.

Without loss of generality we may choose N_i in such a way that they would intersect each other in the simplest way. Namely, each connected component B of $N_i \cap N_j$ would contain only one point of $F_i \cap F_j$ and no point of others F_k and would consist of entire fibers of N_i and N_j, so that the fibers define a structure of bi-disk $D^2 \times D^2$ on B.

To achieve this, one has to make the fibers of the tubular fibration $N_i \to F_i$ at each intersection point of F_i and F_j coinciding with a disk in F_j and then diminish all N_i appropriately.

Now let us extend $X(L)$ to $X(F) = D^4 \smallsetminus \cup_{i=1}^m \operatorname{Int} N_i$. This is a compact 4-manifold. Its boundary contains $X(L)$, the rest of it is a union of pieces of boundaries of N_i with $i = 1, \dots, m$. These pieces are fibered over the corresponding pieces of F_i with fiber S^1.

By the Alexander duality, the orientation of F_i gives rise to a homomorphism $H_1(X(F)) \to \mathbb{Z}$ that maps a homology class to its linking number with F_i. These homomorphisms altogether determine a homomorphism $H_1(X(F)) \to \mathbb{Z}^m$. For any $\zeta = (\zeta_1, \dots, \zeta_m)$, the composition of this homomorphism with the homomorphism

$$\mathbb{Z}^n \to (\mathbb{C}^\times)^m : (n_1, \dots, n_m) \to (\zeta_1^{n_1}, \dots, \zeta_m^{n_m})$$

is a homomorphism $H_1(X(F)) \to (\mathbb{C}^\times)^m$ extending μ. If each F_i has no closed connected components, then this extension is unique. Let us denote it by $\overline{\mu}$.

According to Appendix D.6, in $H_2(X(F); \mathbb{C}_{\overline{\mu}})$ there is a Hermitian intersection form. Denote its signature by $\sigma_\zeta(L)$.

Theorem 2.A. *$\sigma_\zeta(L)$ does not depend on $F_1, \dots, F_m$.*

Proof. Any F_i' with $\partial F_i' = F_i' \cap \partial D^4 = K_i$ is cobordant to F_i. The cobordisms $W_i \subset D^4 \times I$ can be made pairwise transversal. They define a cobordism $D^4 \times I \smallsetminus \cup_i \operatorname{Int} N(W_i)$ between $X(F)$ and $X(F')$. By Theorem *D.B*,

$$\sigma_\zeta(\partial D^4 \times I \smallsetminus \cup_i \operatorname{Int} N(W_i)) = 0.$$

The manifold $\partial D^4 \times I \smallsetminus \cup_i \operatorname{Int} N(W_i)$ is the union of $X(F)$, $-X(F')$ and a *homologically negligible* part $\partial(N(\cup_i \operatorname{Int} W_i))$, the boundary of a regular neighborhood of the cobordism $\cup_i W_i$ between $\cup_i F_i$ and $\cup_i F_i'$. By Theorem *D.A*,

$$\sigma_\zeta(\partial D^4 \times I \smallsetminus \cup_i \operatorname{Int} N(W_i)) = \sigma_\zeta(D^4 \smallsetminus \cup_i F_i) - \sigma_\zeta(D^4 \smallsetminus \cup_i F_i')$$

Hence, $\sigma_\zeta(D^4 \smallsetminus \cup_i F_i) = \sigma_\zeta(D^4 \smallsetminus \cup_i F_i')$. □

2.4. Colored links. In the definition of signature $\sigma_\zeta(L)$ above one needs to numerate the components K_i of L to associate to each of them the corresponding component ζ_i of ζ, but there is no need to require connectedness of each K_i. This leads to a notion of colored link.

An *m-colored link* L is an oriented link in S^3 together with a map (called *coloring*) assigning to each connected component of L a color in $\{1, \dots, m\}$. The sublink L_i is constituted by the components of L with color i for $i = 1, \dots, m$.

For an m-colored link $L = L_1 \cup \dots \cup L_m$ and $\zeta = (\zeta_1, \dots, \zeta_m) \in (S^1)^m$, the signature $\sigma_\zeta(L)$ is defined as above, but each component K_j colored with color i is associated to ζ_i.

2.5. Relations to other link signatures. If $\zeta_i = -1$ for all $i = 1, \dots, m$, then the signature $\sigma_\zeta(L)$ coincides with the Murasugi signature $\xi(L)$ introduced in [11]. If all ζ_i are roots of unity of a degree, which is a power of a prime number and all linking numbers $\operatorname{lk}(L_i, L_j)$ vanish, then $\sigma_\zeta(L)$ coincides with the signature defined by Florens [2].

In the most general case, $\sigma_\zeta(L)$ coincides with the signature defined for arbitrary ζ by Cimasoni and Florens [1] using a 3-dimensional approach, with a version of Seifert surface, C-complex.

3. In higher dimensions.

3.1. Apology for the generalization of higher dimensional links. There is a spectrum of objects considered as generalizations of classical knots and links. The closest generalization of classical knots are pairs (S^n, K), where K is a smooth submanifold diffeomorphic to S^{n-2}. Then the requirements on K are weakened. Say, one may require K to be only homeomorphic to S^{n-2}, not diffeomorphic. Or just a homology sphere of dimension $n - 2$. The codimension is important in order to keep any resemblance to classical knots.

In the same spirit, for the closest higher-dimensional counter-part of classical links one takes a pair consisting of S^n and a collection of its disjoint smooth submanifolds diffeomorphic to S^{n-2}. One allows to weaken the restrictions on the submanifolds. Up to arbitrary closed submanifolds.

I suggest to allow transversal intersections of the submanifolds.

Of course, the main excuse for this is that some results can extended to this setup. Here is a couple of other reasons.

First, in the classical dimension, it is easy for submanifolds to be disjoint. Generically curves in 3-sphere are disjoint. If they intersect, it is a miracle or, rather, has a special cause.

Generic submanifolds of codimension two in a manifold of dimension > 3 intersect. If they do *not* intersect, this is a miracle, or consequence of a special cause.

Second, classical links emerge naturally as links of singular points of complex algebraic curves in $\mathbb{C}^2$. Recall that for an algebraic curve $C \subset \mathbb{C}^2$ and a point $p \in C$ the boundary of a sufficiently small ball B centered at p, the link $(\partial B, \partial B \cap C)$ is well-defined up to diffeomorphism, and it is called the *link of C at p*.

An obvious generalization of this definition to an algebraic hypersurface $C \subset \mathbb{C}^n$ gives rise to a pair (S^{2n-1}, K) with connected K. It cannot be a union of *disjoint* submanifolds of S^{2n-1}.

It would not be difficult to extend the results of this paper to a more general setup. For example, one can replace the ambient sphere with a homology sphere, or even more general manifold. However, one should stop somewhere. The author prefers this early point, because the level of generality accepted here suffices for demonstrating the new opportunities open by a systematic usage of twisted homology. On the other hand, further generalizations can make formulations more cumbersome.

3.2. Colored links. By an *m-colored link of dimension n* we shall mean a collection of m oriented smooth closed n-dimensional submanifolds $L_1, \dots, L_m$ of the sphere S^{n+2} such that any sub-collection has transversal intersection. The latter means that for any $x \in L_{i_1} \cap \dots \cap L_{i_k}$ the tangent spaces $T_x L_{i_1}, \dots, T_x L_{i_k}$ are transverse, that is, $\dim(T_x L_{i_1} \cap \dots \cap T_x L_{i_k}) = n + 2 - 2k$.

3.3. Generic configurations of submanifolds. More generally, an m-colored configuration of transversal submanifolds in a smooth manifold M is a family of m smooth submanifolds $L_1, \dots, L_m$ of M such that any sub-collection has transversal intersection. If M has a boundary, the submanifolds are assumed to be transversal to the boundary, as well as the intersection of any sub-collection. Furthermore, assume that $\partial M \cap L_i = \partial L_i$ for any $i = 1, \dots, m$.

As above, in Section 2.3, for any m-colored configuration L of transversal submanifolds $L_1, \dots, L_m$ in M one can find a collection of their tubular neighborhoods $N_1, \dots, N_m$ which agree with each other in the sense that for any sub-collection $L_{i_1}, \dots, L_{i_\nu}$ the intersection of the corresponding neighborhoods $N_{i_1} \cap \dots \cap N_{i_\nu}$ is neighborhood of the intersection $L_{i_1} \cap \dots \cap L_{i_\nu}$ fibered over this intersection with the corresponding poly-disk fiber.

Denote the complement $M \smallsetminus \cup_{i=1}^m \operatorname{Int} N_i$ by $X(L)$ and call it an *exterior* of L. This is a smooth manifold with a system of corners on the boundary. The differential type of the exterior does not depend on the choice of neighborhoods. Moreover, one can eliminate the choice of neighborhoods and deleting of them from the definition. Instead, one can make a sort of real blowing up of M along $L_1, \dots, L_m$. However, for the purposes of this paper it is easier to stay with the choices.

3.4. Link signatures. Let $L = L_1 \cup \dots \cup L_m$ be an m-colored link of dimension $2n-1$ in S^{2n+1}.

As well known (see, e.g., [7]), for each oriented closed codimension 2 submanifold K of S^{2n+1} there exists an oriented smooth compact submanifold F of D^{2n+2} such that $\partial F = K$. Choose for each L_i such a submanifold of D^{2n+2}, denote it by F_i, and make all the F_i transversal to each other by small perturbations.

As a union of m-colored transversal submanifolds of D^{2n+2}, $F = F_1 \cup \dots \cup F_m$ has an exterior $X(F)$. By the Alexander duality, $H^1(X(F); \mathbb{C}^\times)$ is naturally isomorphic to $H_{2n}(F, L; \mathbb{C}^\times)$. Let $\zeta = (\zeta_1, \dots, \zeta_m) \in (S^1)^m$. Take $\sum_{i=1}^m \zeta_i [F_i] \in H_{2n}(F, L; \mathbb{C}^\times)$ and denote by μ the Alexander dual cohomology class considered as a homomorphism $H_1(X(F)) \to \mathbb{C}^\times$. Denote by $\mathbb{C}_\mu$ the local coefficient system on $X(F)$ corresponding to μ.

According to Appendix D.6, in $H_{n+1}(X(F); \mathbb{C}_\mu)$ there is an intersection form, which is Hermitian, if n is odd, and skew-Hermitian, if n is even. Denote its signature by $\sigma_\zeta(L)$.

Theorem 3.A. *$\sigma_\zeta(L)$ does not depend on $F_1, \dots, F_m$.*

Proof. Any F'_i with $\partial F'_i = F'_i \cap \partial D^{2n+2} = L_i$ is cobordant to F_i. The cobordisms $W_i \subset D^{2n+2} \times I$ can be made pairwise transversal to form m-colored configuration W of transversal submanifolds of $D^{2n+2} \times I$. They define a cobordism $X(W)$ between $X(F)$ and $X(F')$. By Theorem *D.B*,

$$\sigma_\zeta(\partial X(W)) = 0.$$

The manifold $\partial X(W) = \partial D^{2n+2} \times I \smallsetminus \cup_i \operatorname{Int} N(W_i)$ is the union of $X(F)$, $-X(F')$ and a *homologically negligible* part $\partial(N(\cup_i \operatorname{Int} W_i))$, the boundary of a regular neighborhood of the cobordism $\cup_i W_i$ between F and F'. By Theorem *D.A*,

$$\sigma_\zeta(\partial X(W)) = \sigma_\zeta(X(F)) - \sigma_\zeta(X(F'))$$

Hence, $\sigma_\zeta(X(F)) = \sigma_\zeta(X(F'))$. □

4. Span inequalities. Let $L = L_1 \cup \dots, \cup L_m$ be an m-colored link of dimension $2n-1$ in S^{2n+1}. Let $F = F_1 \cup \dots \cup F_m$ be an m-colored configuration of transversal oriented compact $2n$-dimensional submanifolds of D^{2n+2} with $\partial F_i = F_i \cap \partial D^{2n+2} = L_i$. In this section we consider restrictions on homological characteristics of F in terms of invariants of L.

4.1. History. The first restrictions of this sort were found by Murasugi [10] and Tristram [18] for classical (1-colored) links. To m-colored classical links and pairwise disjoint surfaces F_i the Murasugi-Tristram inequalities were generalized by Florens [2]. A further generalization to m-colored classical links and intersecting F_i was found by Cimasoni and Florens [1]. Higher dimensional generalizations for 1-colored links were found by the author [21], [22].

4.2. No-nullity span inequalities. The most general results in this direction are quite cumbersome. Therefore, let me start with weak but simple ones.

Recall that $\sigma_\zeta(L)$ can be obtained from F: for an appropriate local coefficient system $\mathbb{C}_\mu$ on $X(F)$, this is the signature of a Hermitian intersection form defined in $H_{n+1}(X(F); \mathbb{C}_\mu)$. The signature of an Hermitian form cannot be greater than the dimension of the underlying space. In particular,

$$|\sigma_\zeta(L)| \le \dim_{\mathbb{C}} H_{n+1}(X(F); \mathbb{C}_\mu). \tag{1}$$

This can be considered as a restriction on a homological characteristic of F in terms of invariants of L. However, $\dim_{\mathbb{C}} H_{n+1}(X(F); \mathbb{C}_\mu)$ is not a convenient characteristic of F. It can be estimated in terms of more convenient ones.

Let $\zeta = (\zeta_1, \dots, \zeta_m) \in (S^1)^m$. Let $p_1, \dots, p_k \in \mathbb{Z}[t_1, t_1^{-1} \dots, t_m, t_m^{-1}]$ be generators of the ideal of relations satisfied by complex numbers ζ_i. Let d be the greatest common divisor of the integers $p_1(1, \dots, 1)$, $\dots$, $p_k(1, \dots, 1)$, if at least one of these integers does not vanish, and zero otherwise. Cf. Appendix C.6. Let

$$P = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } d > 1 \text{ and } p \text{ is a prime divisor of } d \\ \mathbb{Q}, & \text{if } d = 0 \end{cases}$$

By $C.C$,

$$\dim_{\mathbb{C}} H_{n+1}(X(F); \mathbb{C}_\mu) \le \dim_P H_{n+1}(X(F); P).$$

The advantage of passing to homology with non-twisted coefficients is that we can use the Alexander duality:

$$\begin{aligned} H_{n+1}(X(F); P) = H_{n+1}(D^{2n+2} \smallsetminus F; P) \\ = H^{n+1}(D^{2n+2}, \partial D^{2n+2} \cup F; P) \\ = H^n(\partial D^{2n+2} \cup F; P) = H^n(F, L; P). \end{aligned}$$

Hence,

$$|\sigma_\zeta(L)| \le \dim_P H_n(F, L; P).$$

4.3. General span inequalities. The inequality (1) can be improved. Indeed, the manifold $X(F)$ has a non-empty boundary. Therefore, its intersection form may be degenerate and the right hand side of (1) may be replaced by a smaller quantity, the rank of the form. The rank is known to be the rank of the homomorphism $H_{n+1}(X(F); \mathbb{C}_\mu) \to H_{n+1}(X(F), \partial X(F); \mathbb{C}_\mu)$. Let us estimate this rank.

Lemma 4.A. *For any exact sequence* $\dots \overset{\rho_{k+1}}{\to} C_k \overset{\rho_k}{\to} C_{k-1} \overset{\rho_{k-1}}{\to} \dots$ *of vector spaces and any integers n and r*

$$\operatorname{rk}(\rho_{n+1}) + \operatorname{rk}(\rho_{n-2r}) = \sum_{s=0}^{2r} (-1)^s \dim C_{n-s} \tag{2}$$

Proof. The Euler characteristic of the exact sequence

$$0 \to \operatorname{Im} \rho_{n+1} \hookrightarrow C_n \overset{\rho_n}{\to} C_{n-1} \to \dots \overset{\rho_{n-2r+1}}{\to} C_{n-2r} \to \operatorname{Im} \rho_{n-2r} \to 0$$

is the difference between the left and right hand sides of (2). On the other hand, it vanishes, as the Euler characteristic of an exact sequence. $\square$

Lemma 4.B. *Let X be a topological space, A its subspace, ξ a local coefficient system on X with fiber $\mathbb{C}$. Then for any natural n and $r \le \frac{n}{2}$*

$$\begin{aligned} &\operatorname{rk}(H_{n+1}(X; \xi) \to H_{n+1}(X, A; \xi)) + \operatorname{rk}(H_{n-2r}(X; \xi) \to H_{n-2r}(X, A; \xi)) \\ &= \sum_{s=0}^{2r} (-1)^s b_{n+1-s}(X, A) - \sum_{s=0}^{2r} (-1)^s b_{n-s}(A) + \sum_{s=0}^{2r} (-1)^s b_{n-s}(X) \end{aligned} \tag{3}$$

where $b_k(*) = \dim_{\mathbb{C}} H_k(*; \xi)$

Proof. Apply Lemma *4.A* to the homology sequence of pair (X, A) with coefficients in ξ. $\square$

Theorem 4.C. *For any integer r with $0 \le r \le \frac{n}{2}$*

$$(4)\quad |\sigma_\zeta(L)| + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n-s}(S^{2n+1} \smallsetminus L;\mathbb{C}_\zeta) \le \sum_{s=0}^{2r}(-1)^s \dim H_{n+1+s}(F,L;P) + \sum_{s=0}^{2r}(-1)^s \dim H_{n+s}(F;P)$$

$$(5)\quad |\sigma_\zeta(L)| + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1+s}(S^{2n+1} \smallsetminus L;\mathbb{C}_\zeta) \le \sum_{s=0}^{2r}(-1)^s \dim H_{n-s}(F,L;P) + \sum_{s=0}^{2r}(-1)^s \dim H_{n-s-1}(F;P)$$

where ζ and P are is in Section 4.2

Proof. As mentioned above,

$$(6)\qquad |\sigma_\zeta(L)| \le \operatorname{rk}(H_{n+1}(X(F);\mathbb{C}_\mu) \to H_{n+1}(X(F),\partial X(F);\mathbb{C}_\mu)).$$

By Lemma $4.B$,

$$(7)\quad \operatorname{rk}(H_{n+1}(X(F);\mathbb{C}_\mu) \to H_{n+1}(X(F),\partial X(F);\mathbb{C}_\mu)) \le \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1-s}(X(F),X(L);\mathbb{C}_\zeta) - \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n-s}(X(L);\mathbb{C}_\zeta) + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n-s}(X(F);\mathbb{C}_\zeta).$$

Summing up these inequalities and moving one of the sums from the right hand side to the left, we obtain:

$$(8)\quad |\sigma_\zeta(L)| + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n-s}(X(L);\mathbb{C}_\zeta) \le \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1-s}(X(F),X(L);\mathbb{C}_\zeta) + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n-s}(X(F);\mathbb{C}_\zeta).$$

The left hand sum of (8) coincides with the left hand side of (4). The right hand side can be estimated using Theorem $C.C$:

$$(9)\quad \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1-s}(X(F),X(L);\mathbb{C}_\zeta) + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n-s}(X(F);\mathbb{C}_\zeta) \le \sum_{s=0}^{2r}(-1)^s \dim_P H_{n+1-s}(X(F),X(L);P) + \sum_{s=0}^{2r}(-1)^s \dim_P H_{n-s}(X(F);P).$$

Further,

$$H_{n+1-s}(X(F),X(L);P) = H_{n+1-s}(D^{2n+2} \smallsetminus F, S^{2n+1} \smallsetminus L;P).$$

By the Alexander duality,

$$H_{n+1-s}(D^{2n+2} \smallsetminus F, S^{2n+1} \smallsetminus L; P) = H^{n+1+s}(D^{2n+2}, F; P).$$

By exactness of the pair sequence, $H^{n+1+s}(D^{2n+2}, F; P) = H^{n+s}(F; P)$.

Similarly,

$$\begin{aligned} H_{n-s}(X(F); P) = H_{n-s}(D^{2n+2} \smallsetminus F; P) \\ = H^{n+2+s}(D^{2n+2}, F \cup S^{2n+1}; P) \\ = H^{n+1+s}(S^{2n+1} \cup F; P) = H^{n+1+s}(F, L; P) \end{aligned}$$

The last equality in this sequence holds true if $n + 1 + s < 2n + 1$, that is, $s < n$.

Since P is a field,

$$\dim_P H^{n+s}(F; P) = \dim_P H_{n+s}(F; P), \tag{10}$$

$$\dim_P H^{n+1+s}(F, L; P) = \dim_P H_{n+1+s}(F, L; P). \tag{11}$$

Combining formulas (10), (11) with the calculations above and equalities (9) and (8), we obtain the first desired inequalities (4).

The inequalities (5) are proved similarly. Namely, by Lemma *4.B*

$$\begin{aligned} &\operatorname{rk}(H_{n+1}(X(F); \mathbb{C}_\mu) \to H_{n+1}(X(F), \partial X(F); \mathbb{C}_\mu)) \\ &\le \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+2+s}(X(F), X(L); \mathbb{C}_\zeta) - \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1+s}(X(L); \mathbb{C}_\zeta) \\ &\qquad + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1+s}(X(F); \mathbb{C}_\zeta). \end{aligned} \tag{12}$$

Summing up inequalities (6) and (12) and moving one of the sums from the right hand side to the left, we obtain:

$$\begin{aligned} &|\sigma_\zeta(L)| + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1+s}(X(L); \mathbb{C}_\zeta) \\ &\le \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+2+s}(X(F), X(L); \mathbb{C}_\zeta) + \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1+s}(X(F); \mathbb{C}_\zeta). \end{aligned} \tag{13}$$

After this the same estimates and transformations as in the proof of (4) gives rise to (5). □

4.4. Nullities. The sum in the left hand side of the inequalities (4) is an invariant of the link L. Its special case for classical links with $r = 0$ is known as ζ-nullity and appeared in the Murasugi-Tristram inequalities and their generalizations.

Denote $\sum_{s=0}^{2r}(-1)^s \dim H_{n-s}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu)$ by $n^r_\zeta(L)$ and call it *rth ζ-nullity of L*.

By the Poincaré duality (see Appendix D.3), $H_{n-s}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu)$ is isomorphic to $H^{n+1+s}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu)$. The latter vector space is dual to $H_{n+1+s}(S^{2n+1} \smallsetminus L; \mathbb{C}_{\mu^{-1}})$ and anti-isomorphic to $H_{n+1+s}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu)$, see Appendix D.5. Therefore,

$$n^r_\zeta(L) = \sum_{s=0}^{2r}(-1)^s \dim_{\mathbb{C}} H_{n+1+s}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \tag{14}$$

and $n^r_\zeta(L) = n^r_{\bar\zeta}(L)$. This sum is a part of the left hand side of (5).

Now we can rewrite Theorem *4.C* as follows:

Theorem 4.D. *For any integer r with $0 \le 2r \le n$*

$$|\sigma_\zeta(L)| + n^r_\zeta(L) \le \sum_{s=0}^{2r}(-1)^s \dim H_{n+s+1}(F, L; P) + \sum_{s=0}^{2r}(-1)^s \dim H_{n+s}(F; P) \tag{15}$$

$$|\sigma_\zeta(L)| + n^r_\zeta(L) \le \sum_{s=0}^{2r}(-1)^s \dim H_{n-s}(F, L; P) + \sum_{s=0}^{2r}(-1)^s \dim H_{n-s-1}(F; P) \tag{16}$$

If F_i are pairwise disjoint, than the right hand sides of (15) and (16) are equal due to Poincaré-Lefschetz duality for F, but we do not assume that $F = \cup F_i$ is a manifold, and therefore the inequalities (15) and (16) are not equivalent and we have to keep both of them.

5. Slice inequalities.

Again, as in the preceding section, let $L_1, \dots, L_m \subset S^{2n+1}$ be smooth oriented transversal to each other submanifolds constituting an m-colored link $L = L_1 \cup \dots \cup L_m$ of dimension $2n-1$.

Let $\Lambda_i \subset S^{2n+2}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n+1}, with $\partial\Lambda_i \cap S^{2n+1} = L_i$. In this section we consider restrictions on homological characteristics of $\Lambda = \cup_{i=1}^m \Lambda_i$ in terms of invariants of link L. Of course, some results of this kind can be deduced from the results of the preceding section, but an independent consideration gives better results.

5.1. No-nullity slice inequalities.

The most general results in this direction are quite cumbersome. Therefore, let me start with weak but simple ones.

We will use the same algebraic objects as in the preceding section. In particular, $\zeta = (\zeta_1, \dots, \zeta_m) \in (S^1)^m$, $p_1, \dots, p_k \in \mathbb{Z}[t_1, t_1^{-1} \dots, t_m, t_m^{-1}]$ are generators of the ideal of relations satisfied by complex numbers ζ_i. Integer d is the greatest common divisor of the integers $p_1(1, \dots, 1)$, ..., $p_k(1, \dots, 1)$, if at least one of them does not vanish, and $d = 0$ otherwise. Cf. 4.2 and Appendix C.6. Finally,

$$P = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } d > 1 \text{ and } p \text{ is a prime divisor of } d \\ \mathbb{Q}, & \text{if } d = 0 \end{cases}$$

Let $\mu : H_1(S^{2n+1} \smallsetminus L) \to \mathbb{C}^\times$ be the homomorphism which maps the meridian of L_i to ζ_i. The local coefficient system $\mathbb{C}_\mu$ on $S^{2n+1} \smallsetminus L$ defined by μ extends to $S^{2n+2} \smallsetminus \Lambda$. We will denote the extension by the same symbol $\mathbb{C}_\mu$.

The sphere S^{2n+1} bounds in S^{2n+2} two balls, hemi-spheres S^{2n+2}_+ and S^{2n+2}_- such that $\partial S^{2n+2}_+ = S^{2n+1}$ and $\partial S^{2n+2}_- = -S^{2n+1}$ with the orientations inherited from the standard orientation of S^{2n+2}. In $H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$ there is a (Hermitian or skew-Hermitian) intersection form. Its signature is zero by Theorem *D.B*, because Λ bounds a configuration of pairwise transversal submanifolds $\Delta = \Delta_1 \cup \dots \cup \Delta_m$ in D^{2n+3} and $\mathbb{C}_\mu$ extends over $D^{2n+3} \smallsetminus \Delta$.

Theorem 5.A. *Under the assumption above,*

$$2|\sigma_\zeta(L)| \le \dim_P H_n(\Lambda; P). \tag{17}$$

Proof. The intersection form on $H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$ restricted to the images of $H_{n+1}(S^{2n+2}_+ \smallsetminus \Lambda; \mathbb{C}_\mu)$ and $H_{n+1}(S^{2n+2}_- \smallsetminus \Lambda; \mathbb{C}_\mu)$ has signatures $\sigma_\zeta(L)$ and $-\sigma_\zeta(L)$, respectively. Therefore the dimension of each of the images is at least $|\sigma_\zeta(L)|$.

The images are obviously orthogonal to each other with respect to the intersection form, because their elements can be realized by cycles lying in disjoin open hemi-spheres. Hence

$$2|\sigma_\zeta(L)| \le \dim_{\mathbb{C}} H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu).$$

On the other hand, by Theorem *C.C*,

$$\dim_{\mathbb{C}} H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu) \le \dim_P H_{n+1}(S^{2n+2} \smallsetminus \Lambda; P) = \dim_P H_n(\Lambda; P).$$

Summing up these two inequalities, we obtain the desired one. □

5.1.1. *General slice inequalities.*

Theorem 5.B. *Under assumptions above*

$$\begin{aligned}(18)\quad & 2|\sigma_\zeta(L)| + 2n^r_\zeta(L) \\ & \le \sum_{s=0}^{2r} (-1)^s \dim_P H_{n-s}(\Lambda \smallsetminus L; P) + \sum_{s=-2r+1}^{2r-1} (-1)^s \dim_P H_{n-s}(\Lambda; P)\end{aligned}$$

Lemma 5.C. *Let j be the inclusion $S^{2n+1} \smallsetminus L \to S^{2n+2} \smallsetminus \Lambda$. Then*

$$\begin{aligned}(19)\quad & 2|\sigma_\zeta(L)| + 2\,\mathrm{rk}(j_* : H_{n+1}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \to H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)) \\ & \le \dim_{\mathbb{C}} H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)\end{aligned}$$

Proof. Denote by $i^\pm$ the inclusion $S^{2n+2}_\pm \smallsetminus \Lambda \to S^{2n+2} \smallsetminus \Lambda$. Observe that the space $H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$ has a natural filtration:

$$\begin{aligned}(20)\quad & j_* H_{n+1}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \\ & \subset i^+_* H_{n+1}(S^{2n+2}_+ \smallsetminus \Lambda; \mathbb{C}_\mu) + i^-_* H_{n+1}(S^{2n+2}_- \smallsetminus \Lambda; \mathbb{C}_\mu) \\ & \subset H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)\end{aligned}$$

The inclusion homomorphisms

$$j_* : H_{n+1}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \to H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$$

and the boundary homomorphism

$$\partial : H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu) \to H_n(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu)$$

of the Mayer-Vietoris sequence of the triad $(S^{2n+2} \smallsetminus \Lambda; S^{2n+2}_+ \smallsetminus \Lambda, S^{2n+2}_- \smallsetminus \Lambda)$ are dual to each other with respect to the intersection forms:

$$j_*(a) \circ b = a \circ \partial(b) \ \text{ for any } a \in H_{n+1}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \text{ and } b \in H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu).$$

Since the intersection forms are non-singular, it follows that $\mathrm{rk}\, j_* = \mathrm{rk}\, \partial$.

By exactness of the Mayer-Vietoris sequence, the rank of ∂ is the dimensions of the top quotient of the filtration (20), while the rank of j_* is the dimension of the smallest term $j_* H_{n+1}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu)$ of this filtration.

The middle term of the filtration contains the subspaces $i^+_* H_{n+1}(S^{2n+2}_+ \smallsetminus \Lambda; \mathbb{C}_\mu)$ and $i^-_* H_{n+1}(S^{2n+2}_- \smallsetminus \Lambda; \mathbb{C}_\mu)$. Their intersection is the smallest term, which is orthogonal to both of the subspaces. Therefore the dimension of the quotient of the middle term of the filtration by the smallest term is at least $2|\sigma_\zeta(L)|$

The dimension of the whole space $H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$ is the sum of the dimensions of the factors. We showed above that the top and lowest factor have the same dimensions equal to $\operatorname{rk} j_*$ and that the dimension of the middle factor is at least $2|\sigma_\zeta(L)|$. $\square$

Lemma 5.D. *For any exact sequence* $\dots \overset{\rho_{k+1}}{\to} C_k \overset{\rho_k}{\to} C_{k-1} \overset{\rho_{k-1}}{\to} \dots$ *of vector spaces and any integers n and t*

$$\operatorname{rk}(\rho_n) - \operatorname{rk}(\rho_{n+2t}) = \sum_{s=0}^{2t-1} (-1)^s \dim C_{n+s} \tag{21}$$

Proof. The Euler characteristic of the exact sequence

$$0 \to \operatorname{Im} \rho_{n+2t} \hookrightarrow C_{n+2t-1} \overset{\rho_{n+2t-1}}{\to} C_{n+2t-2} \to \dots \overset{\rho_{n+1}}{\to} C_n \to \operatorname{Im} \rho_n \to 0$$

is $\operatorname{rk}(\rho_n) - \sum_{s=0}^{2t-1} (-1)^s \dim C_{n+s} - \operatorname{rk}(\rho_n + 2t)$, that is the difference between the left and right hand sides of (21). On the other hand, it vanishes, as the Euler characteristic of an exact sequence. $\square$

Lemma 5.E. *Let X be a topological space, A its subspace, ξ a local coefficient system on X with fiber $\mathbb{C}$. Then for any natural n and integer r*

$$\operatorname{rk}(H_{n+1}(A;\xi) \to H_{n+1}(X;\xi)) - \operatorname{rk}(H_{n+2+2r}(X;\xi) \to H_{n+2+2r}(X,A;\xi)) \tag{22}$$

$$= \sum_{s=0}^{2r} (-1)^s b_{n+1+s}(A) - \sum_{s=0}^{2r} (-1)^s b_{n+2+s}(X,A) + \sum_{s=0}^{2r-1} (-1)^s b_{n+2+s}(X)$$

where $b_k(*) = \dim_{\mathbb{C}} H_k(*;\xi)$.

Proof. Apply Lemma *5.D* to the homology sequence of pair (X, A) with coefficients in ξ. $\square$

Lemma 5.F. *For any integer r with $0 \le r \le \frac{n}{2}$*

$$\begin{aligned} 2|\sigma_\zeta(L)| + 2n^r_\zeta(L) &\\ \le 2 \sum_{s=0}^{2r} (-1)^s \dim_{\mathbb{C}} H_{n+2+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) &\\ + \sum_{s=-2r+1}^{2r-1} (-1)^s \dim_{\mathbb{C}} H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu) & \end{aligned} \tag{23}$$

Proof. By Lemma *5.E* applied to the pair $(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L)$, we obtain

$$\begin{aligned}(24)\quad & \operatorname{rk}(j_* : H_{n+1}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \to H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)) \\ & \geq \sum_{s=0}^{2r} (-1)^s H_{n+1+s}(S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \\ & - \sum_{s=0}^{2r} (-1)^s \dim_{\mathbb{C}} H_{n+2+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \\ & + \sum_{s=0}^{2r-1} (-1)^s \dim_{\mathbb{C}} H_{n+2+s}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)\end{aligned}$$

From this inequality and inequality (19) we obtain

$$\begin{aligned}(25)\quad & 2|\sigma_\zeta(L)| + 2n^r_\zeta(L) \\ & \leq 2 \sum_{s=0}^{2r} (-1)^s \dim_{\mathbb{C}} H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \\ & - 2 \sum_{s=0}^{2r-1} (-1)^s \dim_{\mathbb{C}} H_{n+s+2}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu) \\ & + \dim_{\mathbb{C}} H_{n+1}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)\end{aligned}$$

From this and the Alexander duality (which states that $H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$ is isomorphic to $H_{n+1-s}(S^{2n+2} \smallsetminus \Lambda; \mathbb{C}_\mu)$) the desired inequality follows. □

Lemma 5.G.

$$\begin{aligned}(26)\quad & \sum_{s=0}^{2r} (-1)^s \dim_{\mathbb{C}} H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \\ & \leq \sum_{s=0}^{2r} (-1)^s \dim_P H_{n-s}(\Lambda \smallsetminus L; P)\end{aligned}$$

Proof. By Theorem *C.C*

$$\begin{aligned}(27)\quad & \sum_{s=0}^{2r} (-1)^s \dim_{\mathbb{C}} H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; \mathbb{C}_\mu) \\ & \leq \sum_{s=0}^{2r} (-1)^s \dim_P H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; P).\end{aligned}$$

By Poincaré duality (cf. Appendix D.3), $H_{n+1+s}(S^{2n+2} \smallsetminus \Lambda, S^{2n+1} \smallsetminus L; P)$ is isomorphic to $H^{n+1-s}(S^{2n+2} \smallsetminus S^{2n+1}, \Lambda \smallsetminus L; P)$. The latter is isomorphic to $H^{n-s}(\Lambda \smallsetminus L; P)$. By the universal coefficients formula, $H^{n-s}(\Lambda \smallsetminus L; P)$ is isomorphic to $H_{n-s}(\Lambda \smallsetminus L; P)$. □

Lemma 5.H.

$$\sum_{s=-2r+1}^{2r-1}(-1)^s\dim_{\mathbb{C}} H_{n+1+s}(S^{2n+2}\smallsetminus\Lambda;\mathbb{C}_\mu) \le \sum_{s=-2r+1}^{2r-1}(-1)^s\dim_P H_{n-s}(\Lambda;P) \tag{28}$$

Proof. By Theorem *C.C*

$$\sum_{s=-2r+1}^{2r-1}(-1)^s\dim_{\mathbb{C}} H_{n+1+s}(S^{2n+2}\smallsetminus\Lambda;\mathbb{C}_\mu) \le \sum_{s=-2r+1}^{2r-1}(-1)^s\dim_P H_{n+1+s}(S^{2n+2}\smallsetminus\Lambda;P). \tag{29}$$

By Poincaré duality, $H_{n+1+s}(S^{2n+2}\smallsetminus\Lambda;P)$ is isomorphic to $H^{n+1-s}(S^{2n+2},\Lambda;P)$. From the sequence of pair (S^{2n+2},Λ) it follows that $H^{n+1-s}(S^{2n+2},\Lambda;P)$ is isomorphic to $H^{n-s}(\Lambda;P)$. By the universal coefficient formula, $H^{n-s}(\Lambda;P)$ is isomorphic to $H_{n-s}(\Lambda;P)$. □

Proof of Theorem *5.B*. Sum up the inequalities of the last three Lemmas. □

Appendix Appendix. Twisted homology.

Appendix A. Twisted coefficients and chains.

Appendix A.1. ***Local coefficient system.*** Let X be a topological space, and ξ be a $\mathbb{C}$-bundle over X with a fixed flat connection.

Here by a *connection* we mean operations of *parallel transport*: for any path s in X connecting points x and y the parallel transport T_s is an isomorphism from the fiber $\mathbb{C}_x$ over x to the fiber $\mathbb{C}_y$ over y, such that the parallel transport along product of paths equals the composition of parallel transports along the factors. In formula: $T_{uv}=T_v\circ T_u$. A connection is flat, if the parallel transport isomorphism does not change when the path is replaced by a homotopic path.

A flat connection in a bundle ξ over a simply connected X gives a trivialization of ξ.

Another name for ξ is a *local coefficient system* with fiber $\mathbb{C}$.

Appendix A.2. ***Monodromy representation.*** Recall that for a path-connected locally contractible X (and in more general situations, which would not be of interest here) it is defined by the *monodromy reprensentation* $\pi_1(X,x_0)\to\mathbb{C}^\times$, where $\mathbb{C}^\times=\mathbb{C}\smallsetminus 0$ is the multiplicative group of $\mathbb{C}$. The monodromy representation assigns to $\sigma\in\pi_1(X,x_0)$ a complex number ζ such that the parallel transport isomorphism along a loop which represents σ is multiplication by ζ.

Since $\mathbb{C}^\times$ is commutative, a homomorphism $\pi_1(X,x_0)\to\mathbb{C}^\times$ factors through the abelianization $\pi_1(X,x_0)\to H_1(X)$. Thus a local coefficient system with fiber $\mathbb{C}$ is defined also by a homology version $\mu: H_1(X)\to\mathbb{C}^\times$ of the monodromy representation, which can be considered also as a cohomology class belonging to $H^1(X;\mathbb{C}^\times)$.

The local coefficient system defined by a monodromy representation $\mu : H_1(X) \to \mathbb{C}^\times$ is denoted by $\mathbb{C}^\mu$. Sometimes instead of μ we will write data which defines μ, for example the images under μ of generators of $H_1(X)$ selected in a special way.

Appendix A.3. ***Twisted singular chains.*** Homology groups $H_n(X;\xi)$ of X with coefficients in ξ is a classical invariant studied in algebraic topology. It is an immediate generalization of $H_n(X;\mathbb{C})$. Hence it is quite often ignored in textbooks on homology theory, I recall the singular version of the definition.

Recall that a singular p-dimensional chain of X with coefficients in $\mathbb{C}$ is a formal finite linear combination of singular simplices $f_i : T^p \to X$ with complex coefficients.

A singular chain of X with coefficients in ξ is also a formal finite linear combination of singular simplices, but each singular simplex $f_i : T^p \to X$ appears in it with a coefficient taken from the fiber $\mathbb{C}_{f_i(c)}$ of ξ over $f_i(c)$, where c is the baricenter of T^p. Of course, all the fibers of ξ are isomorphic to $\mathbb{C}$. So, a chain with coefficients in ξ can be identified with a chain with coefficients in $\mathbb{C}$, provided the isomorphisms $\mathbb{C}_{f_i(c)} \to \mathbb{C}$ are selected. But they are not.

All singular p-chains of X with coefficients in ξ form a complex vector space $C_p(X;\xi)$.

The boundary of such a chain is defined by the usual formula, but one needs to bring the coefficient from the fiber over $f_i(c)$ to the fibers over $f_i(c_i)$, where c_i is the baricenter of the ith face of T^p. For this, one may use translation along the composition with f_i of any path connecting c to c_i in T^p: since T^p is simply connected and the connection of ξ is flat, the result does not depend on the path.

These chains and boundary operators form a complex. Its homology is called *homology with coefficients in* ξ and denoted by $H_p(X;\xi)$.

Homology with coefficients in the local coefficient system corresponding to the trivial monodromy representation $1 : H_1(X) \to \mathbb{C}^\times$ coincides with homology with coefficients in $\mathbb{C}$.

Appendix A.4. ***Twisted cellular chains.*** It is possible to calculate the homology with coefficients in a local coefficient system using cellular decomposition. Namely, a p-dimensional cellular chain of a cw-complex X with coefficients in a local coefficient system ξ is a formal finite linear combination of p-dimensional cells in which a coefficient at a cell belongs to the fiber over a point of the cell. It does not matter which point is this, because fibers over different points in a cell are identified via parallel transport along paths in the cell: any two points in a cell can be connected in the cell by a path unique up to homotopy.

In order to describe the boundary operator, let me define the *incidence number* $(z\sigma_x : \tau)_y \in \mathbb{C}_y$ where σ is a p-cell, τ is a $(p-1)$-cell, $z \in \mathbb{C}_x$, $x \in \sigma$, $y \in \tau$. The boundary operator is then defined by the incidence numbers:

$$\partial(z\sigma) = \sum_\tau (z\sigma_x : \tau)_y \tau.$$

Let $f : D^p \to X$ be a characteristic map for σ. Assume that a point y in $(p-1)$-cell τ is a regular value for f. This means that y has a neighborhood U in τ such that $f^{-1}(U) \subset S^{p-1} \subset D^p$ is the union of finitely many balls mapped by f homeomorphically onto U. Connect $f^{-1}(x) \in D^p$ with all the points of $f^{-1}(y)$ by straight paths. Compositions of these paths with f are paths $s_1, \dots s_N$ connecting

x with y. Then put

$$(z\sigma : \tau)_y = \sum_{i=1}^{N} \varepsilon_i T_{s_i}(z)$$

where T_{s_i} is a parallel transport operator and $\varepsilon_i = +1$ or -1 according to whether f preserves or reverses the orientation on the ith ball out of N balls constituting $f^{-1}(U)$.

Appendix B. Twisted acyclicity.

Appendix B.1. ***Acyclicity of circle.*** According to one of the most fundamental properties of homology, the dimension of $H_0(X;\mathbb{C})$ is equal to the number of path-connected components of X. In particular, $H_0(X;\mathbb{C})$ does not vanish, unless X is empty.

This is not the case for twisted homology. A crucial example is the circle S^1. Let $\mu : H_1(S^1) \to \mathbb{C}^\times$ maps the generator $1 \in \mathbb{Z} = H_1(S^1)$ to $\zeta \in \mathbb{C}^\times$.

Theorem B.A. Twisted acyclicity of circle. *$H_*(S^1;\mathbb{C}^\mu) = 0$, iff $\zeta \neq 1$.*

Proof. The simplest cw-decomposition of S^1 consists of two cells, one-dimensional σ_1 and zero-dimensional σ_0. One can easily see that $\partial\sigma_1 = (\zeta - 1)\sigma_0$. Hence $\partial : C_1(S^1;\mathbb{C}^\mu) \to C_0(S^1;\mathbb{C}^\mu)$ is an isomorphism, iff $\zeta \neq 0$. □

Appendix B.2. ***Vanishing of twisted homology.***

Corollary B.B. *Let X be a path connected space and $\mu : H_1(S^1 \times X) \to \mathbb{C}^\times$ be a homomorphism. Denote by ζ the image under μ of the homology class realized by a fiber $S^1 \times$ point. Then $H_*(S^1 \times X;\mathbb{C}^\mu) = 0$, if $\zeta \neq 0$.*

Proof. Since $H_1(S^1 \times X) = H_1(S^1) \times H_1(X)$, the homomorphism μ can be presented as product of homomorphisms $\mu_1 : H_1(S^1) \to \mathbb{C}^\times$ and $\mu_2 : H_1(X) \to \mathbb{C}^\times$ which can be obtained as compositions of μ with the inclusion homomorphisms. Thus $\mathbb{C}^\mu = \mathbb{C}^{\mu_1} \otimes \mathbb{C}^{\mu_2}$, and we can apply Künneth formula

$$H_n(S^1 \times X;\mathbb{C}^\mu) = \sum_{p=0}^{n} H_p(S^1;\mathbb{C}^{\mu_1}) \otimes H_{n-p}(X;\mathbb{C}^{\mu_2})$$

and refer to Theorem $B.A$. □

Corollary B.C. *Let B be a path connected space, $p : X \to B$ a locally trivial fibration with fiber S^1. Let $\mu : H_1(X) \to \mathbb{C}^\times$ be a homomorphism. Denote by ζ the image under μ of homology class realized by a fiber of p. Then $H_*(X;\mathbb{C}^\mu) = 0$, if $\zeta \neq 0$.*

Proof. It follows from Theorem $B.A$ via the spectral sequence of fibration p. □

Appendix C. Estimates of twisted homology.

Appendix C.1. ***Equalities underlying the Morse inequalities.***

Lemma C.A. *For a complex* $C : \dots \to C_i \xrightarrow{\partial_i} C_{i-1} \to$ *of finite dimensional vector spaces over a field* F

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C) = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F C_s - \operatorname{rk} \partial_{r-1} - \operatorname{rk} \partial_{2n+r}. \tag{C.30}$$

Proof. First, prove inequality (C.30) for $n = 0$. Since $H_s(C) = \operatorname{Ker} \partial_s / \operatorname{Im} \partial_{s+1}$, we have $\dim_F H_s(C) = \dim \operatorname{Ker} \partial_s - \dim_F \operatorname{Im} \partial_{s+1}$. Further, $\dim_F \operatorname{Im} \partial_{s+1} = \operatorname{rk} \partial_{s+1}$, and $\dim_F \operatorname{Ker} \partial_s = \dim_F C_s - \operatorname{rk} \partial_s$. It follows

$$\dim_F H_s(C) = \dim_F C_s - \operatorname{rk} \partial_s - \operatorname{rk} \partial s + 1 \tag{C.31}$$

This is a special case of (C.30) with $n = 0$, $r = s$.

The general case follows from it: make alternating summation of (C.31) for $s = r, \dots, 2n + s$. □

Appendix C.2. ***Algebraic Morse type inequalities.***

Lemma C.B. *Let* P *and* Q *be fields,* R *be a subring of* Q *and let* $h : R \to P$ *be a ring homomorphism. Let* $C : \dots \to C_p \to C_{p-1} \to \dots \to C_1 \to C_0$ *be a complex of free finitely generated* R*-modules. Then for any* n *and* r

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_Q H_s(C \otimes_R Q) \le \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_P H_s(C \otimes_h P)$$

Thus, the greater ranks of differentials, the smaller

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_F H_s(C).$$

Proof. Choose free bases in modules C_i. Let M_i be the matrix representing $\partial_i : C_i \to C_{i-1}$ in these bases. The same matrix represents the differential ∂_i^Q of $C \otimes_R Q$. The matrix obtained from M_i by replacement the entries with their images under h represents the differential ∂_i^P of $C \otimes_h P$. The minors of the latter matrix are the images of the former one under h. Consequently, the $\operatorname{rk} \partial_i^Q \ge \operatorname{rk} \partial_i^P$.

By Lemma $C.A$

$$\sum_{s=r}^{2n+r} (-1)^{s-r} \dim_Q H_s(C \otimes_R Q) = \sum_{s=r}^{2n+r} (-1)^{s-r} \dim_Q C_s \otimes_R Q - \operatorname{rk} \partial_{r-1}^Q - \operatorname{rk} \partial_{r+2n}^Q \tag{C.32}$$

and

$$\text{(C.33)}\quad \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(C\otimes_h P)=$$

$$\sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P C_s\otimes_h P-\operatorname{rk}\partial^P_{r-1}-\operatorname{rk}\partial^P_{r+2n}$$

Compare the right hand sides of these equalities. The dimensions $\dim_P C_s\otimes_h P$, $\dim_Q C_s\otimes_R Q$ are equal to the rank of free R-module C_s. Since, as it was shown above, $\operatorname{rk}\partial^Q_i\ge\operatorname{rk}\partial^P_i$, the right hand side of (C.33) is smaller than the right hands side of (C.33). ☐

Probably, the simplest application of Lemma *C.B* gives well-known upper estimation of the Betti numbers with rational coefficients by the Betti numbers with coefficients in a finite field. It follows from the universal coefficients formula.

Appendix C.3. ***Application to twisted homology.***

Theorem C.C. *Let X be a finite cw-complex, and $\mu: H_1(X)\to\mathbb{C}^\times$ be a homomorphism. If $\operatorname{Im}\mu\subset\mathbb{C}^\times$ generates a subring R of $\mathbb{C}$ and there is a ring homomorphism $h: R\to Q$, where Q is a field, such that $h\mu(H_1(X))=1$, then we can apply Lemma C.B and get an upper estimation for dimensions of twisted homology groups in terms of dimensions of non-twisted ones.*

$$\text{(C.34)}\qquad \sum_{s=r}^{2n+r}(-1)^{s-r}\dim_Q H_s(X;\mathbb{C}^\mu)\le\sum_{s=r}^{2n+r}(-1)^{s-r}\dim_P H_s(X;P)$$

Here are several situations in which the assumptions of this theorem are fulfilled.

Appendix C.4. ***Estimates by untwisted $\mathbb{Z}/p\mathbb{Z}$ Betti numbers.*** Let $H_1(X)$ be generated by g and $\zeta=\mu(g)$ be an algebraic number. Assume that p is the minimal integer polynomial with relatively prime coefficients which annihilates ζ. Assume also that $g(1)$ is divisible by a prime number p. Then for R we can take $\mathbb{Q}[\zeta]\subset\mathbb{C}$, for P the field $\mathbb{Z}/p\mathbb{Z}$, and for h the ring homomorphism $\mathbb{Q}[\zeta]\to\mathbb{Z}/p\mathbb{Z}$ mapping $\zeta\mapsto 1$.

Here is a more general situation: Let $H_1(X)$ be generated by $g_1,\dots g_k$, and $\zeta_i=\mu(g_i)$ be an algebraic number for each i. Assume that p_i is the minimal integer polynomial with relatively prime coefficients which annihilates ζ_i. Assume also that the greatest common divisor of $g_1(1),\dots,\ g_k(1)$ is divisible by a prime number p. Then for R we can take $\mathbb{Q}[\zeta_1,\dots,\zeta_k]\subset\mathbb{C}$, for P the field $\mathbb{Z}/p\mathbb{Z}$, and for h the ring homomorphism $\mathbb{Q}[\zeta_1,\dots,\zeta_k]\to\mathbb{Z}/p\mathbb{Z}$ mapping $\zeta_i\mapsto 1$ for all i.

Appendix C.5. ***Estimates by rational Betti numbers.*** Let $H_1(X)$ be generated by g and $\zeta=\mu(g)$ be transcendent. Then for R we can take the ring $\mathbb{Z}[\zeta,\zeta^{-1}]$, for Q the field $\mathbb{Q}(\zeta)$, for P the field $\mathbb{Q}$, and for h the ring homomorphism $\mathbb{Z}[\zeta]\to\mathbb{Q}$ which maps ζ to 1.

Appendix C.6. ***The most general estimates.*** Let $H_1(X)$ be generated by $g_1,\dots g_k$ and $\zeta_i = \mu(g_i)$. Laurent polynomials with integer coefficients annihilated by $\zeta_1,\dots,\zeta_m$ form an ideal in the ring $\mathbb{Z}[t_1,t_1^{-1}\dots,t_m,t_m^{-1}]$. Let $p_1,\dots,p_k$ be generators of this ideal. Let d be the greatest common divisor of the integers $p_1(1,\dots,1)$, ..., $p_k(1,\dots,1)$, if at least one of them is not 0. Otherwise, let $d=0$

In other words, consider the specialization homomorphism

$$S:\mathbb{Z}[t_1,t_1^{-1}\dots,t_m,t_m^{-1}]\to\mathbb{C}:t_i\mapsto\zeta_i.$$

Let K be the kernel of S, and let d be the generator of the ideal which is the image of K under the homomorphism

$$\mathbb{Z}[t_1,t_1^{-1}\dots,t_m,t_m^{-1}]\to\mathbb{Z}:t_i\mapsto 1.$$

Then for R we can take the ring $\mathbb{Z}[\zeta_1,\zeta_1^{-1},\dots,\zeta_k,\zeta_k^{-1}]$. For Q we can take the quotient field of R, but since both Q and its quotient field are contained in $\mathbb{C}$, let us take $Q=\mathbb{C}$.

If $d>1$, then we can take for P the field $\mathbb{Z}/p\mathbb{Z}$ with any prime p which divides d. If $d=0$, then let $P=\mathbb{Q}$. The case $d=1$ is the most misfortunate: then our technique does not give any non-trivial estimate. For $d>1$ or $d=0$ we have the inequality (C.34).

Appendix D. Twisted duality.

Appendix D.1. ***Cochains and cohomology.*** Cochain groups $C^p(X;\xi)$ (which are vector spaces over $\mathbb{C}$) and cohomology $H^p(X;\xi)$ are defined similarly: p-cochain with coefficients in ξ is a function assigning to a singular simplex $f:T^p\to X$ an element of $\mathbb{C}_{f(c)}$, the fiber of ξ over $f(c)$.

This can be interpreted as the chain complex of the local coefficient system $\operatorname{Hom}(\mathbb{C},\xi)$ whose fiber over $x\in X$ is $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}_x)$. More generally, for any local coefficient systems ξ and η on X with fiber $\mathbb{C}$ there is a local coefficient system $\operatorname{Hom}(\xi,\eta)$ constructed fiber-wise with the parallel transport defined naturally in terms of the parallel transports of ξ and η. If the monodromy representations of ξ and η are μ and ν, respectively, then the monodromy representation of $\operatorname{Hom}(\xi,\eta)$ is $\mu^{-1}\nu:H_1(X)\to\mathbb{C}^\times:x\mapsto\mu^{-1}(x)\nu(x)$.

Similarly, for any local coefficient systems ξ and η on X with fiber $\mathbb{C}$ there is a local coefficient system $\xi\otimes\eta$. If $\mu,\nu:H^1(X)\to\mathbb{C}^\times$ are homomorphisms, then $\mathbb{C}^\mu\otimes\mathbb{C}^\nu$ is the local coefficient system $\mathbb{C}^{\mu\nu}$ corresponding to the homomorphism-product $\mu\nu:H^1(X)\to\mathbb{C}^\times:x\mapsto\mu(x)\nu(x)$.

If $\nu=\mu^{-1}$ (that is $\mu(x)\nu(x)=1$ for any $x\in H^1(X)$), then $\mathbb{C}^\mu\otimes\mathbb{C}^\nu$ is the non-twisted coefficient system with fiber $\mathbb{C}$.

In contradistinction to non-twisted case, there is no way to calculate $H_n(X;\xi\otimes\eta)$ in terms of $H_*(X;\xi)$ and $H_*(X;\eta)$. Indeed, both $H_*(S^1;\mathbb{C}^\mu)$ and $H_*(S^1;\mathbb{C}^{\mu^{-1}})$ vanish, unless $\mu:H_1(S^1)\to\mathbb{C}^\times$ is trivial, but $H_0(S^1;\mathbb{C}^\mu\otimes\mathbb{C}^{\mu^{-1}})=H_0(S^1;\mathbb{C})=\mathbb{C}$.

Appendix D.2. ***Multiplications.*** Usual definitions of various cohomological and homological multiplications are easily generalized to twisted homology. For this one needs a bilinear pairing of the coefficient systems. (Recall that in the case of non-twisted coefficient system a pairing of coefficient groups also is needed.) For local coefficient systems ξ, η and ζ with fiber $\mathbb{C}$ on X, a pairing $\xi \oplus \eta \to \zeta$ is a fiber-wise map which is bilinear over each point of X. Given such a pairing, there are pairings

$$\smile: H^p(X;\xi) \times H^q(X;\eta) \to H^{p+q}(X;\zeta),$$

$$\frown: H^{p+q}(X;\xi) \times H^q(X;\eta) \to H^p(X;\zeta),$$

etc.

A pairing $\xi \oplus \eta \to \zeta$ of local coefficients systems can be factored through the universal pairing $\xi \oplus \eta \to \xi \otimes \eta$.

Since $\mathbb{C}^\mu \otimes \mathbb{C}^{\mu^{-1}}$ is a non-twisted coefficient system with fiber $\mathbb{C}$, this gives rise to a non-singular pairing

$$C_p(X;\mathbb{C}^{\mu^{-1}}) \otimes C^p(X;\mathbb{C}^\mu) \to \mathbb{C}$$

which induces a non-singular pairing

$$\frown: H_p(X;\mathbb{C}^{\mu^{-1}}) \otimes H^p(X;\mathbb{C}^\mu) \to \mathbb{C}$$

Thus, the vector spaces $H_p(X;\mathbb{C}^{\mu^{-1}})$ and $H^p(X;\mathbb{C}^\mu)$ are dual.

Appendix D.3. ***Poincaré duality.*** Let X be an oriented connected compact manifold of dimension n. Then $H_n(X, \partial X)$ is isomorphic to $\mathbb{Z}$ and the orientation is a choice of the isomorphism, or, equivalently, the choice of a generator of $H_n(X, \partial X)$. We denote the generator by $[X]$.

Let $\mu : H_1(X) \to \mathbb{C}^\times$ be a homomorphism. There are the Poincaré-Lefschetz duality isomorphisms

$$[X] \frown: H^p(X;\mathbb{C}^\mu) \to H_{n-p}(X, \partial X;\mathbb{C}^\mu),$$

$$[X] \frown: H^p(X, \partial X;\mathbb{C}^\mu) \to H_{n-p}(X;\mathbb{C}^\mu)$$

Similarly to the case of non-twisted coefficients, there are non-singular pairings: the cup-product pairing

$$\smile: H^p(X;\mathbb{C}^\mu) \times H^{n-p}(X, \partial X;\mathbb{C}^{\mu^{-1}}) \to H^n(X;\mathbb{C}) = \mathbb{C}$$

and intersection pairing

$$\circ : H_p(X;\mathbb{C}^\mu) \times H_{n-p}(X, \partial X;\mathbb{C}^{\mu^{-1}}) \to \mathbb{C} \tag{D.35}$$

However, the local coefficient systems of the homology or cohomology groups involved in a pairing are different, unless $\operatorname{Im} \mu \subset \{\pm 1\}$.

Appendix D.4. ***Conjugate local coefficient systems.*** Recall that for vector spaces V and W over $\mathbb{C}$ a map $f : V \to W$ is called semi-linear if $f(a + b) = f(a) + f(b)$ for any $a, b \in V$ and $f(za) = \overline{z} f(a)$ for $z \in \mathbb{C}$ and $a \in V$. This notion extends obviously to fiber-wise maps of complex vector bundles. If ξ and η local coefficient systems of the type that we consider, then fiber-wise semi-linear bijection $\xi \to \eta$ commuting with all the transport maps is called a *semi-linear equivalence* between ξ and η.

For any local coefficient system ξ with fiber $\mathbb{C}$ on X there exists a unique local coefficient system on X which is semi-linearly equivalent to ξ. It is denoted by $\overline{\xi}$ and called *conjugate* to ξ. If $\xi = \mathbb{C}^\mu$, then $\overline{\xi}$ is $\mathbb{C}^{\overline{\mu}}$, where $\overline{\mu}(x) = \overline{\mu(x)}$ for any $x \in H_1(X)$.

Appendix D.5. ***Unitary local coefficient systems.*** A homomorphism $\mu : H_1(X) \to \mathbb{C}^\times$ is called *unitary* if $\operatorname{Im}\mu \subset S^1 = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. In S^1 the inversion $z \mapsto z^{-1}$ coincides with the complex conjugation: if $|z| = 1$, then $z^{-1} = \overline{z}$. Therefore if $\mu : H_1(X) \to \mathbb{C}^\times$ is unitary, then $\overline{\mathbb{C}^\mu} = \mathbb{C}^{\mu^{-1}}$ and there exists a *semi-linear* equivalence $\mathbb{C}^\mu \to \mathbb{C}^{\mu^{-1}}$.

This semi-linear equivalence induces semi-linear equivalence

$$H_k(X; \mathbb{C}^\mu) \to H_k(X; \mathbb{C}^{\mu^{-1}})$$

and similar semi-linear equivalences in cohomology and relative homology and cohomology.

Combining a semi-linear isomorphism

$$H_{n-p}(X, \partial X; \mathbb{C}^\mu) \to H_{n-p}(X, \partial X; \mathbb{C}^{\mu^{-1}})$$

of this kind with the intersection pairing (D.35) we get a *sesqui-linear* pairing

$$\circ : H_p(X; \mathbb{C}^\mu) \times H_{n-p}(X, \partial X; \mathbb{C}^\mu) \to \mathbb{C} \tag{D.36}$$

(Sesqui-linear means that it is linear on the first variable, and semi-linear on the second one.) This pairing is non-singular, because the bilinear pairing (D.35) is non-singular, and (D.36) differs from it by a semi-linear equivalence on the second variable.

Appendix D.6. ***Intersection forms.*** Let X be an oriented connected compact smooth manifold of even dimension $n = 2k$ and $\mu : H_1(X) \to \mathbb{C}^\times$ be a unitary homomorphism. Combining the relativisation homomorphism

$$H_{n-p}(X; \mathbb{C}^\mu) \to H_{n-p}(X, \partial X; \mathbb{C}^\mu)$$

with the pairing (D.36) for $p = k$ define sesqui-linear form

$$\circ : H_k(X; \mathbb{C}^\mu) \times H_k(X; \mathbb{C}^\mu) \to \mathbb{C} \tag{D.37}$$

It is called the *intersection form* of X.

If k is even, this form is *Hermitian*, that is $\alpha \circ \beta = \overline{\beta \circ \alpha}$. If k is odd, it is *skew-Hermitian*, that is $\alpha \circ \beta = -\overline{\beta \circ \alpha}$.

The difference between Hermitian and skew-Hermitian forms is not as deep as the difference between symmetric and skew-symmetric bilinear forms. Multiplication by $i = \sqrt{-1}$ turns a skew-Hermitian form into a Hermitian one, and the original form can be recovered. In order to recover, just multiply the Hermitian form by $-i$.

The intersection form (D.37) may be singular. Its radical, that is the orthogonal complement of the whole $H_k(X; \mathbb{C}^\mu)$, is the kernel of the relativisation homomorphism $H_k(X; \mathbb{C}^\mu) \to H_k(X, \partial X; \mathbb{C}^\mu)$. It can be described also as the image of the inclusion homomorphism

$$H_k(\partial X; \mathbb{C}^{\mu\, \mathrm{in}_*}) \to H_k(X; \mathbb{C}^\mu),$$

where in_* is the inclusion homomorphism $H_1(\partial X) \to H_1(X)$.

Appendix D.7. ***Twisted signatures and nullities.*** As well-known for any Hermitian form on a finite-dimensional space V there exists an orthogonal basis in which the form is represented by a diagonal matrix. The diagonal entries of the matrix are real. The number of zero diagonal entries is called the *nullity*, and the difference between the number of positive and negative entries is called the *signature* of the form. These numbers do not depend on the basis.

For a skew-Hermitian form by nullity and signature one means the nullity and signature of the Hermitian form obtained by multiplication of the skew-Hermitian form by i.

For a compact oriented $2k$-manifold X and a homomorphism $\mu : H_1(X) \to \mathbb{C}$ the signature and nullity of the intersection form

$$\circ : H_k(X; \mathbb{C}^\mu) \times H_k(X; \mathbb{C}^\mu) \to \mathbb{C}$$

are denoted by $\sigma_\mu(X)$ and $n_\mu(X)$, respectively, and called the *twisted* signature and nullity of X.

The classical theorems about the signatures of the symmetric intersection forms of oriented compact $4k$-manifolds are easily generalized to twisted signatures:

Theorem D.A. Additivity of Signature. *Let X be an oriented compact manifold of even dimension. If A and B are its compact submanifolds of the same dimension such that $A \cup B = X$, $\operatorname{Int} A \cap \operatorname{Int} B = \varnothing$ and $\partial(A \cap B) = \varnothing$, then for any $\mu : H_1(X) \to \mathbb{C}^\times$*

$$\sigma_\mu(X) = \sigma_{\mu\,\mathrm{in}_*}(A) + \sigma_{\mu\,\mathrm{in}_*}(B)$$

where in *denotes an appropriate inclusion.*

Theorem D.B. Signature of Boundary. *Let X be an oriented compact manifold of odd dimension. Then $\sigma_{\mu\,\mathrm{in}_*}(\partial X) = 0$ for any $\mu : H_1(X) \to \mathbb{C}^\times$.*

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